Linear Algebra
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Lecture 7.2 - Diagonal Matrices

So in the last week, we discussed what is meant by the rank of a matrix and the determinant of a matrix and also, we saw some of its nice properties. We also discussed the particularly nice factorization of a given matrix into, say a product of three other matrices where the first and the third one were product of elementary matrices, and the middle one or the second one was a block matrix with the first block being the identity and the remaining blocks being the zero matrices. This factorization was particularly useful in calculating the rank and many times the determinant of our given matrix.

However, when it comes to looking at the powers of a given matrix, it is not useful to have this kind of a factorization, because there are elementary matrices involved in that. And it is quite bothersome to deal with the powers of those elementary matrices as well. So to tackle this problem or to address the powers, there is a different factorization more useful factorization, in this case, it is called the diagonalization. So, in this week, we will discuss what is meant by the eigenvalue, eigenvectors of a given linear transformation and also diagonalizability. But before we come to all that, let us have a brief interlude where we discuss and recall some nice properties of the diagonal matrices. So let us begin by recalling what a diagonal matrix is.

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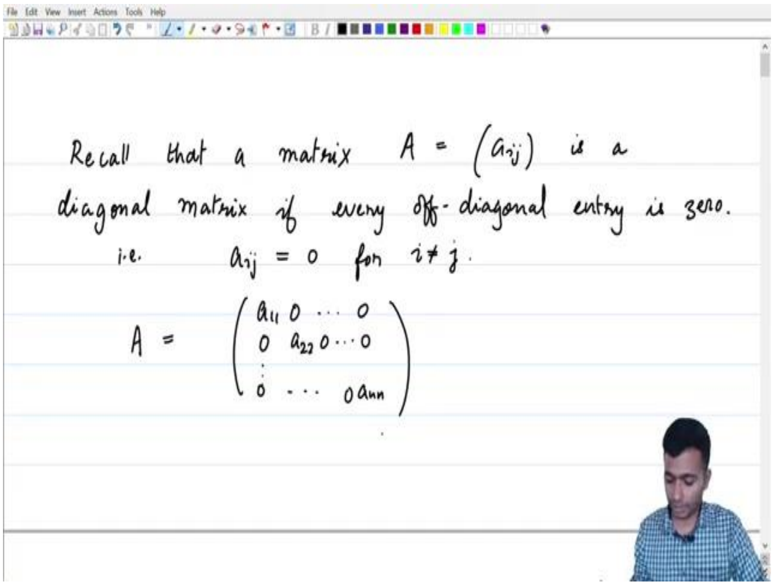

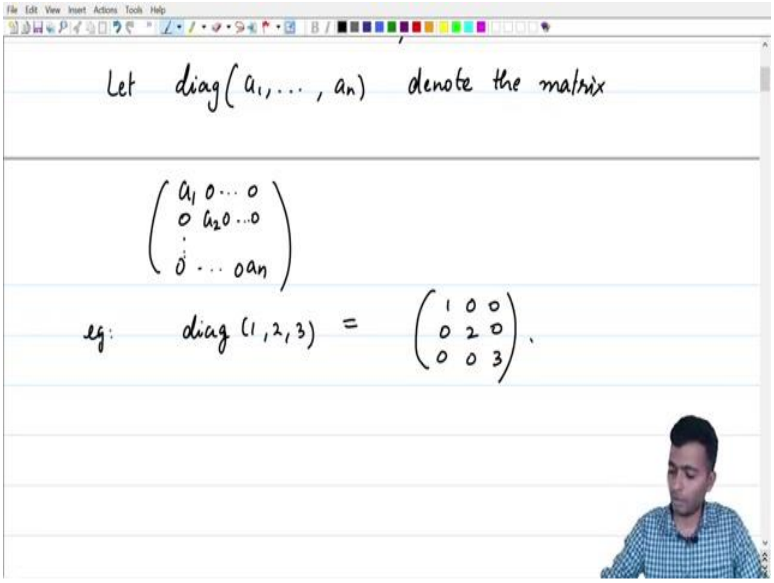

Recall that a matrix A equal to a i j is a diagonal matrix if every entry which is not in the diagonal is necessarily 0, if every off diagonal entry is 0, i. e a i j is equal to 0 for i not equal to j. So that means A will have a particularly nice form, it will be something of this type, a 11 0 0 0 0 0, a 2 2 0 dot dot dot 0 0 0 a n n. So every entry which is not in the diagonal is 0, could happen that the entries in the diagonal also could have certain 0 entries.

So for example, 0 matrix is a diagonal matrix. It is not demanded that the diagonal should be non-zero. The only thing that is being demanded is that the off-diagonal entries are 0. So we shall denote by diag, the short form expression for this, so let diag a 1 to a n, denote the matrix which is given by let us see, a 1 0 dot dot dot 0 0, a 2 0 dot dot dot 0. So we can capture just the diagonals and diagonal entries and abbreviate the expression as diag of a 1 to a n. So, in particular, if diag of say 1, 2, 3 is going to be a 3 cross 3 matrix, which will be 1 0 0, 0 2 0, 0 0 3.

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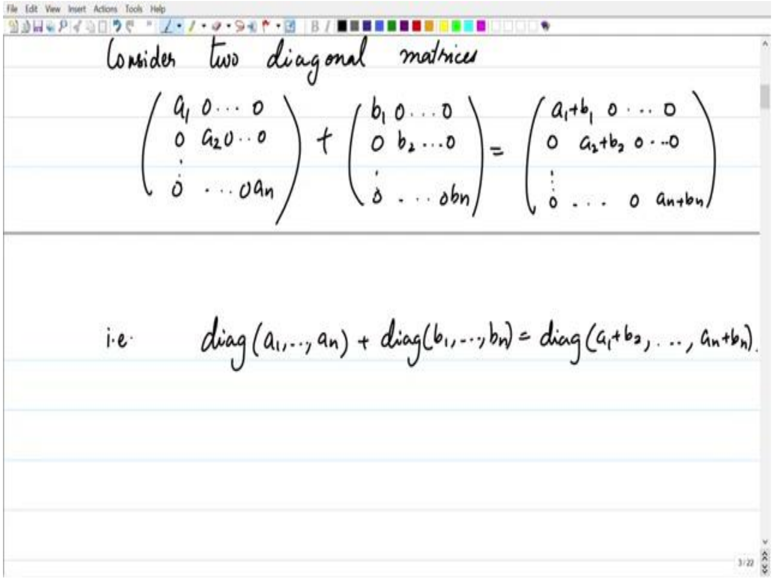

So let us look at some operations with diagonal matrices. So consider two diagonal matrices. Let us look at say a 1, 0, 0, 0, a 2 0, so on and let us add it to one another diagonal matrix. This is assumed to be b 1, 0 0. This will give us by the very definition of matrix addition, a 1 plus b 1, 0 0 0, a 2 plus b 2 0 dot dot dot 0, a n plus b n. So if I am to use the notation that we have just introduced, diagonal diag of a 1 to a n plus diag of b 1 to b n is equal to diag of a 1 plus b 1, dot dot, a n plus b n, the matrix addition is particularly nice.

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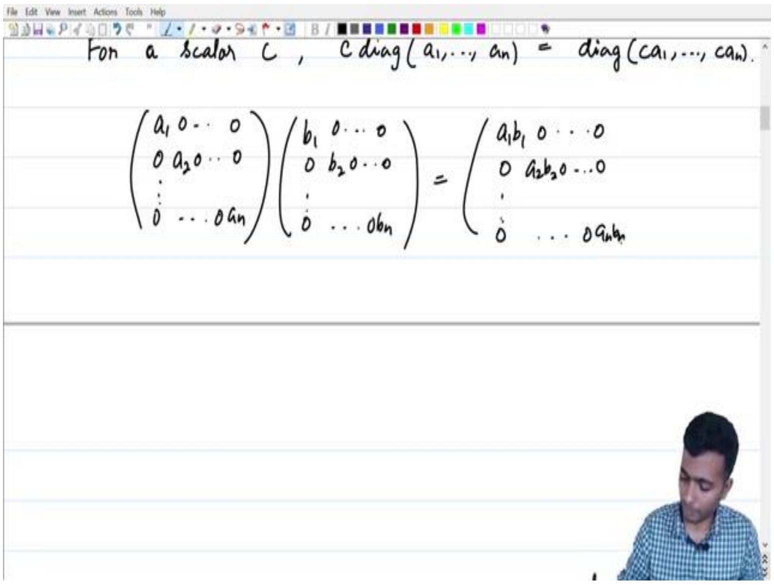

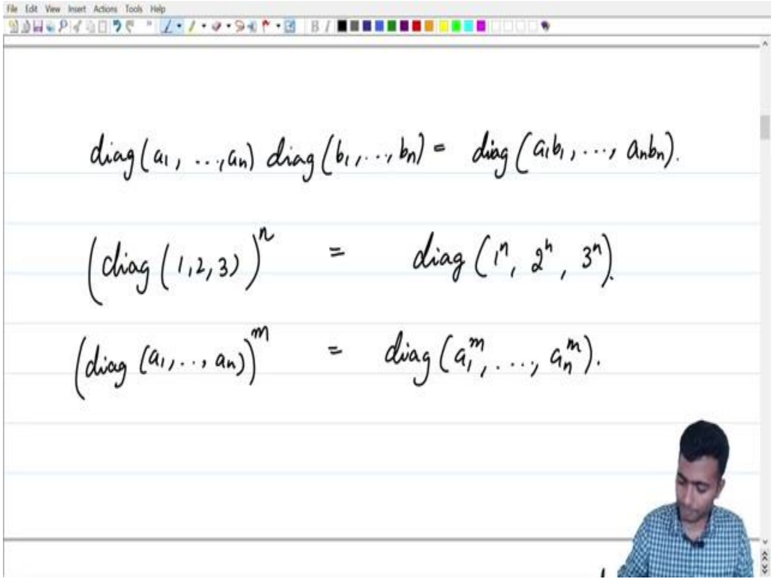

So, check for a scalar. So, let me leave it now as an exercise for a scalar C, C times the diagonal of a 1 to a n will be equal to the diagonal of C a, 1 up to C a n. So it is not just the matrix addition, which behaves well, the scalar multiplication is also particularly nice. In fact, not just these 2 operations, if you take two diagonal matrices and multiply them, that also is particularly well behaved in the expressions that we have developed. For example, if you look at a diagonal matrix, when multiply to another diagonal matrix, I leave it to you to check that this is going to be just the component wise product here.

So if I were to write it again in the short notation, this will be a 1 to a n times the diagonal matrix with entries b 1 to b n, it will just be equal to the diagonal matrix with entries a 1 b 1 up to a n b n. Which is good because if you look at the diagonal matrix of say 1, 2, 3 earlier and raise it to a power n, this is just going to be equal to the corresponding 1 to the power n, 2 to the power n and 3 to the power n by what we have just noted. So in particular, this is true for any n cross n matrix, so diagonal of a 1, a 2 up to diagonal of a 1 to a n, if you multiply this matrix with itself n times, this will just give us a 1 to the power n. Okay, too many n's working around, so let me just make it m, this will just be m and this will just be m.

So, that is good. Let us look at what happens when we look at polynomial expressions of our diagonal matrix.

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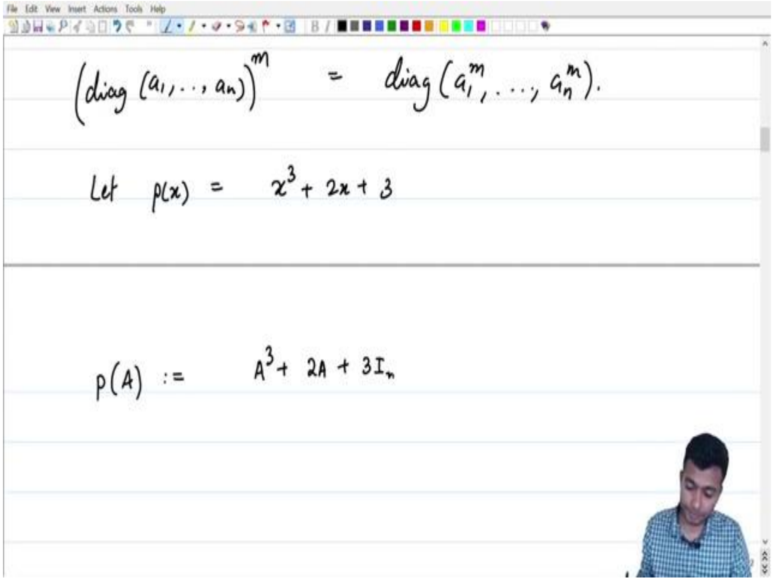

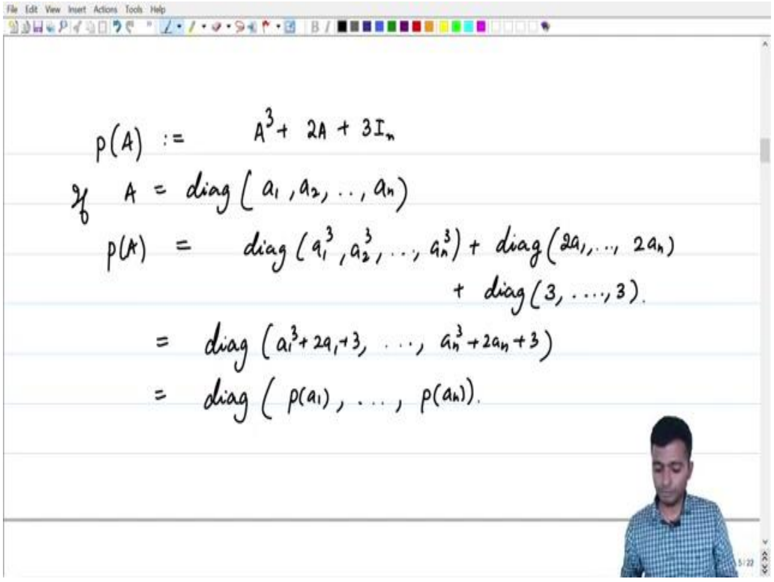

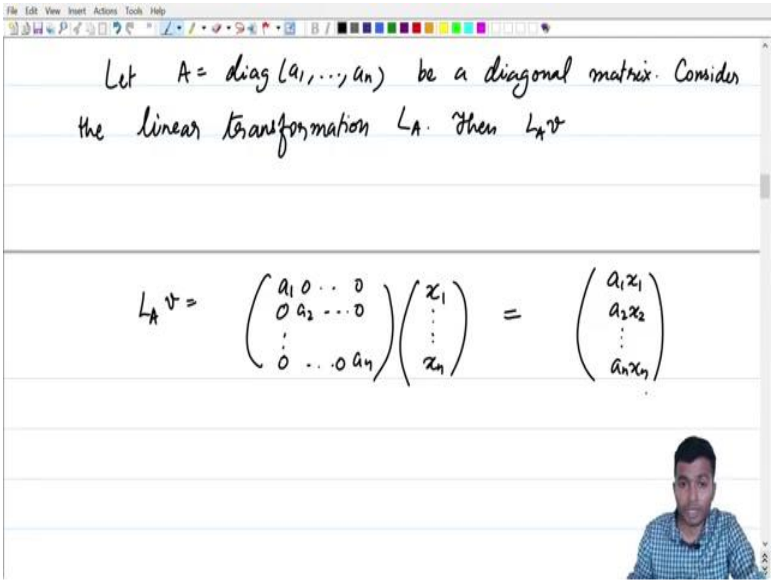

So, let us look at it with an example. Let p of x be something like say, x to the power 3 plus 2 x plus 1, or maybe not, let us call it 3. Then P of A is defined as for any matrix any square matrix A, P of A is defined as A cube plus 2 times A plus 3 times the identity matrix. So where identity matrix is of size n, where n is the size of our matrix A. So, if A is a matrix of the type say diagonal of say a 1, a 2 a 3, let us just maybe a n, let us look at a diagonal matrix of size n, then what is going to be A to the power 3? Then I would say that the expression for P of A will be diagonal of a 1 to the power 3, a 2 to the power 3, a n to the power 3 plus 2 times, so I will just pull in the 2 times diagonal here to write it like this, diagonal of 2 a 1, 2 a n plus the diagonal consisting of 3, all 3s, right?

And this is going to be diagonal of a 1 cube plus 2 a 1 plus 3 a n cube plus 2 a n plus 3 but whatever we have just seen, and this is precisely P of a 1 up to P of a n. So the polynomial expression of a diagonal matrix will give us a matrix which is a diagonal matrix again, whose entries are corresponding polynomial expressions of the diagonal entries from a. So, again, with respect to polynomial expressions, evaluation of p of a will be easily done with diagonal matrices as compared to normal matrix.

If you sit down with a normal matrix A and start looking at polynomial expressions then forget polynomial expressions and focus on simple powers. Very soon it starts becoming quite complicated, but that is not the case here. That is what was observed.

So, another proposition to note, maybe lemma is that, you recall the factorization that was being that was being referred to in the beginning of this video, we could write any matrix as the product of 3 m or….let us forget the factorization. If you go back to the previous week, we had seen one of the results, which said that the rank of a matrix is equal to the number of non-zero rows in the row echelon form. And here that manifests in telling that the rank of our diagonal matrix is just equal to the number of non-zero entries. So let A be a diagonal matrix. So let me refer you to the last week's lectures to conclude that the rank of A is equal to the number of its non-zero entries. So, now let us look at the linear transformation corresponding to a given matrix. So, let A be a diagonal matrix.

So, let A equal to say diag of a 1 to a n be a diagonal matrix. So, we would like to see, so, we would like to see the impact of that on the corresponding linear transformation L A, L subscript A, consider the linear transformation L subscript A. Let us see what the action of L A on typical element of R n is going to be, then L A of x will just be L A of V for some vector V in R into R in R n would be the following:

L A of V will just be equal to a 1 0 to 0 0 a 2 a n, this will be matrix multiplied to x 1 to x n, where x 1 to x n is the column vector corresponding to V with respect to the standard basis. And this if you observe, this is just a 1 x 1, a 2 x 2, a n x n. So, if you notice carefully, what it does is that it dilates the first coordinate by a 1, it dialates the second coordinate by a 2 and so on, dilates the nth coordinate by a n. In particular, if you look at the impact of L A on the standard basis, let us see.

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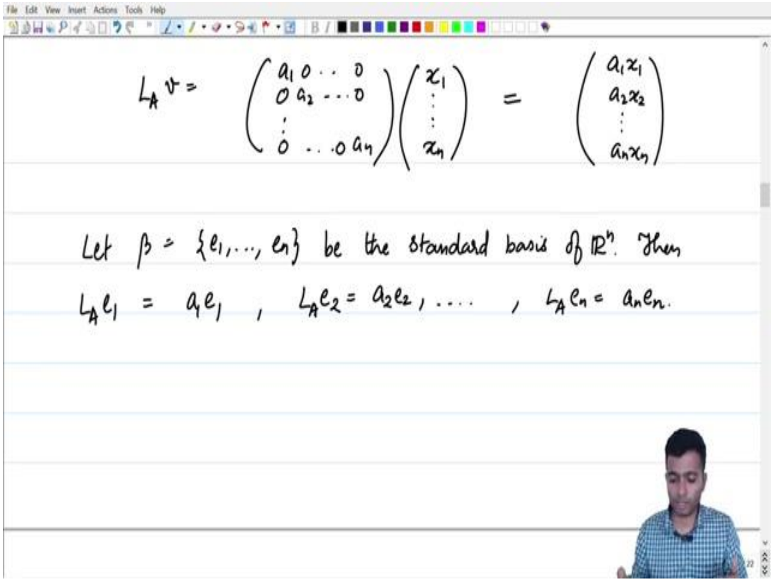

So, let beta equal to e 1 to e n be the standard basis of R n, then L A of e 1 by what we just noted will just turn out to be equal to a 1 times e 1. L A times e 2 is what is called as a 2 times e 2. L A times e n is equal to a n times e n. So, in this setting a i's are what are called as Eigenvalues and e i's are what are called as Eigenvectors. So this is one of the first examples of eigenvalues and eigenvectors which we have not yet defined, which will be the content of your next video.