**Linear Algebra**
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**Lecture 7.3 - Eigenvectors and eigenvalues**

So, when we discussed diagonal matrices, we observed that given a diagonal matrix A if we consider the corresponding linear transformation L A, then L A dilates each of the vectors in the coordinate basis in the standard basis, and it dilutes it by the corresponding value in the diagonal. So, in this video, we will explore this phenomenon in much greater detail for in much greater depth for an arbitrary linear transformation from a vector space to itself. So, let us begin by considering a linear operator T on a vector space V.

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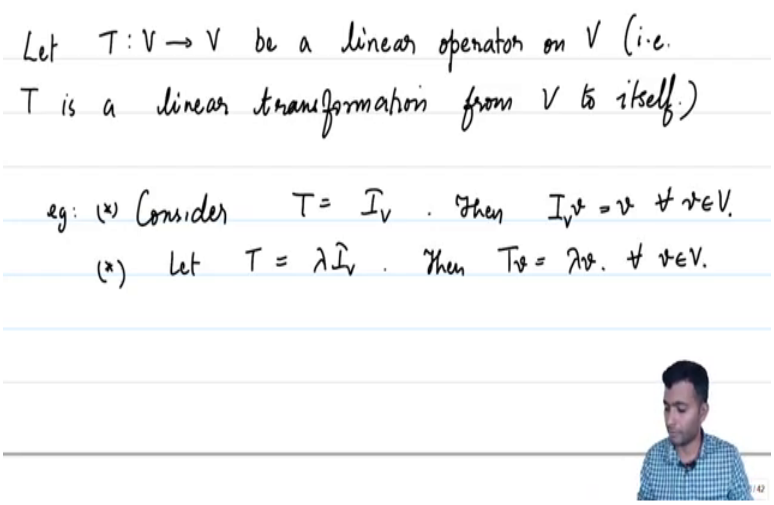


So let T from V to itself be a linear operator on V. Recall that a linear operator is just a linear operator on V is just a linear transformation from V to itself. That is, T is a linear transformation from V to itself. So, for example, T being the identity map is the simplest example. So for example, consider T equal to the identity map of V, then I v of V is equal to V for all v in capital V. Next simple example will be a multiple of lambda I v. So, this is one example, let us consider another example. Now, let T be equal to lambda times I v, then T v is equal to Lambda I v on V acting on V, which is equal to lambda times v.

So, notice that our first example where the linear methods, linear operator is the identity map, it dilates the vector space, every vector in the vector space by 1, or in other words it leaves it fixed. And the second example, every vector is dilated by lambda.

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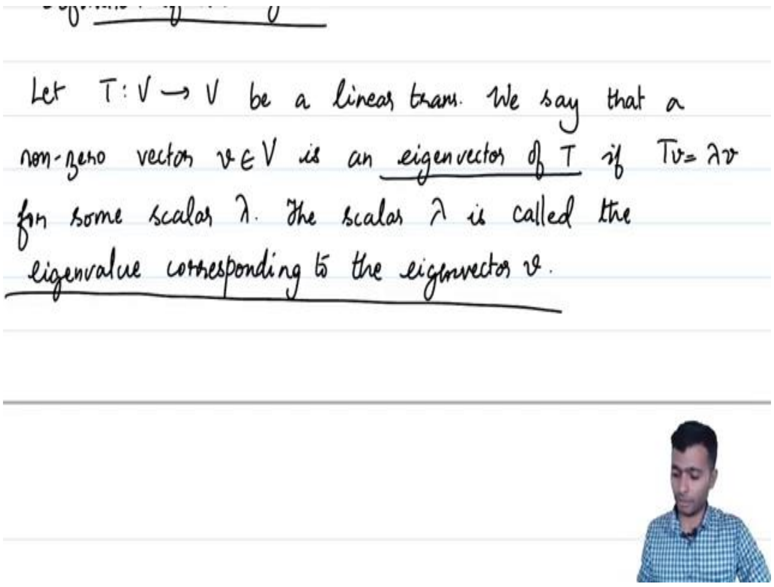


So, if we consider an arbitrary linear transformation, this need not be the case, it need not dilate every vector. However, there are some special vectors in the vector space, which might get dilated by a given linear transformation and such vectors have a special name, they are called the eigenvector. So, let me now give a definition, this is the definition of an eigenvector. So, let T from V to itself be a linear operator on V or a linear transformation from V to itself. We say that a nonzero vector, notice that we are imposing a condition of the vector being nonzero. Nonzero vector v in capital V is an Eigenvector of T if T v is equal to lambda v for some scalar lambda.

So, if the vector v is getting dilated by some lambda, then V is set to be an eigenvector provided V is not the 0 vector, the scalar lambda is called the eigenvalue corresponding to v, it is called the so, let me just underline this eigenvector of T, this is called the eigenvalue corresponding to the eigenvector V. So we have defined two objects here, one is the notion of an eigenvector, which is basically a nonzero vector in the given vector space, which is dilated by our linear operator, and the second one is the eigenvalue, which is the degree to which it is getting dilated. Or in other words, it is the scalar lambda such that T v is equal to lambda times V.

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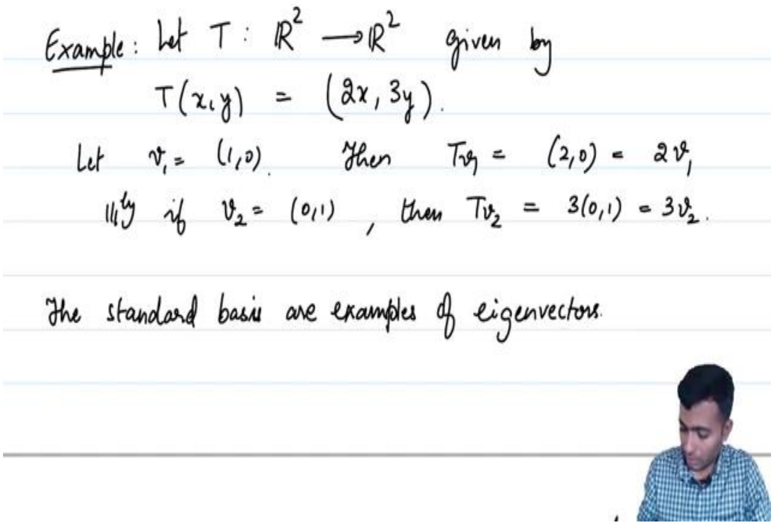


So let us look at a few examples, so maybe a good example would be, let us consider some simple example say from R 2 itself. Let T be not given by, T of say x, y is equal to 2 x and 3 y. So we have already seen some examples here. We will come back to that maybe, but let us just focus on this particular example, T of x, y is 2 x, 3 y. You notice the coordinate basis are eigenvectors, so let v 1 be equal to 1, 0 then T v 1 is equal to T of 1, 0 which is 2, 0, which is equal to 2 times v 1. Similarly, if v 2 is equal to 0, 1, then T v 2 is equal to 0, 3 which is 3 times 0, 1 which is equal to 3 v. So the coordinate basis here are eigenvectors so, the standard basis are examples of eigenvectors.

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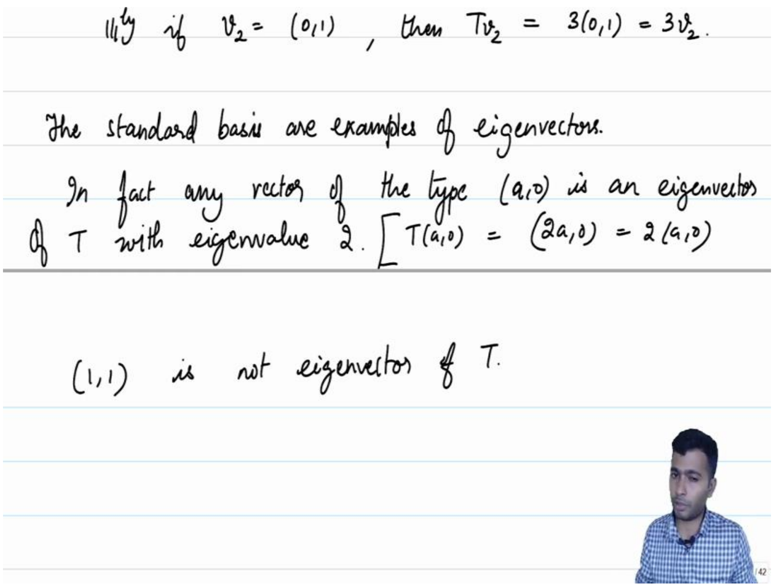

You will observe carefully is a linear map, there are plenty of examples of eigenvectors. Say any vector, in fact any of the type a, 0 is an eigenvector of T. In fact, any vector of the type a, 0 is an eigenvector of T with eigenvalue 2. Also notice that not every vector is an eigenvector of T. So, 1, 1 for example, is not an eigenvector of T. So, why is a, 0 is an eigenvector? So let me just put it in square brackets why this is the case, T of a, 0, this is just 2 a, 0 which is just two times a, 0 that is all. And why is 1, 1 not an eigenvector? T of 1, 1 is two times 1, 2 and three times 1, 3 so it is 2, 3, which is not a scalar multiple of 1, 1.

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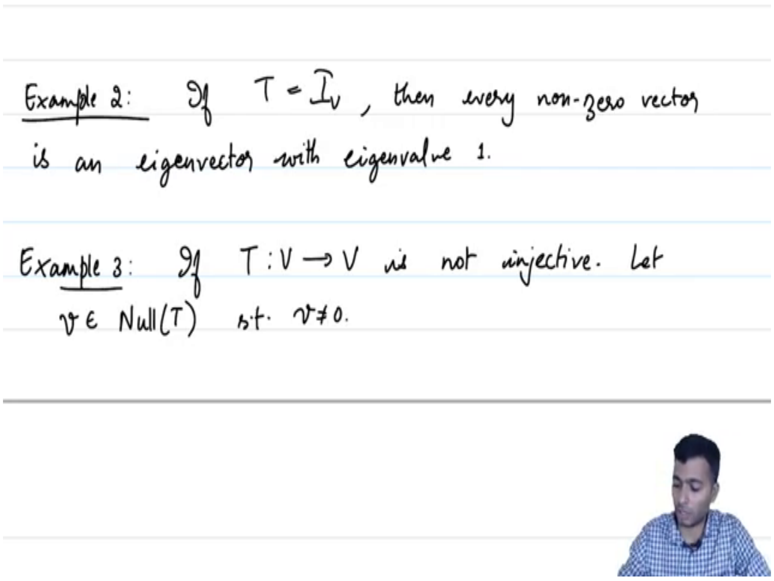


Let us look at more examples. Let us put numbers. This is example 2, if T is the identity map that is the first example maybe we should have considered, then every vector is an Eigen, every nonzero vector. Notice that the definition of an eigenvector we have imposed this condition, every nonzero vector is an eigenvector with eigenvalue 1. Similarly, with the dilation lambda times I v, the eigenvalue there will just turn out to be lambda. So, every vector, another example, every vector in the null space of our given linear transformation T which is if T from V to V is not injective, what happens then the null space of T has nonzero vectors. Let V be in N of T or the null space of T or maybe let me write null of T so that there is no confusion such that v is not equal to the 0 vector so, the 0 here is the 0 vector.

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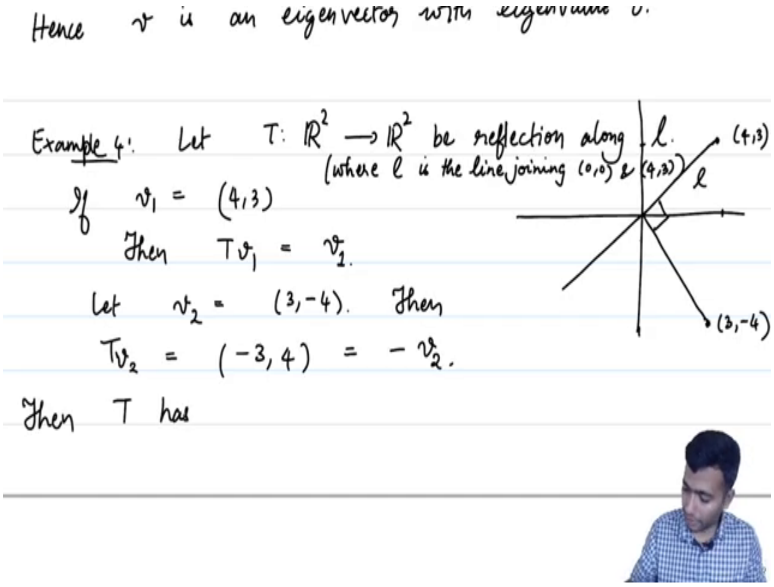


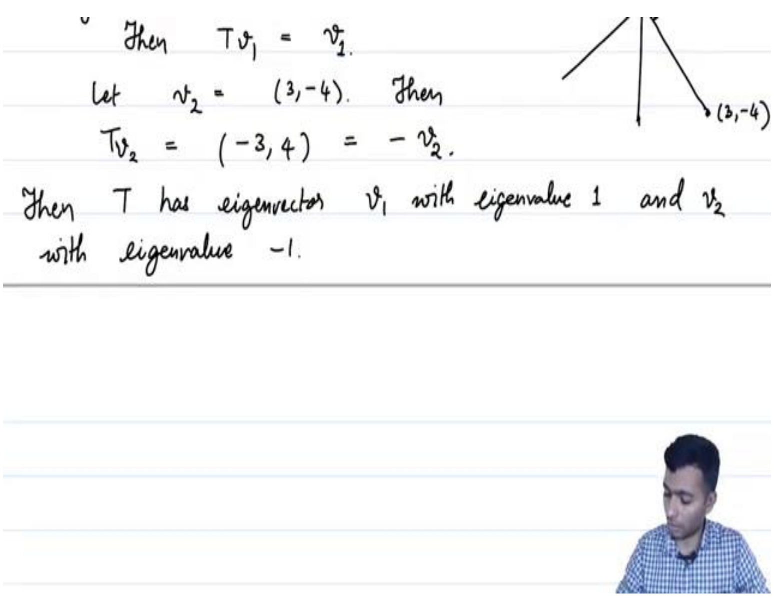


And what does T do to our V? Then T v is equal to 0, the 0 vector of v that is nothing but the scalar 0 times our vector v. So again, your job is to keep track of which 0 is where. So this 0 is in the vector space V, this 0 is in the real numbers, it is a scalar. So that means hence v which is a nonzero vector is hence an eigenvector and v is an eigenvector and what is the corresponding eigenvalue, eigenvector with eigenvalue 0? We have only demanded that the eigenvector should be nonzero. We have not demanded that the eigenvalue cannot be the 0 scalar, we have not demanded that at all, so yeah, so V is an Eigenvector with eigenvalue 0.

Maybe a good exercise to think over would be to show that a vector or linear transformation is invertible if all eigenvalues are nonzero. Let us look at more examples, so next would be maybe example 4, I think. So consider this linear map, let T be the map from R 2 to R 2, which is given by a reflection along a line. So let me just draw it for you, suppose this is our Cartesian coordinates. So let this be 4 and let this be 3, so this is our 4, 3 and let us look at the line joining the points. Let us draw the line and then this is our point 4, 3 and let us look at the reflection along this particular line, which is joining 0 and 4, 3. So, any point here is mapped to a point corresponding point here.

So in particular, let us look at this line, so this line so 4, 3. 3 minus 4 would be a perpendicular so this will be going like this. This point, this turns out to be 3 minus 4, this is perpendicular. And what will T do to 4, 3? Notice that if v 1 is equal to 4, 3, then T v 1, if you reflect the vector 4, 3 along the line joining 0 to 4, 3 it does not do anything to it, it just fixes it so, this is equal to our v 1. What about the vectors, say v 2. So, let v 2 be the perpendicular vector which is 3 minus 4, and if you reflect it, it will go to the other direction, the other direction it will just be minus 3, 4. Then T v 2 is equal to minus 3, 4, which is equal to minus of 3, minus 4 which is minus of v 2.

So v 1 and v 2 then this let T be a reflection along l, where l is the line joining where l is the line joining 0, 0 and 4, 3. So then T has Eigenvectors v 1 and v 2, notice that v 1 and v 2 both are nonzero as eigenvectors v 1 with eigenvalue 1 and v 2 with eigenvalue minus 1. So let us come back to this example later. We will revisit this example, so example 2 have at the back of your mind while studying eigenvalues and eigenvectors. So next, so we have defined what an eigenvector and an eigenvalue is for a linear transformation. So, linear transformations and matrices are very closely related and you would like to define a corresponding or similar notion for matrices as well so, let us do that next.

So, we do not consider 0 vector to be the eigenvector, always keep that in mind because in the definition itself, we are incorporating that an eigenvectors should be a nonzero vector, because there is no eigenvalue which can be associated to the 0 vector, every scalar will turn out to be an eigenvalue and we do not want that. So let us now look at what is meant by the notion of an eigenvector and eigenvalue for a matrix for an n cross n matrix. So let us start with an n cross n matrix.

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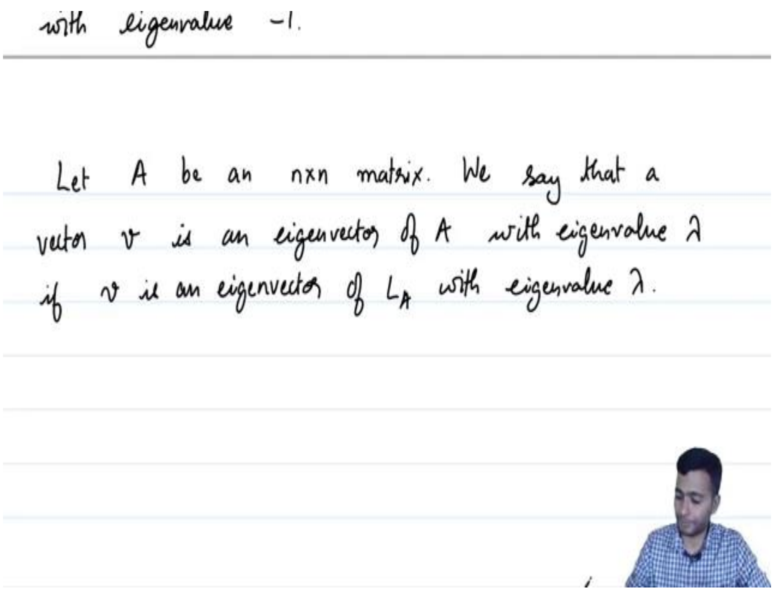


Let A be an n cross n matrix with real entries, of course, we say that vector V in R n is an eigenvalue of A, sorry, eigenvector of A with eigenvalue lambda if v is an eigenvector of the linear transformation L A with eigenvalue lambda. So, if V is an eigenvector of L A with eigenvalue lambda. So, for all practical purposes, we do not distinguish between the matrix A and the linear operator L A.

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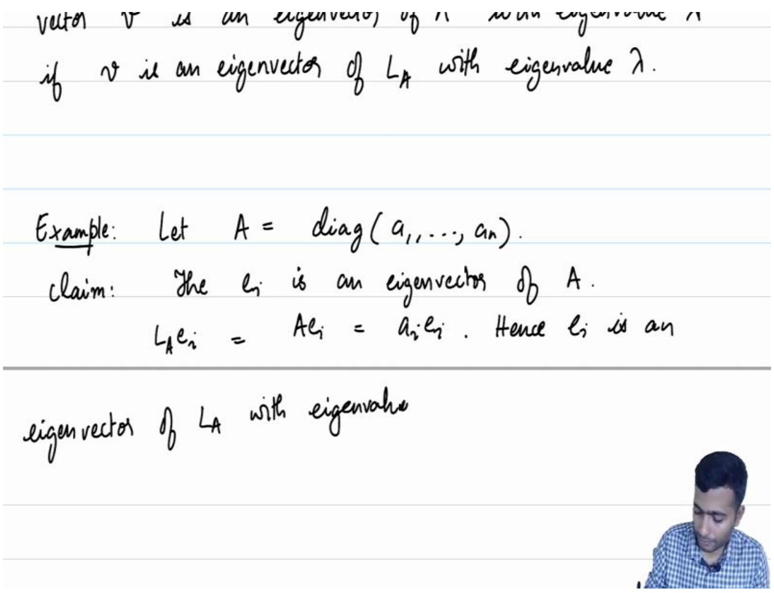


Yes, I should probably introduce one more example which is something which you have already seen. Let us consider a matrix, so this is at a good place we will be looking at this example. So, let A be a diagonal matrix say a 1 to a n, then my claim is that each of the eigenvectors of A or each of the standard basis vectors is an eigenvector of A, then e i, let us just see what L A does to e i, then my claim is then e i is an eigenvector of A.

So we should check that it is an eigenvector of L A, so L A e i if you notice, this is just A e i, which is equal to, we have already done this, this is going to be a i times e i. Yes, it is indeed the eigenvector of A with so hence, e i is an eigenvector of L A with eigenvalue a i.

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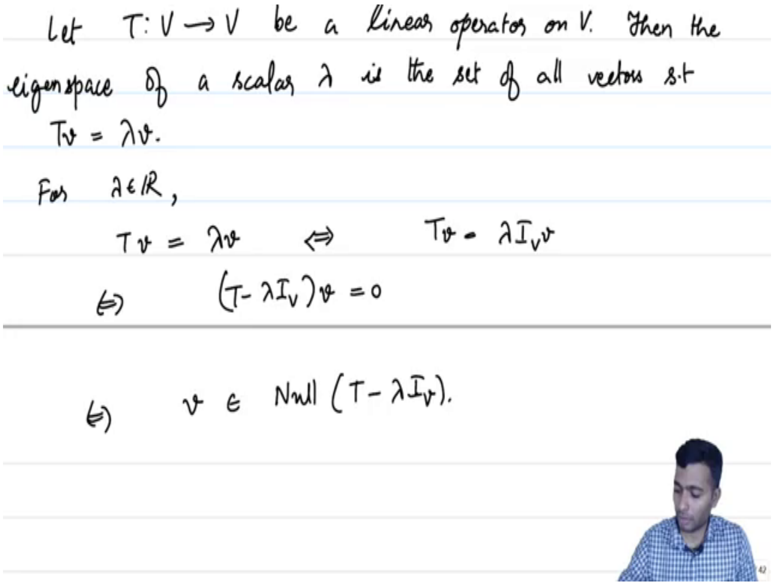


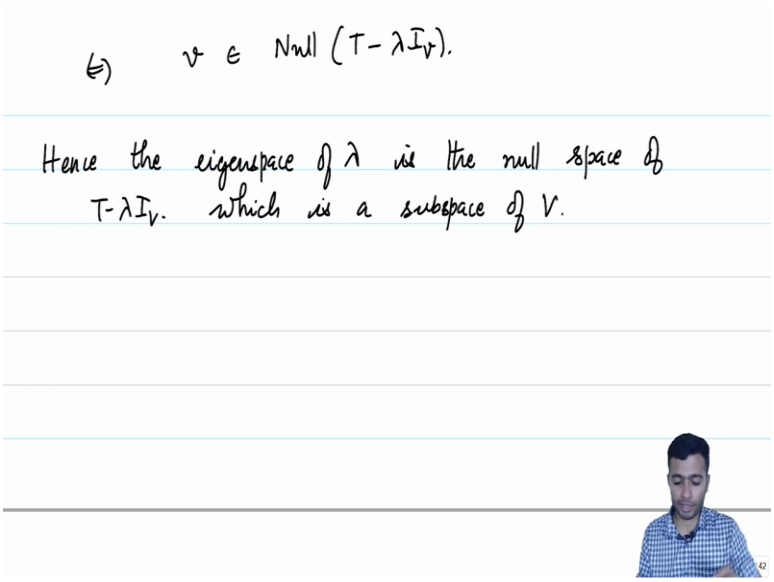

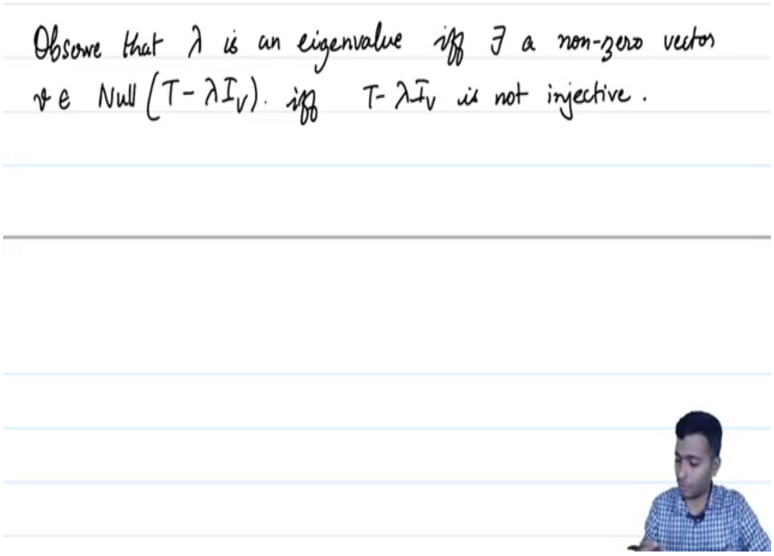


Next let us give ourselves a definition of what is meant by the Eigen space corresponding to lambda. So let, definition of an Eigen space, we have already seen what an eigenvector is and what an eigenvalue is, let us look at what an Eigen space is. So, let T from V to itself be a linear operator, I will slowly start using this term more frequently operator on V that means it is a linear transformation from V to itself, then Eigen space of a scalar lambda is the set of vectors such that T v is equal to lambda v. So, notice that every eigenvector of T with eigenvalue lambda is in the Eigen space of lambda apart from 0, 0s are obviously there, but every eigenvector corresponding to the eigenvalue lambda or every eigenvector which has eigenvalue lambda will also be in the Eigen space of T. So, let us try to see more about the Eigen space of lambda say for example.

So, if lambda is a scalar so, for lambda in R let us see what it means to say that T v is equal to lambda v. T v is equal to lambda v can be rewritten as this is if and only if T v is equal to lambda I v v. And by the operation of linear transformations, the vector addition of linear transformations this is if and only if T minus lambda I v of v is equal to 0. So, I e this if and only if v belongs to the null space of T minus lambda I v.

So, v is hence, v is in the Eigen space of lambda. So, rather Eigen space of lambda is just the null space. So, let me write it in a more refined manner, the Eigen space of lambda is the null space of T minus lambda I v, so in particular, the Eigen space of lambda is a subspace of V. So lambda is an eigenvalue if there is at least one nonzero vector in the null space of T minus lambda I v. So, also observe that lambda is an eigenvalue if and only if there exists a nonzero vector v in the null space of T minus lambda times I v. But this is the same as telling that T minus I v is not injective. So this is if and only if T minus lambda I v is not injective.

So lambda is an eigenvalue of our given linear transformation T if and only if T minus I v is not injective, or if T minus lambda I v is not invertible, that is an alternate definition we can keep to check whether something is an eigenvalue. This is at times useful, for example, let us consider one of the examples we already looked into.

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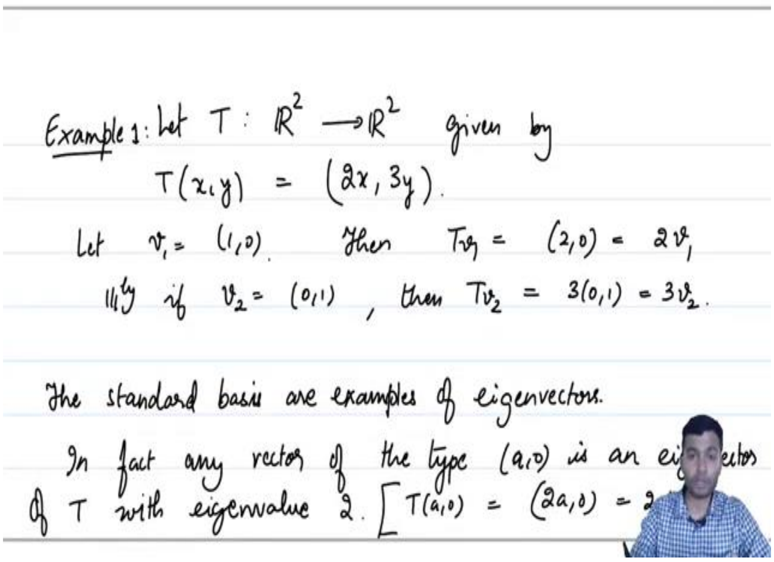


Let us maybe consider the first example that might be, let me put a number. So recall that the first example was T of x, y equal to 2 x, 3 y so, let us come back to this example. So, consider example 1 again revisited.

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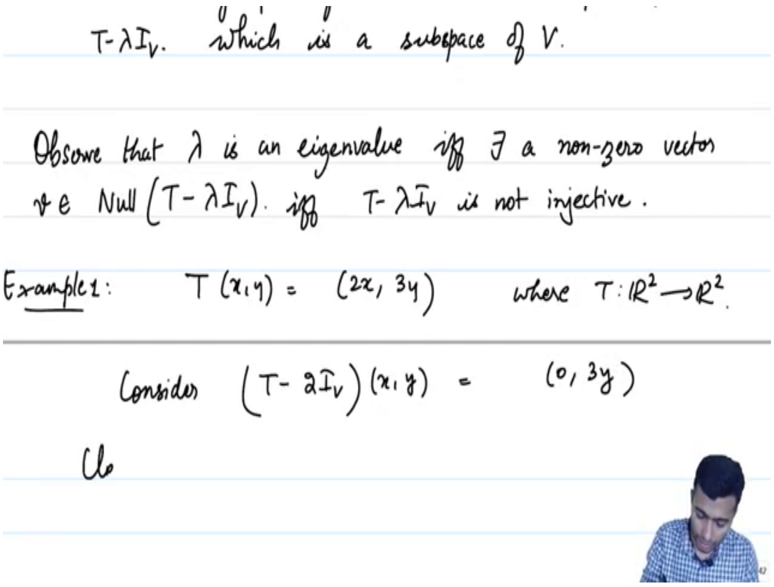


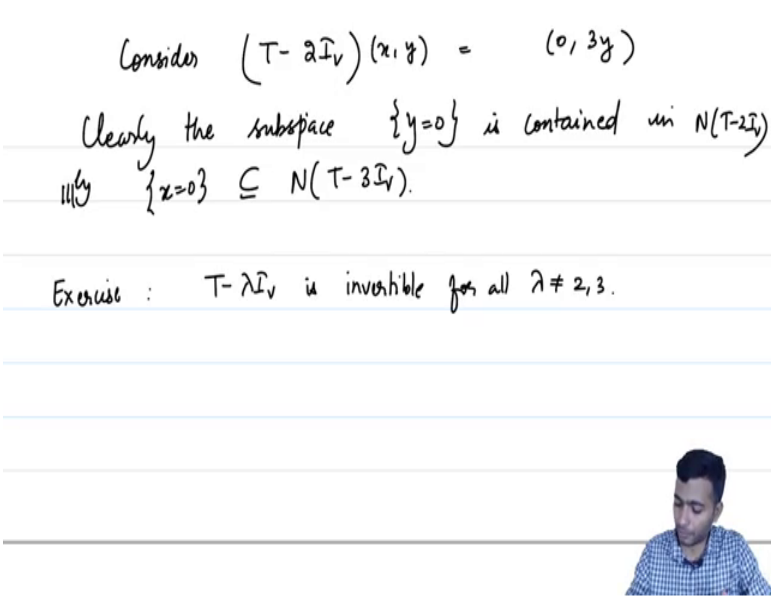


So, T of x, y is equal to 2 x, 3 y. So, we know that both 2 and 3 are eigenvalues of T and that is quite straightforward because consider T minus 2 times, so this is where T is from R 2 to R 2. So consider T minus 2 times I v, and we would like to see whether its Eigen space or rather its null space is just the 0 vector or there are more. But we already know that if you consider T minus 2 I v of say x, y, this is just going to be equal to 0, 3 y.

And clearly, the x axis or the subspace, let me put it like this. The subspace y equal to 0, which is a one-dimensional subspace is contained in the null space right here, and now T minus 2 I v. Similarly, x is equal to 0 is contained in the null space of T minus 3 times the identity map. So yes, this also tells us that 2 and 3 are eigenvalues. This also tells us that T does not have any other eigenvalue, why is that the case? Because consider T minus lambda I v, let me just leave it as an exercise for you to check that T minus lambda I v is invertible for all lambda, which is not equal to 2 or 3 and therefore, it cannot be not injective, it has to be therefore injective because it is invertible therefore, the null space of T minus lambda will just have the 0 vector, therefore it cannot be a eigenvalue.

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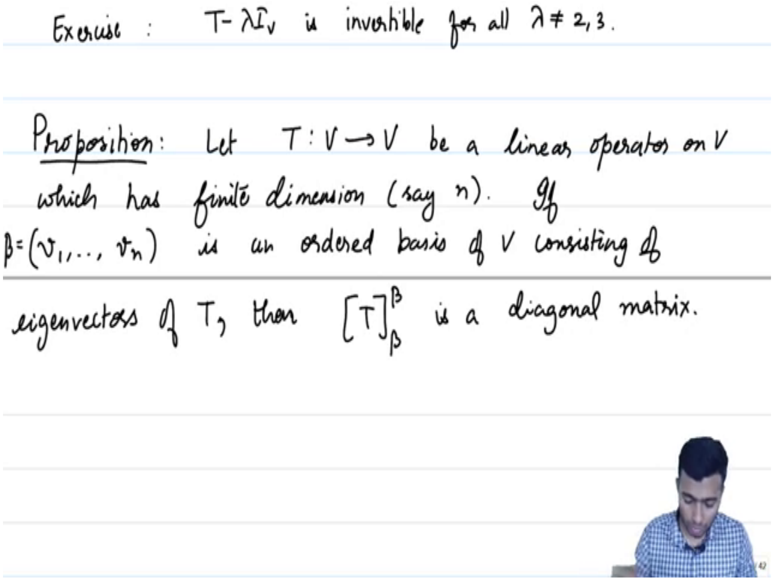


Next let us discuss the relationship between diagonal matrices and eigenvectors. So we have already seen that if we have a diagonal matrix, the coordinate base is turned out to be eigenvectors or other words the matrix of the linear transformation corresponding to it is a diagonal matrix. So let us make it more formal here, so let us put it into a theorem, maybe a proposition. This proposition states that linear operator on a vector space V is having a diagonal matrix if we have a basis, which consists of eigenvectors. So let us start with a linear map from V to itself. So let T from V to V be a linear operator on V, which is of dimension n, let us say which has finite dimension let us say n. Then if v 1 to v n is an ordered basis of V consisting of eigenvectors of T.

So let us call it beta. Beta equal to v 1, v 2, up to v n and ordered basis of V consisting of eigenvectors of T then, so let us remove this then. What do we have as a conclusion? Then the matrix of T with respect to the basis beta will be a diagonal matrix, then the matrix of T with respect to beta is a diagonal matrix, the converse is also true, I write it down.

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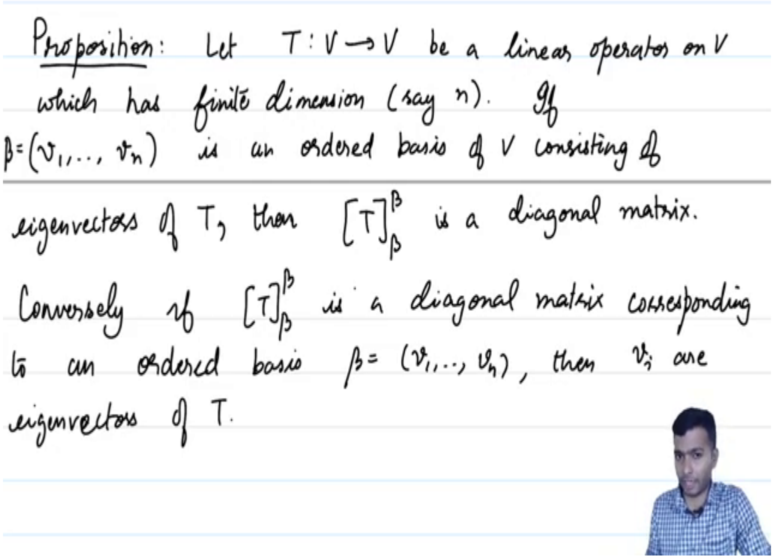


Conversely, if the matrix of a linear transformation corresponding to a basis beta is diagonal, then the basis vectors in beta are eigenvectors of T. Conversely, if the matrix of linear transformation the matrix of T beta is a diagonal matrix corresponding to an ordered basis beta, which is say v 1 to v n, then v i are eigenvectors of T. So, the proposition tells us that if we have a linear operator with a basis of eigenvectors of T, then with respect to this basis the matrix of the linear transformation will be a diagonal matrix. In fact, we will see that the matrix will have as its diagonal entries the eigenvalues. And converse is also true that if you have a matrix which is a diagonal matrix with respect to some basis, then the vectors in the basis will be eigenvectors of T.

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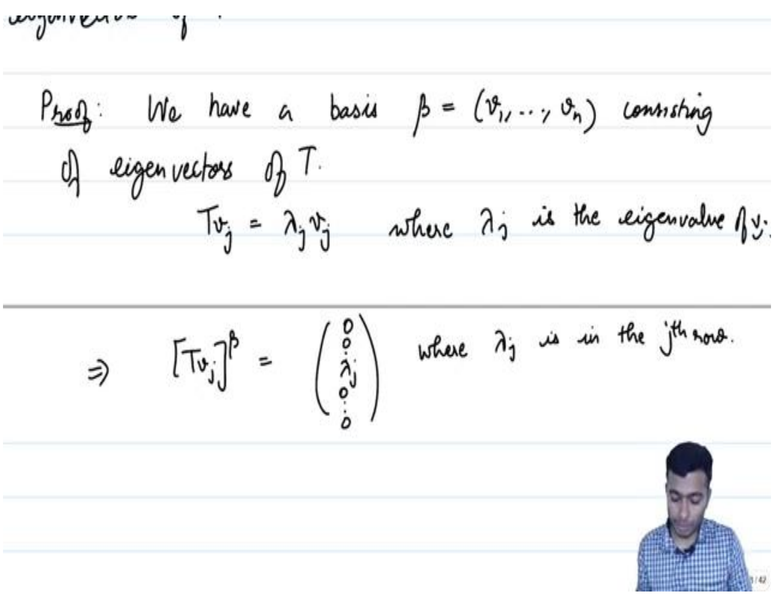

Let us give a quick proof of this. It is going to be actually quite short. So let us see what the first statement says, the first statement says that we have a basis consisting of eigenvectors of V. So, given we have a basis beta which is say v 1 to v n consisting of eigenvectors of T. Let us see what is the matrix of T with respect to beta, but to do that, we have to look at what is T of v j. So, what is T of v j? T of v j is some lambda j times v j, where lambda j is the eigenvalue of v j.

Remember that each of the v j's are eigenvectors of T so, what does that mean? This implies that T v j beta is just equal to 0 0, there is a lambda j in the jth column 0 dot dot dot 0, where lambda j is in the jth row. But T v j beta will just turn out to be the jth column of the matrix of T.

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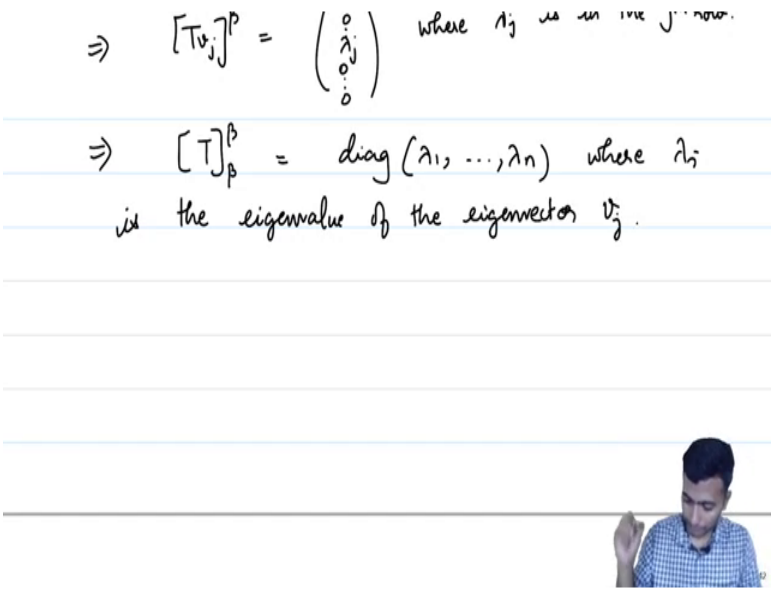


And putting this placing this together, we have T beta beta will just turn out to be the diagonal matrix of lambda 1, lambda 2 up to lambda n, where lambda i is the eigenvalue of the eigenvector v j. Let us next prove the converse to this proposition.

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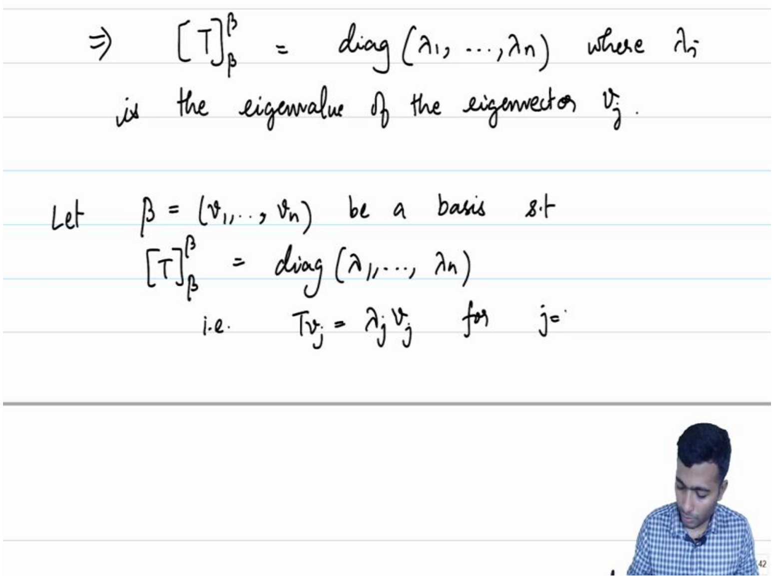


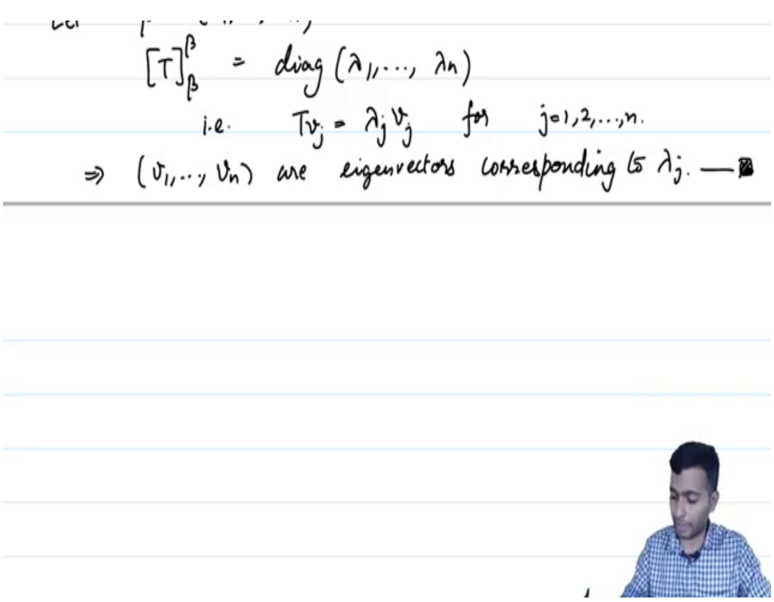


The converse is telling us that if we have a diagonal matrix, so, let beta equal to v 1 to v n be a basis such that T beta is a diagonal matrix. The basis such that you just actually go back in the previous argument and we will get it as equal to say diagonal of lambda 1 to lambda n. But what does that mean? By very definition this just implies, let me leave it for you to check that T v j is then equal to lambda j v j for j equal to 1 to n. This just tells us that v 1 to v n are eigenvectors corresponding to lambda j so we have proved the result. So, we have observed that any linear transformation, if it has a matrix, which is diagonal then there is a basis consisting of the eigenvectors and vice versa, there is a basis consisting of eigenvectors of our given linear transformation the matrix is also a diagonal matrix.

So, this motivates definition of that of diagonalizability. So, we say that a linear transformation is diagonalizable if we can get hold of a basis with respect to which the matrix of T is a diagonal matrix.

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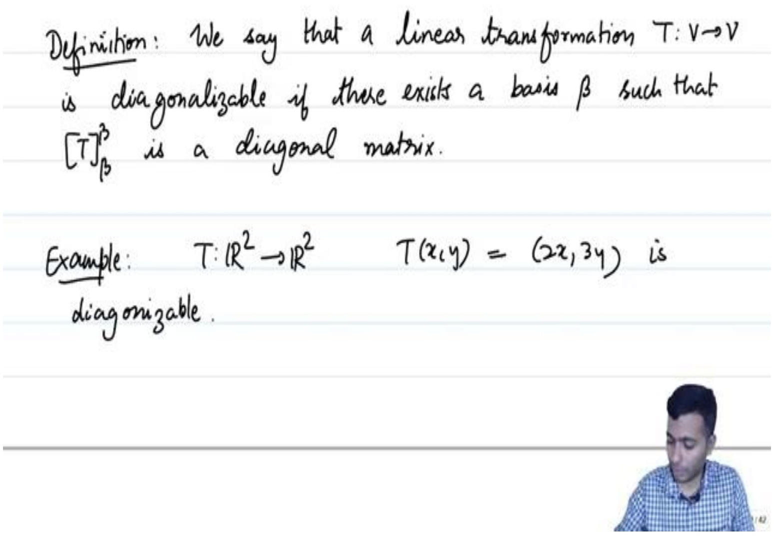


So, let us give a definition, we say that a linear transformation, linear transformation T from V to itself is diagonalizable if there exists a basis beta with respect, such that the matrix T beta beta is a diagonal matrix. So, one of the most straightforward examples is the linear transformation corresponding to a diagonal matrix, they have to be diagonalizable. So, example in fact, example one is diagonalizable, T from R 2 to R 2 such that T of x, y is equal to say 2 x and 3 y, this is diagonalizable by the very definition, why? Because what will be our beta here? Our beta will just turn out to be the standard basis.

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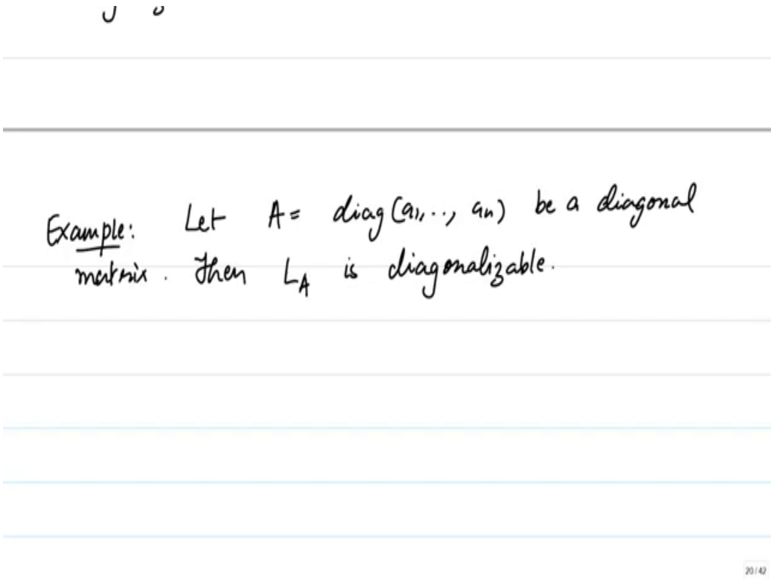


In fact, let us start with a diagonal matrix. So, let A equal to diagonal of a 1 to a n be a diagonal matrix then L A is diagonalizable, again with respect to the standard basis of R n. It is an n cross n matrix which is a diagonal matrix, so with respect to the standard basis, the matrix of L A is a diagonal matrix and by definition, this is going to be a diagonalizable linear transformation. So let us look at one more example we had considered.

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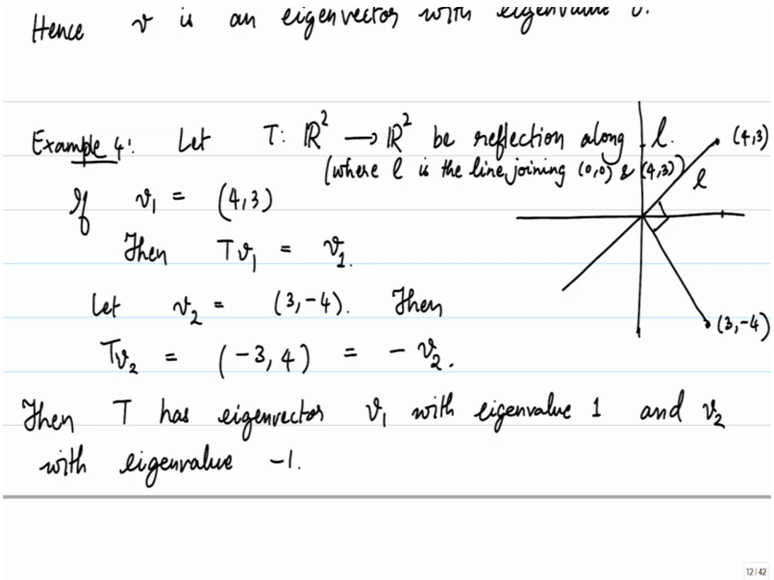

Example: Let $A = \text{diag}(a_1, \ldots, a_n)$ be a diagonal matrix. Then $L_A$ is diagonalizable.

Let us revisit example 4 above.

$T: \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection along $l$ which is the line joining 0 and (4, 3)

Recall that (4,3) & (3, −4) are eigenvectors

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the line joining 0 and (4,3)

Recall that (4,3) & (3, −4) are eigenvectors
with eigenvalues 1 and −1 respectively.

Check that (4,3) & (3, −4) are linearly independent.

Let $\beta = ((4,3), (3,-4))$.

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Let us revisit one of the examples which we had promised to revisit, which is this example 4, which is basically the reflection along the line joining 0 to 4, 3. So I write it down, so let us consider, let us revisit example 4. What was our T? T was a map from R 2 to itself given by reflection along l, which is the line joining the origin to 4, 3. Let me not write 2, which is the origin line joining 0 and 4, 3 so infinite line, so we do not want to consider this segment, it is a line and you reflect every vector along this particular line. So we had noticed that we had two eigenvectors for this linear map T.

So, recall that 4, 3, the vector 4, 3 and 3, minus 4 are eigenvectors with eigenvalues 1 and minus 1 respectively. But we also know that or I will leave it as an exercise for you to check that 4, 3 and 3 minus 4 are linearly independent. What can we say about set of two vectors in

R 2 which are linearly independent, it should necessarily be a basis. So, let Beta be equal to set 4, 3 and 3, minus 4.

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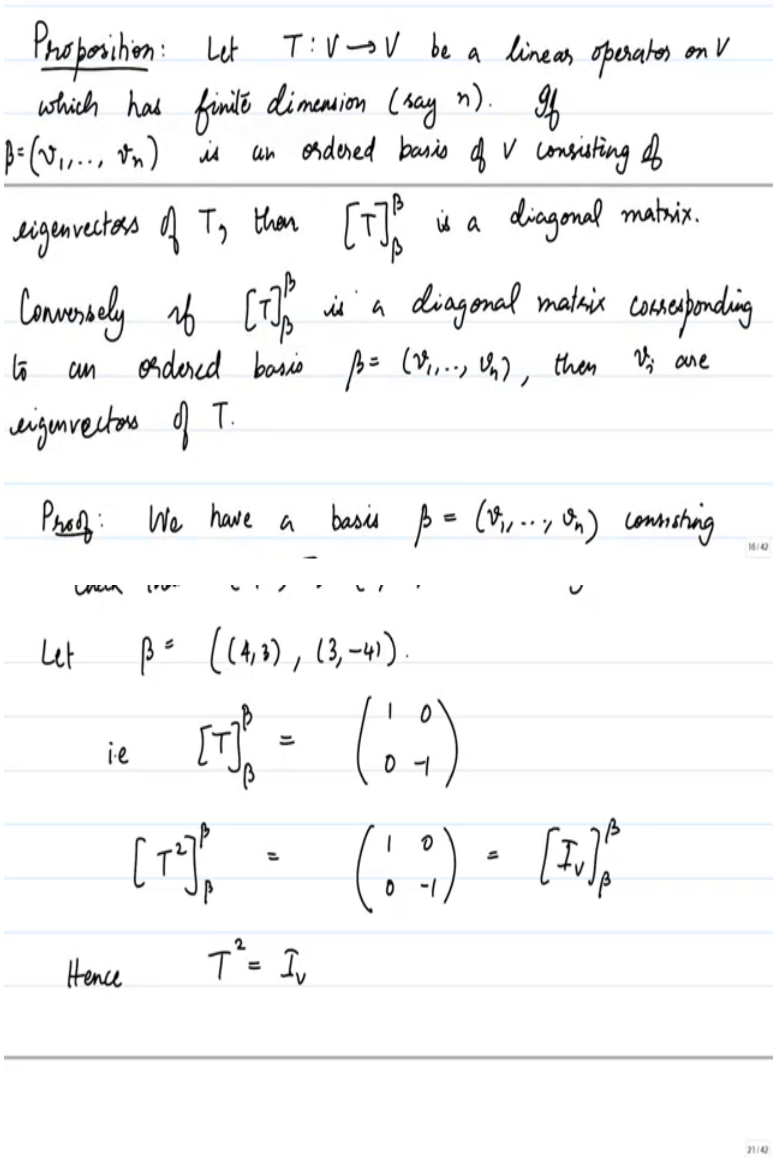


Let us try to see or let us just jump up to look at what we did to as a proposition, we have obtained a basis of T which has eigenvectors and which has every vector as an eigenvector.

So this means that T with respect to beta is equal to 1, 0, 0, minus 1. So, this particular form is quite nice because if you now consider T square, what is going to be T square? If you notice, this is going to be again beta with respect to beta, this will just turn out to be the product of this matrix with itself which is going to be the identity matrix, which is the identity matrix of the identity with respect to the basis beta. And hence we have obtained hence the matrix, the linear transformation T when multiplied by with itself will give you back the identity. So it is an inverse of itself that is what we have just proved. So, if we can get hold of basis which has eigenvectors then it is quite useful as you can notice, we can say a lot more than what meets the i directly.

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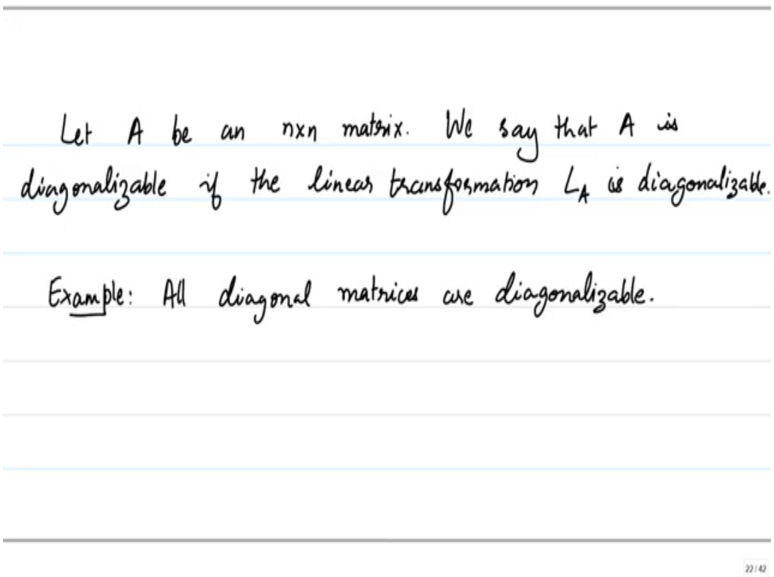


So, we have just defined what is meant by diagonalizable for a linear transformation, we would also like to do the same for a matrix. So let A be an arbitrary n cross n matrix, let A be an n cross n matrix. So we will say that A is diagonalizable if the corresponding linear transformation is diagonalizable, so we say that A is diagonalizable if the linear transformation L A is diagonalizable. So, notice that A to begin with need not be a diagonal matrix, A could be some arbitrary matrix. And what is L A? L A is the linear transformation corresponding to A. So, if you look at the standard basis and look at the matrix of L A with respect to the standard basis, we will get back A, but A need not be a diagonal matrix to begin with.

However, if you consider the linear transformation L A and if we could get hold of some basis of R n with respect to which our linear transformation L A is a diagonal matrix, then we say that A is also diagonalizable or then we say that A is diagonalizable. So, needless to say, example, all diagonal matrices are already diagonalizable with respect to the standard basis you look at the matrix of L A, all diagonal matrices are diagonalizable.

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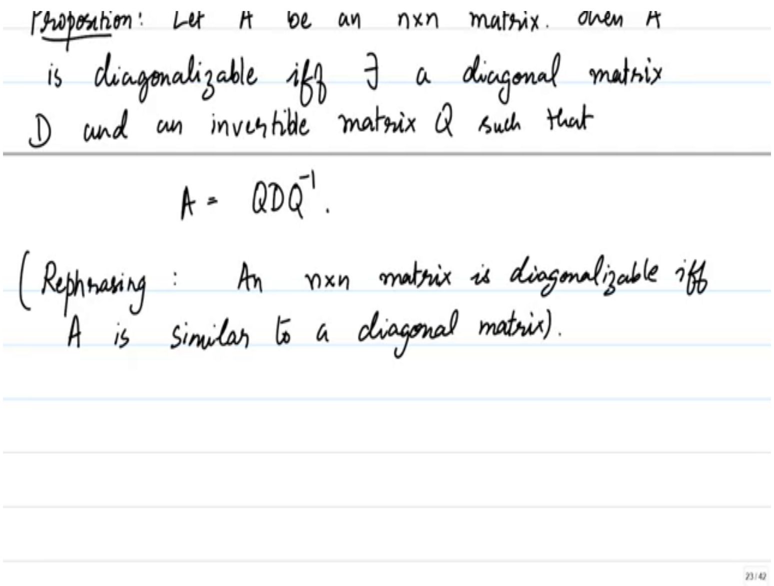


So, let us now look at a necessary and sufficient condition on when we can say that a matrix is diagonalizable, let us capture in the next proposition. So proposition, so let A be an n cross n matrix, then A is diagonalizable if and only if we can get hold of our diagonal matrix D and an invertible matrix Q such that A is Q D Q inverse, if and only if there exist a diagonal matrix D and an invertible matrix Q such that A is equal to Q D Q inverse.

So, notice that A is equal to Q D Q inverse tells us that A is similar to D, but D is a diagonal matrix, so this is rephrasing: an n cross n matrix is diagonalizable if and only if it is similar to a diagonal matrix, diagonalizable if and only if A is similar, we call the definition of similar we say that two matrices are similar, A and B are similar if A is equal to something like Q B Q inverse where Q is some invertible matrix, so A is similar to a diagonal matrix. So, this is a good characterization to keep in mind so, let us give a proof of this proposition.

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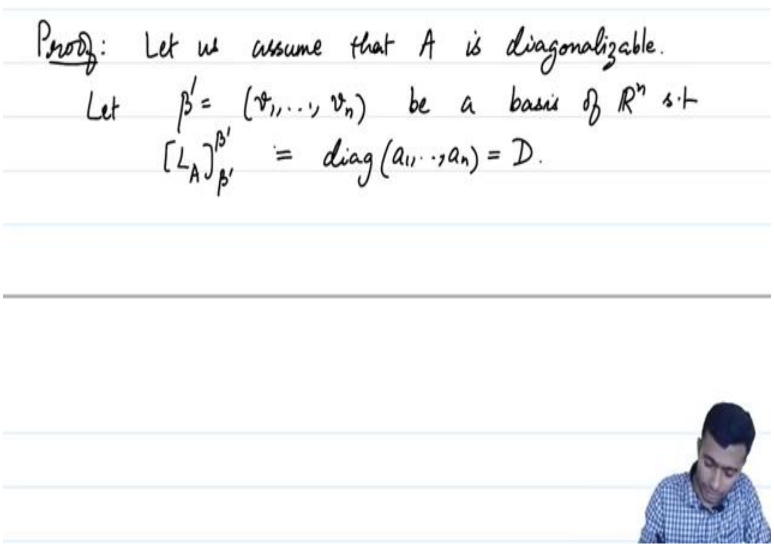


So, suppose A is diagonalizable. I have already stated the proposition, I was writing proof. So, let us look at a proof of the statement. So, we have already assumed let us assume that A is diagonalizable. So, let us assume that A is diagonalizable, what does that mean? That means, that the matrix L A that matrix is diagonal with respect to some basis. So let beta equal to say v 1 to v n be a basis, let me call it beta prime. Beta let us keep it for the standard basis so, let beta prime be a basis of R n such that L A beta prime beta prime is equal to a diagonal matrix or let us say this is diagonal of a 1 to a n, let us call this D.

So, we have assumed that A is diagonalizable by definition, L A is diagonal matrix. L A is a linear transformation, which has a diagonal matrix with respect to some basis, let us call that beta prime. So with respect to beta prime L A has the matrix representation given by D.

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$$[L_A]_{\beta'}^{\beta'} = diag(a_1, \dots, a_n) = D.$$

$$L_A = I_{\mathbb{R}^n} L_A I_{\mathbb{R}^n}$$ where $I_{\mathbb{R}^n}$ is the identity linear trans. in $\mathbb{R}^n$.

Let $\beta$ be the standard basis of $\mathbb{R}^n$.

$$A = [L_A]_\beta^\beta = [I L_A I]_\beta^\beta = [I]_{\beta'}^{\beta} [L_A]_{\beta'}^{\beta'} [I]_\beta^{\beta'}$$

$$L_A = I_{\mathbb{R}^n} L_A I_{\mathbb{R}^n}$$ where $I_{\mathbb{R}^n}$ is the identity linear trans. in $\mathbb{R}^n$.

Let $\beta$ be the standard basis of $\mathbb{R}^n$.

$$A = [L_A]_\beta^\beta = [I L_A I]_\beta^\beta = [I]_{\beta'}^{\beta} [L_A]_{\beta'}^{\beta'} [I]_\beta^{\beta'}$$

Let $Q = [I]_{\beta'}^{\beta}$, then $Q^{-1} = [I]_\beta^{\beta'}$

Hence $A = QDQ^{-1}$

Example: All diagonal matrices are diagonalizable.

Proposition: Let $A$ be an $n \times n$ matrix. Then $A$ is diagonalizable iff $\exists$ a diagonal matrix $D$ and an invertible matrix $Q$ such that

$$A = QDQ^{-1}.$$

(Rephrasing: An $n \times n$ matrix is diagonalizable iff $A$ is similar to a diagonal matrix).

Proof: Let us assume that $A$ is diagonalizable

But then what is L A? L A is just I composed with L A composed with I, where I is the identity matrix. So, L A is just I v, L A I v, where I v is the…So let me just plot the I R n where I R n is the identity matrix, identity linear transformation in R n. And now let us look at the basis the matrix of L A with respect to beta, L A, beta beta, where beta is the standard basis. This is equal to A, let beta be the standard basis and hence by definition L A beta beta is nothing but A, let us just write it now as L A I, L A I from beta to beta.

Now let us write this to be equal to I L A I from beta to beta prime, beta prime to beta prime, beta prime to beta by the very definition of or by the consequence of how the matrices behave with respect to the composition.

Let us call Q to be the matrix so, let Q be the matrix, I beta prime beta, then this is a change of basis matrix then Q inverse is nothing but I beta beta prime. This is something which we have already seen and therefore, A is nothing but Q. What is the matrix of L A with respect to beta prime? Recall that beta prime was exactly that basis with respect to which L A was a diagonal matrix. So, this is Q D Q inverse and that is precisely what we had set out to prove. Recall what we had written, the proposition is diagonalizable if there is a matrix which is diagonal D and an invertible matrix Q such that A is equal to Q D Q.

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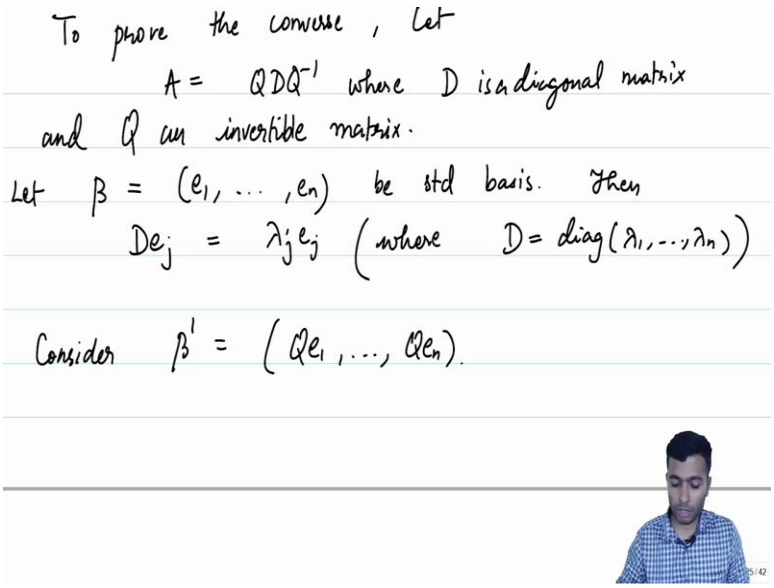

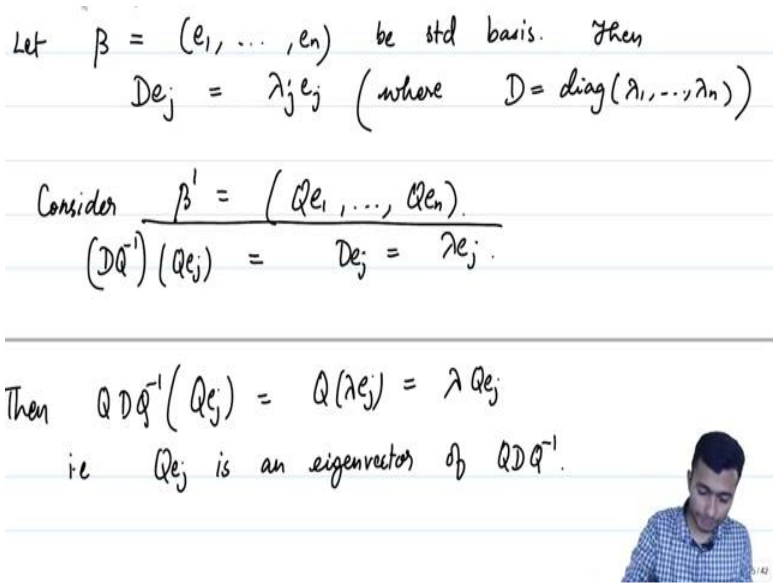


Now, let us look at the converse, we have only shown one side of the proposition. So, to prove the converse let A be equal to Q D Q inverse where D is a diagonal matrix and Q is an invertible matrix. So, we also know that the fact that D is a diagonal matrix tells us that D e j is lambda j e j where Lambda j is the jth entry along the diagonal. So, what we will do is let us consider the following vectors, beta be equal to, so this is the standard basis. So, let beta be the standard basis, then what do we know about D e j? By definition D e j is something like lambda j times e j, where lambda j is obtained in the lambda 1 to lambda n.

Let us now consider beta prime where beta prime is given by Q e 1 up to Q e n. And let us notice how beta prime, how A behaves on beta prime. So, notice that D Q inverse of Q e j will just be equal to D e j which is equal to lambda e j. So, what is going to be Q then Q D Q inverse of Q e j is going to be Q of lambda e j, but this is a linear map, this is going to be lambda times Q e j. That means Q e j is eigenvector for Q D Q inverse. Q e j is an eigenvector of Q D Q inverse. So what do we have now, beta prime is a set, so recall that beta prime is a set consisting of eigenvectors of Q D Q inverse.

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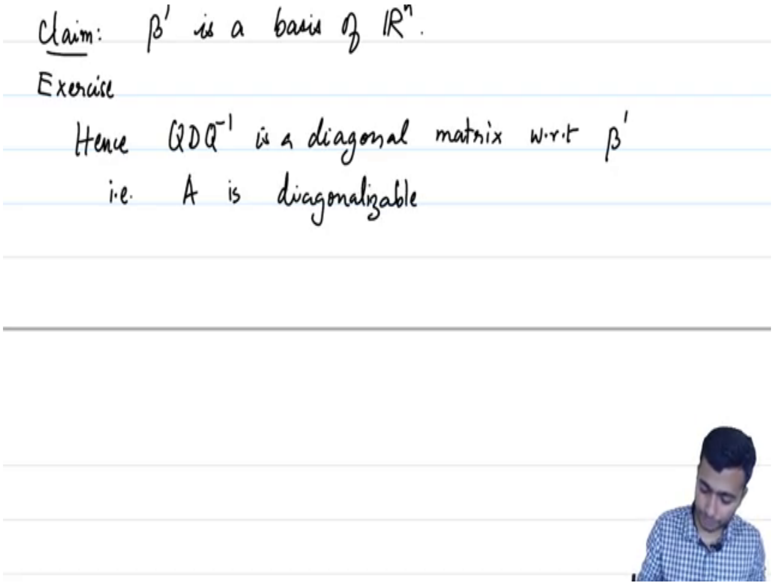


But, let me put a claim down, this beta prime is a basis of R n, if you prove this claim then we are done because then we would have obtained the basis of R n which consists exclusively of eigenvectors of our given matrix or given linear transformation whichever way you want to look at it. But then what is beta prime? Beta prime is the image of a basis under an invertible linear transformation, I leave this as an exercise for you to check at this time that if you look at the image under an invertible linear transformation, then that will turn out to be a basis. So, let me just leave it as an exercise for you to check this part.

And with this we have obtained a basis with respect to which the matrix of Q D Q inverse is diagonal, hence Q D Q inverse is a diagonal matrix with respect to beta prime which is the same as saying that A is diagonalizable.

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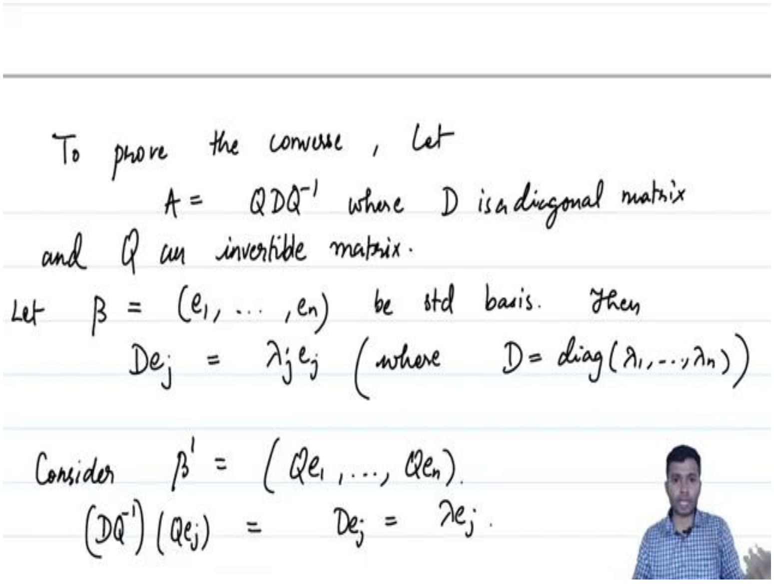


So, if we actually look at this proposition carefully, it is telling us that a given matrix is diagonalizable if and only if it is similar to a diagonal matrix. And the previous proposition was telling us that some linear transformation is diagonalizable if we can get hold of a basis consisting of eigenvectors. So putting these together, we can explicitly say what our D and what our Q is going to be. So, let us just write down a proposition explicitly mentioning what our D and Q j are.

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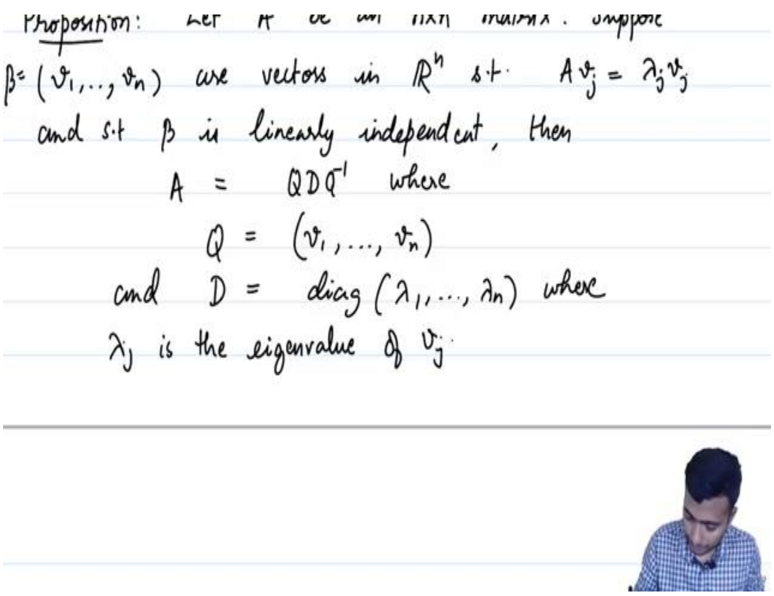


So, proposition, so let A be n cross n matrix. Suppose, v 1 to v n are vectors or it is an ordered set, are vectors in R n such that A v j is equal to lambda j v j and such that they are linearly independent and let us call it beta and such that beta is linearly independent. Then A is equal to Q D Q inverse, where Q is the matrix obtained by inserting the vectors v 1, v 2 up to v n and D is a matrix obtained by putting in the corresponding eigenvalues. So we can very explicitly compute eigenvalue of v j, let us give a proof of this.

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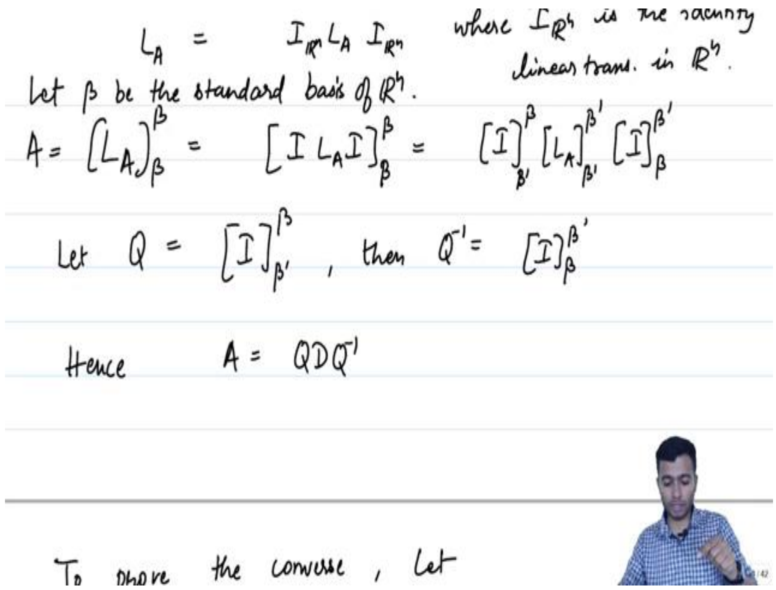


So we have already done all the hard work. Let us just go back and see what we had noticed. We had noticed that we will get an equation of this type, we will get A is Q D Q inverse where Q is I beta prime beta, so let us just redo it.

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Proof: From the proof of the previous proposition,

$$A = [L_A]_\beta^\beta = [I]_{\beta'}^\beta [L_A]_{\beta'}^{\beta'} [I]_\beta^{\beta'}$$

$$[L_A]_{\beta'}^{\beta'} = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

$$A = [L_A]_\beta^\beta = [I]_{\beta'}^\beta [L_A]_{\beta'}^{\beta'} [I]_\beta^{\beta'}$$

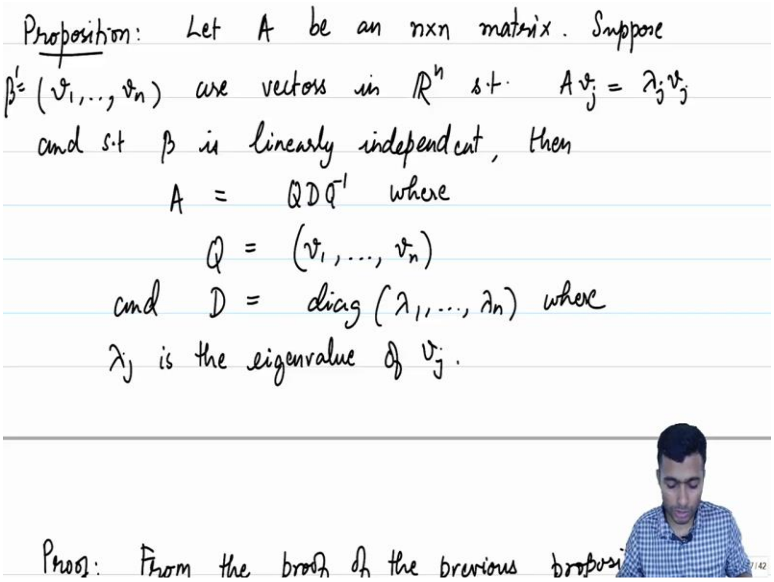
$[L_A]_{\beta'}^{\beta'} = \text{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_j$ are

s.t. $Av_j = \lambda_j v_j$

The $j^{th}$ column of the change of basis matrix $[I]_{\beta'}^\beta$ is the column vector of $v_j$.

Therefore

$$Q := [I]_{\beta'}^\beta = (v_1, \ldots, v_n).$$




Let us recall that from the proof of the previous proposition we have L A beta beta which is our matrix A this is equal to, what do we call, let us call this beta prime. The proposition, let us call the ordered basis to be beta prime. So, notice that this is linearly independent, forces it to be a basis, because it is in linearly dependent vectors in a vector space of dimension. So, this will just turn out to be equal to L A beta beta prime beta prime beta. No, no, no, this will be I beta prime beta and I beta beta prime. But what is L A beta prime beta? L A beta prime beta is just beta prime beta prime is just diagonal of lambda 1 to lambda n where lambda I is such that A v j is equal to lambda j v j, where lambda j are such that A v j is equal to lambda j v j.

And what remains is to check for what is I beta prime beta. So, this is the change of basis matrix from beta prime to beta. So, what will be the jth column of this matrix, the jth column will be I of v j, where v j is the jth vector in the ordered basis beta prime. So, I of v j is just equal to in the jth column of this change of basis matrix, let me write it again, let me write it afresh. The jth column of the change of basis matrix I beta prime beta is the column vector of v j and therefore, I beta prime beta which is let us say Q is nothing but v 1 to v n. So in the next video, let us discuss techniques for computing the eigenvalues of a given linear transformation.