Linear Algebra
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Lecture 7.1 - Problem Session

So this video is the problem session based on the material that was covered in week 3 and week 4 of this course. As usual, it is meant to supplement the problems that were given in assignments and I hope that you have worked on the assignment problems. So let us begin by checking for whether the following maps which I will be writing down in a moment are linear.

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Problem 1: Check whether the following maps are linear

(i) : $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (x+z,\ 2x+y,\ 3x+y+z).$$

(ii) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ given by

$$T(p(x)) = x\,p(x) + p'(x).$$

So problem one, check whether the following maps are linear. So the first one, it is a map front R 3 to R 3 given by T of x y z is equal to x plus z, 2 x plus y, 3 x plus y plus z. The second map that we will be considering is a map T from P 2 of R to P 3 of R given by T of say P of x this is equal to x times p of x plus p prime of x, let us check whether the 2 maps given here are linear. So, looking for a given map checking for whether a given map is linear or not is quite straightforward, if you recall the simpler condition that we had found for checking such cases. So, let me just recall that for you.

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Solution: Recall that $T: V \to W$ is a linear transformation if $T(v_1 + cv_2) = T(v_1) + cT(v_2)$ $\forall$ $v_1, v_2 \in V$ and $c \in \mathbb{R}$

(i) $T: \mathbb{R}^3 \to \mathbb{R}^3$ where $T(x,y,z) = (x+z, 2x+y, 3x+y+0$

Problem 1: Check whether the following maps are linear

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if $T(v_1 + cv_2) = T(v_1) + cT(v_2)$ $\forall$ $v_1, v_2 \in V$ and $c \in \mathbb{R}$

(i) $T: \mathbb{R}^3 \to \mathbb{R}^3$ where $T(x,y,z) = (x+z, 2x+y, 3x+y+z)$

Let (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

Then

$(x_1, y_1, z_1) + c(x_2, y_2, z_2) = (x_1 + cx_2, y_1 + cy_2, z_1 + cz_2)$

$T\big((x_1, y_1, z_1) + c(x_2, y_2, z_2)\big) = T(x_1 + cx_2, y_1 + cy_2, z_1 + cz_2)$

$$(x_1, y_1, z_1) + c(x_2, y_2, z_2) = (x_1 + cx_2,\ y_1 + cy_2,\ z_1 + cz_2)$$

$$\begin{aligned}
T\big((x_1, y_1, z_1) + c(x_2, y_2, z_2)\big) &= T(x_1 + cx_2,\ y_1 + cy_2,\ z_1 + cz_2) \\
&= \big(x_1 + cx_2 + z_1 + cz_2,\ 2(x_1 + cx_2) + y_1 + cy_2,\ 3(x_1 + cx_2) + y_1 + cy_2 + z_1 + cz_2\big) \\
&= \big(x_1 + z_1 + c(x_2 + z_2),\ 2x_1 + y_1 + c(2x_2 + y_2),\ 3x_1 + y_1 + z_1 + c(3x_2 + y_2 + z_2)\big) \\
&= (x_1 + z_1,\ 2x_1 + y_1,\ 3x_1 + y_1 + z_1) + c(x_2 + z_2,\ 2x_2 + y_2,\ 3x_2 + y_2 + z_2) \\
&= T(x_1, y_1, z_1) + c\,T(x_2, y_2, z_2) \quad \blacksquare
\end{aligned}$$

So solution, recall that a function f or function T from V to W is a linear transformation if T of v 1 plus c v 2 is T of v 1 plus c times t v 2 where v 1 and v 2 are in capital V and C is a scalar. So, recall that T from capital V to W is linear transformation, this is the compact condition that we had found a T of v 1 plus c v 2 is equal to T of v 1 C T of v 2 for all v 1, v 2 in capital V and C in R the scalars. So, we will apply this to, apply this particular proposition we had proved in v 3 to check whether our given maps are linear or not. So, for the first one, so what was the map? T was from R 3 to R 3, where T of x y z is equal to x plus z, 2x plus y, and 3 x plus y plus z let us go back and check it alternate x plus z, 2 x plus y, 3 x plus y plus z.

All right, so let us take 2 vectors in R 3 and the scalar so let x 1 y 1 z 1 comma x 2 y 2 and z 2 be two elements in R 3. And suppose C is in the inner scalars, a real number and what is v 1 plus c v 2 in our case, this is just going to be x 1, y 1 z 1, plus c times x 2 y 2 z 2 is equal to x 1 plus cx 2 y 1 plus cy 2 z 1 plus cz 2. So hence, T of x 1 y 1 z 1 plus c times x 2 y 2 z 2 this by this observation is just equal to T of x 1 plus c x 2 comma y 1 plus c y 2 plus z 1 plus c z 2 but then what is our definition of T of x, y, z? Let me just show it to you, it is written just above. It is x plus z, 2 x plus y, and 3 x plus y.

So here x plus z will just turn out to be equal to x 1 plus c x 2 plus z 1 plus c z 2. That is the first coordinate and the second coordinate was…Notice it is 2 x plus y so whether it is going to be equal to two times x 1 plus c x 2 which is our element, which is our x coordinate here, plus y 1 plus c y 2. And finally, this is going to be the third coordinate, which is 3 x plus y plus z. So 3 times x 1 plus c x 2 plus y 1 plus c y 2 plus z 1 plus c z. Let us split it and write it down. So this is just going to be x 1 plus z 1, plus c times x 2 plus z 2 as the first coordinate, and then the second one will be 2 x 1 plus y 1 plus c times 2 x 2 plus y 2.

And the third coordinate is 3 x 1 plus y 1 plus z 1, these are all real numbers, so we can do all these manipulations freely, plus c times 3 x 2 plus y 2 plus z 2. Now, if you look at this particular vector in R 3, I may as well be able to write it as x 1 plus z 1, 2 x 1 plus y 1, 3 x 1 plus y 1 plus z 1, this is one vector, plus c times x 2 plus y 2, 2 x 2 plus y 2, 3 x 2 plus y 2 plus z 2, which is nothing but T of x 1 y 1 z 1 plus c times T of x 2 y 2 z 2 and that establishes that the given map is a linear transformation.

So let us now look at the second problem. The second problem telling us that T of x is x times p of x plus p prime of x the derivative of p that is the second term featuring here. And it is from p3 to itself, or rather p 2 to p 3 of x rather.

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$T(x, y, z) = (x+z,\ 2x+y,\ 3x+y+z).$

(ii) $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ given by

$T(p(x)) = x\,p(x) + p'(x).$

Solution: Recall that $T: V \to W$ is a linear transformation if $T(v_1 + cv_2) = T(v_1) + cT(v_2)$ $\forall\ v_1, v_2 \in V$ and $c \in \mathbb{R}$

(i) $T: \mathbb{R}^3 \to \mathbb{R}^3$ where $T(x, y, z) = (x+z,\ 2x+y,\ 3x+y+z)$

$$= T(x_1, y_1, z_1) + c\,T(x_2, y_2, z_2) \quad \text{———} \ \blacksquare$$

(ii) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ given by

$$T(p(x)) = x p(x) + p'(x)$$

Let $p_1, p_2 \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$

$T(p_1 + cp_2) =$

Let $p_1, p_2 \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$

$$\begin{aligned} T(p_1 + cp_2) &= x\left(p_1(x) + c\,p_2(x)\right) + \left(p_1(x) + c\,p_2(x)\right)' \\ &= x p_1(x) + c\, x p_2(x) + p_1'(x) + c\,p_2'(x) \\ &= \left(x p_1(x) + p_1'(x)\right) + c\left(x p_2(x) + p_2'(x)\right) \\ &= T(p_1) + c\,T(p_2) \quad \text{———} \ \blacksquare \end{aligned}$$

So let us look at the second problem. T from P 2 of x, or P 2 of R to P 3 of R, this is given by p of p of x is equal to x times p of x. Notice that this raises the degree by 1 plus p prime of x minus p prime of x, plus p prime of x. So we will do the same trick as above, let P 1, P 2 be two polynomials of degree less than or equal to 2 and suppose C be a real number, an element from the field of scalars. What is T of P 1 plus c P 2? This is equal to, by the very definition, this is going to be x times P 1 of x plus c times P 2 of x.

We are just not bother writing off x, but that is understood, let me write it down p 1 of x plus c times p 2 of x plus p 1 of x plus c times p 2 of x the derivative of this particular polynomial, but we know that this derivative behaves well if you look at the derivatives of the sum of the polynomials, this is going to be equal to, so let me just write it like this c times x p 2 x p 2 of x plus this is p 1 prime of x plus c times p 2 prime of x, but the constant times polynomial if you differentiate this is just the constant times the derivatives, this is c times p 2 prime of x, and let us now regroup as earlier. So this is x of p 1 of x plus p 1 prime of x, which is basically T p 1 of x plus c times x p 2 of x plus p 2 prime of x. This is nothing but T of p 1 plus c times T of p 2 thus establishing that both the examples that we consider were linear transformations.

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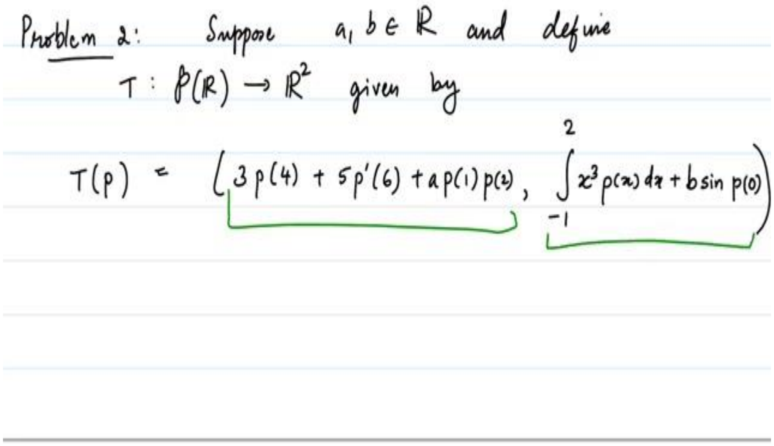

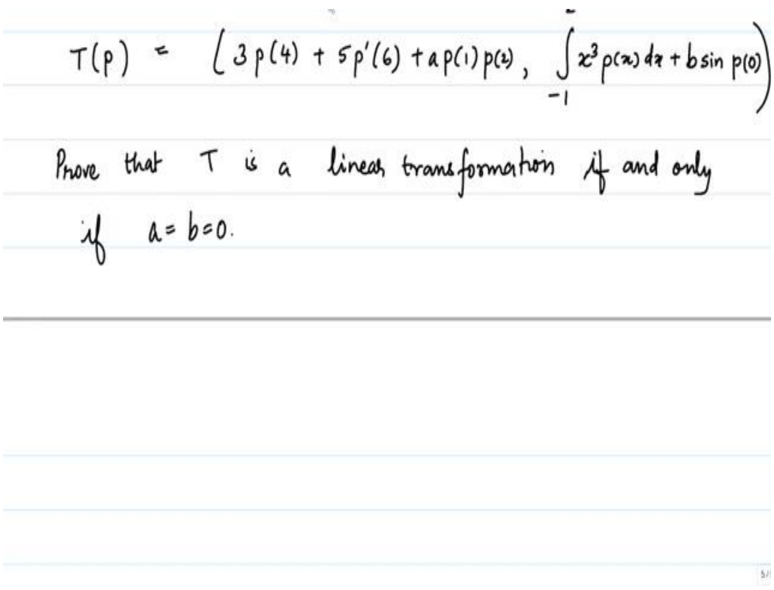

The next problem is quite similar. It is a problem which tries to find the necessary and sufficient condition for a given map to be a linear transformation or that a particular map. So let us see what the problem is, problem 2, suppose a and b are scalars, such that and let us now define map T from p of R to R 2 recall that p of R is the vector space of all polynomials with coefficients in real numbers.

So T is a map from P of R to R 2, as of now we are just defining a function given by T of p, so remember p is a polynomial. So this is just going to be equal to 3 times the evaluation of p at 4 plus 5 times p prime of 6, plus p of 1, maybe a times p of 1 into p of 2. This is the first coordinate in the image, and how about the second one? The second one is the integral minus 1 to 2 x cube, p of x dx plus b times sine of p of 0. So notice that whatever I have just underlined is a real number. You look at all the products and sums, it is just going to give us back some real numbers. Similarly, this will also turn out to be a real number.

So, indeed this is a map from T of R into R2 and the question, is this particular map T a linear transformation? So, or rather it is to prove that, the problem is to prove that T is a linear transformation if and only if a is equal to b is equal to 0. So let us give a proof of this, let me just try to keep this particular….so proving this particular problem has two sides. One is when a is equal to 0 and b is equal to 0, we have to show that T is a linear transformation. That is quite straightforward. Let me not spend time on it, maybe I can do it first.

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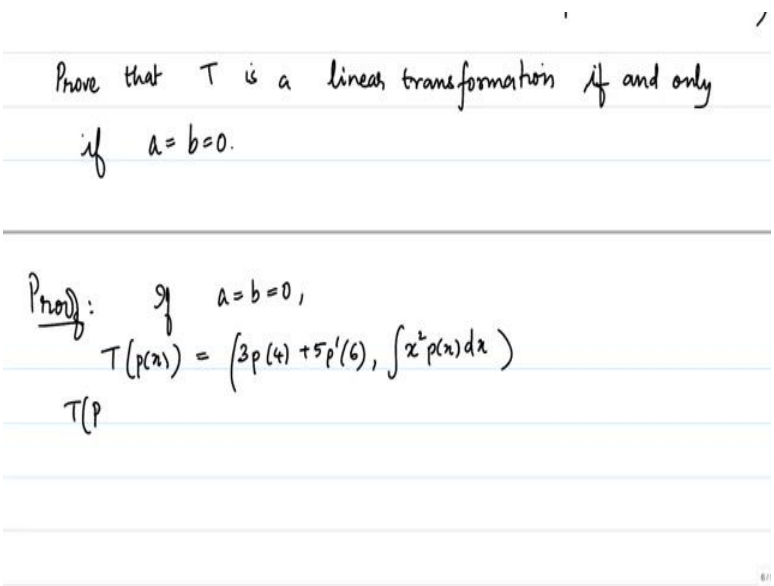

$$T(p(x)) = \left(3p(4) + 5p'(6), \int_{-1}^{1} x^2 p(x)\,dx\right)$$

$$T(p_1 + cp_2) = \left(3(p_1(4) + cp_2(4)) + 5(p_1'(6) + c p_2'(6)), \int_{-1}^{2} x^2(p_1 + cp_2)\,dx\right)$$

$$\int (x^2 p_1 + c x^2 p_2)\,dx = \int x^2 p_1\,dx + c\int x^2 p_2\,dx$$

$$= \left(3p_1(4) + 5p_1'(6), \int_{-1}^{2} x^2 p_1(x)\,dx\right) + c\left(3p_2(4), 5p_2'(6) + \int_{-1}^{2} x^2 p_2(x)\,dx\right)$$

$$= T(p_1) + c\,T(p_2)$$

It will take almost no time to check that, if a is equal to b is equal to 0, then let us see what happens, then T of p of x, this will be equal to, let us keep this here so that formula can be taken easily. This is going to be 3 times p of 4, plus 5 times p prime of 6, the b is 0 and hence there will be having, there will be no extra term, and the next one will be x square p of x dx. This is exactly whatever it will turn out to be. Let us look at what is T of p 1 plus c p 2. That is going to be equal to 3 times p 1 of 4 plus c times p 2 of 4.

Again derivative are (())(15:01) and this is going to be p 1 prime of 6 plus c times p 2 prime anything 2, there is a packet, big packet here of 6, that is the first term. And the second term is going to be p 1 plus c p 2 of x dx. And splitting and taking advantage of the vector addition that we know this is just 3 p 1 of 4 plus 5 p 1 prime of 6 comma integral of x square of p 1 plus c p 2 is just going to be equal to…

So let me just write that down here in green, this is just going to be equal to integral of x square p 1 plus c times x square p 2 dx, which is equal to integral of x square p 1 dx plus c times integral of x square p 2 dx and that helps us in writing this as p 1 of x dx plus the see, now we can be pulled out in common 3 times p 2 of 4, 5 times p 2 prime of 6 integral minus 1 to 2 all those are minus 1 to 2 p x square times p 2 of x dx and we are done because this is equal to T of p 1 and this is T of p 2. So, yes, if b is equal to a is equal to 0, then E is trivially, not trivially but yeah, after these computations, it is a linear transformation.

The problem rises or rather the more difficult side comes in the forward direction wherein if you have the assumption that T is a linear transformation, we want to prove that a is equal to b is equal to 0.

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If T is a linear transformation

$$T(p_1+p_2) - T(p_1) - T(p_2) = 0 \quad \forall \; p_1, p_2 \in \mathcal{P}(\mathbb{R})$$

$$\parallel$$

$$\Big(3\big(p_1(4)+p_2(4)\big) + 5\big(p_1'(6)+p_2'(6)\big) + a\big((p_1(1)+p_2(1))(p_1(2)+p_2(2))\big), \int x^2(p_1+p_2)\,dx + b\sin\big(p_1(0)+p_2(0)\big)\Big)$$

If

$$T(p_1+p_2) - T(p_1) - T(p_2) = 0 \quad \forall \; p_1, p_2 \in \mathcal{P}(\mathbb{R})$$

$$\parallel$$

$$\Big(\boxed{3\big(p_1(4)+p_2(4)\big)} + \boxed{5\big(p_1'(6)+p_2'(6)\big)} + a\big((p_1(1)+p_2(1))(p_1(2)+p_2(2))\big), \boxed{\int x^2(p_1+p_2)\,dx} + b\sin\big(p_1(0)+p_2(0)\big)\Big)$$

$$-\Big(\boxed{3p_1(4)} + \boxed{5p_1'(6)} + a\,p_1(1)p_1(2), \boxed{\int x^2 p_1(x)\,dx} + b\sin p_1(0)\Big)$$

$$-\Big(\boxed{3p_2(4)} + \boxed{5p_2'(6)} + a\,p_2(1)p_2(2), \boxed{\int x^2 p_2(x)\,dx} + b\sin p_2(0)\Big)$$

So in order to do that, let us do one thing. So if T is linear transformation, what do we know? If it is a linear transformation, we know that T of it is additive T of p 1 plus p 2 is T of p 1 plus p 2 for all p 1 p 2, then T of p 1 plus p 2 minus T of p1 minus T of p 2 is equal to 0 for all p 1 comma p 2 in P of R. For all polynomials, the additive property is getting satisfied. Similarly, the scalar property satisfied. Let us focus on the additive property itself. What is this going to manifest as, in the definition that we have, this is just going to be equal to 3 times p 1 of 4, plus p 2 of 4 plus 5 times p 1 prime of 6 plus p 2 prime of 6 plus a times p 1 plus p 2.

So p 1 of 1 plus p 2 of 1 into p 1 of 2 plus p 2 of 2 and to write it down maybe, this is the first one, I have to write the second variable as well, that is just straightforward. That is going to be x square times p 1 plus p 2 dx plus b times sine of p 1 of 0 plus p 2 of 0, so this will be the value of p of p 1 plus p 2. Let us look at what is the other two. Minus T of p 1 we know what it is, that is already given to us that is p 1 of 4, 5 p 1 prime of 6 plus a times p 1 of 1, p 1 of 2 comma integral of x square p 1 of x dx plus b times sin p 1 of 0.

There is one more term which is T of p 2 which will be very similar, p 2 of 4 plus 5 p 2 prime of 6 plus a times p 2 of 1, p 2 of 2 and x square p 2 of x dx plus p sin p 2 of 0, that is it. That is what our, this is the big expression that we are dealing with. So let me now put boxes so the three times this term will cancel off with this and this. Let me now use a different color, this term cancels off with this term and this term. They will come to what is written here separately, this will cancel, maybe I should use a different color. Red can be used to represent this cancels off with this, plus this and there is a minus, if you notice there is a minus here so all these will cancel off.

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$$* = \Big(a\big(p_1(1)p_2(2) + p_2(1)p_1(2)\big),\ b\sin\big(p_1(0)+p_2(0)\big) - b\sin p_1(0) - b\sin p_2(0)\Big)$$

$$= (0,0) \quad \forall\ p_1, p_2 \in \mathcal{P}(\mathbb{R})$$

If $p_1(x) \equiv \pi/2,\quad p_2(x) = -\pi/4$

$$\Rightarrow\ * = \left(-\frac{a\pi^2}{4},\ b\left(\frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}}\right)\right) = (0,0)$$

only if $a=0$ & $b=0$

Hence T is a linear transformation only if $a=b=0$.

Problem: If $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that $T(1,0,3) = (1,6)$ and $T(1,2,0) = (-1,0)$. Then what is $T(5,4,9)$?

So let us just write down what is left, what is left will be after further cancellations, which I will allow you to check, there will be a p 1 of 1 p 1 of 2 is going to get cancelled. So what will remain is a times p 1 of 1, p 2 of 2 plus p 2 of 1 p 1 of 2, that will be the only term which is left in the first coordinate. What about the second one? b times sin of p 1 of 0 plus p 2 of 0 minus b times sin of p 1 of 0 minus b of sin of p 2 of 0. So, what was our demand? Our demand was that this is equal to 0 for all choices of P 1 and P 2, so, this should be equal to 0 comma 0 for all p 1 comma p 2 in p of R.

Only if this happens will be a linear transformation but notice if for example, if p let us pick something p 1 to be the polynomial identically equal to maybe pi by 2 and p 2 of x to be identically equal to minus of pi by 4, if you think about what will happen here, let us focus on what will happen? This will give you then this implies star, let me put star, star is equal to maybe the first term will be what? Pi square by 8 into 2 so by 4 minus of pi square by 4 and the second term will be there will be a out, so there will be a b times sin of pi by 2 minus pi by 4, pi by 4 which is 1 by root 2, 1 by root 2 minus p 1 is pi by 2, which is 1 which is again subtracted by maybe plus 1 by root 2, so something like this.

And this is equal to 0 comma 0, only if notice this is only equal is 0, if a is equal to 0, and b is equal to 0, only if this happens will this be equal to 0 for this particular choice of p 1 p 2 and hence this forces what we were trying to prove. Hence T is additive and in particular, a linear transformation only if a is equal to b is equal to 0. So the next problem deals with calculating the value of a linear transformation at a vector if the value of T at specific vectors are given to you. So let us just write down the problem and look at it more carefully.

So if T from say R 3 to R 2 is a linear transformation, such that we know the value of T at certain points such that T of say 1, 0, 3 is equal to 1, 6 and T of 1, 2, 0 is equal to minus of 1, 0. So, at 1, 0, 3 we know that it is mapped to 1, 6 and 1 to 0 is mapped to say minus 1, 0. Then the problem demands that we calculate the value and what is T of 5, 4, 9, this is what is expected.

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what is $T(5,4,9)$?

<u>Solution:</u> If $(5,4,9) = a(1,0,3) + b(1,2,0) \longrightarrow (*)$

Then $T(5,4,9) = aT(1,0,3) + bT(1,2,0)$

Then $(*) \Rightarrow (5,4,9) = (a+b, 2b, 3a)$

$\Rightarrow a = 3, \ b = 2$

what is $T(5,4,9)$?

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Then $T(5,4,9) = aT(1,0,3) + bT(1,2,0)$

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$\Rightarrow a = 3, \ b = 2$

Hence $T(5,4,9) = 3(1,6) + 2(-1,0)$

$= (1,18)$

So what is the only information that is given to us? Only information that is given to us is that T is a linear transformation, we do not know anything else about the map T, right? So maybe look, let us look at a solution and see how to go about proving this problem or solving this problem. The only thing that we know is that our given linear map is a linear transformation, nothing else. So in other words, we will be forcing ourselves to use just that simple property to get hold of 5, 4, 9. So to do that, notice 5, 4, 9 is kind of special. So if this is equal to say a times 1, 0, 3 plus b times 1, 2, 0 suppose we can write it like this, then what is T of 5, 4, 9?

T of 5, 4, 9 is just out to be equal to a times T of 1, 0, 3 plus b times T of 1, 2, 0. Both of these values we know and therefore, let us see what is a and b, if we can find a and b then we are through, right? So if 5, 4, 9 is equal to or let me write it like this, it is already written down. So let us call it a star and star implies 5, 4, 9 is equal to a plus b, 2 b, 3 a. This implies that 3 a is 9, which implies a is equal to 3 and 2 b is 4, which implies b is equal to 2, and a plus b is 5. This is a consistent system. Yes, check that for a equal to 3 and b equal to 2, 5, 4, 9 is a times 1, 0, 3 plus b times 1, 2, 0.

Therefore, we know how to calculate T of 5, 4, 9. Hence T of 5, 4, 9 is equal to 3 times, what is T of 1, 0, 3? T of 1, 0, 3 we have written is 1, 6. 3 times 1, 6 plus 2 times this was minus 1, 0. What was minus 1, 0? T of 1, 2, 0 is minus 1, 0. We have just exploited this to write whatever we just wrote below. So if you notice this is something which you can easily calculate, this is 3 minus 2, 1, 18 plus 0. So this is exactly the value of T of 5, 4, 9. All right, so we have solved the problem. But let us spend a couple of minutes what we did, to realize what we did here. If you recall, we solved or that we proved a theorem, which stated that we do not need to define a linear transformation on every vector, if we define the linear transformation on a basis, it automatically extends to a linear transformation on V.

In other words, if V 1 we took V n as a basis, and if we mentioned what T of V 1, T of V 2 and T of V n these n vectors are, then T of V has been fixed for every V. But here, notice that we are in R 3, R 3 has dimension 3 and therefore any bases should have 3 vectors. But here we have only considered the value of T at two points, namely 1, 0, 3 and 1, 2, 0. We know that two vectors cannot be spanning set in R 3. So what have we done here? So what we have done here is we have considered the subspace generated by 1, 0, 3 and 1, 2, 0. Let us call that say W.

And we checked whether 5, 4, 9 is in the span of this or is in W, then we looked at T restricted to W and we now know that 1, 0, 3 and 1, 2, 0, we should check that they are linearly independent and then hence they form basis. Therefore, this map is exactly like we had defined in the usual case where we define them on the basis and it extends to every vector in the vector space. Yes, this is a special case applied to a subspace and T is restricted to a subspace.

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$= (1, 18)$

Problem 4: Prove that if $T: \mathbb{R}^4 \to \mathbb{R}^2$ is a linear transformation s.t. $Null(T) = \{(x_1, x_2, x_3, x_4) : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$

Then prove that T is surjective.

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Then prove that T is surjective.

Proof: By dimension theorem

$$4 = \dim(\mathbb{R}^4) = \dim(null(T)) + \dim(R(T)).$$

$Null(T) = \{(x_1, x_2, x_3, x_4) : x_1 = 5x_2 \ \& \ x_3 = 7$

Then prove that T is surjective.

Proof: By dimension theorem

$$4 = \dim(\mathbb{R}^4) = \dim(null(T)) + \dim(R(T)). \to 1$$

$Null(T) = \{(x_1, x_2, x_3, x_4) : x_1 = 5x_2 \ \& \ x_3 = 7x_4\}$

Check that $\beta = \{(5, 1, 0, 0), (0, 0, 7, 1)\}$ is a basis of $Null(T)$.

$$\Rightarrow \quad \dim(Null(T)) = 2$$

a basis of Null(T).

$\Rightarrow \quad \dim(\text{Null}(T)) = 2$

Then by $(*)$ $\quad 4 = 2 + \dim(R(T))$

$\dim(R(T)) = 2.$

Since $R(T)$ is a subspace of $\mathbb{R}^2$ of dimension

Then by $(*)$ $\quad 4 = 2 + \dim(R(T))$

$\dim(R(T)) = 2.$

Since $R(T)$ is a subspace of $\mathbb{R}^2$ of dimension 2, then

$R(T) = \mathbb{R}^2$

$\Rightarrow$ T is surjective $\quad$ —— $\square$

So the next problem is an application of the classical dimension theorem which we have put. So this is problem 4. So prove that if T from R 4 to R 2 is a linear transformation such that we know what its null space is, such that the null space of T is the set of all say x 1, x 2, x 3, x 4 such that x 1 is equal to 5 x 2 and say x 3 is equal to 7 x 4, then prove that T is surjective. So notice that we do not even know explicitly what T is and yet, we are able to say something about the surjectivity of the linear transformation. That is at least that is what the problem demands us to prove here or establish here.

So let us give a proof of this. So in order to do that, let me initially invoke and remind you about the dimension theorem. By the dimension theorem, what is the domain of E, dimension of the domain which is R 4 this is equal to the dimension of the null space of T plus the dimension of the range of T and we know exactly what is the left hand side, R 4 has dimension 4. Now, let us look at what is the null space of T that is given to us. So, the null space of T, this is the set of all x 1, x 2, x 3, x 4 such that x 1 is equal to 5 x 2 and x 3 is equal to 7 x 4.

So, I will not elaborate on this. I will just say that, that from whatever we have seen in the first two weeks, check that B which is given by what will it be, let us say 5, 1, 0, 0 and 0, 0, 7, 1 will turn out to be a basis. So, you should check this out. Look at the, one can look at the expression here and guess what would be the basis. I know that it should be surjective and hence I know that this is going to be a basis. So, it is your job to go back and check that this is indeed a basis. But this implies that dimension of, so this is a check for you. This implies that the dimension of null of T is equal to 2.

What does it mean for the dimension of null of T to be equal to 2? This will turn out to be equal to 2 in that case. So now what happens to star? Then by star 4 is equal to 2 plus dimension of R of T. In other words, dimension of R of T is equal to 2 but where is R of T? Notice that R of T is a subset of R 2. In fact, it is a subspace of R 2. Since R of T is a subspace of R 2, which is a dimension 2 vector space of dimension 2 itself, what is forced? Then R of T is equal to R 2 because if we have a subspace, which has dimension equal to the dimension of the vector space then the subspace is necessarily equal to vector space. But this is precisely what we mean by saying that T is surjective, that is what we have set up.

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Problem 5: Suppose V is a finite dimensional vector space and $T: V \to W$ be a linear transformation. Prove that there exists a subspace U of V s.t $U \cap \text{null}(T) = \{0\}$ and $R(T) = \{Tu : u \in U\}$.

Solution: Since V is finite dimensional, by dim theorem $\dim(R(T))$ is finite

Let $n = \dim(R(T$

$\dim(R(T))$ is finite

Let $n = \dim(R(T))$

& $\beta = (w_1, \ldots, w_n)$ be a basis of $R(T)$.

Let v_j be vectors in V s.t

$Tv_j = w_j$.

Let $U = \text{span}(v_1, \ldots, v_n)$

Claim: U is the required subspace.

and $T: V \rightarrow W$ be a linear transformation. Prove that there exists a subspace U of V s.t $U \cap \text{null}(T) = \{0\}$ and $R(T) = \{Tu : u \in U\}$.

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Let $n = \dim(R(T))$

& $\beta = (w_1, \ldots, w_n)$ be a basis of $R(T)$.

Let v_j be vectors in V s.t

$Tv_j = w_j$.

The next problem let me call it 5, problem 5 it states following, this is again up to that something exists. So suppose V is a finite dimensional vector space and T from V to W be a linear transformation. Prove that there exists a subspace U of V such that U intersected with null space of T is equal to the 0 set and such that and R of T is equal to T u for u in capital U. So, if you are to recall the concept of an isomorphism, what this says is that there is a subspace U of V such that when T is restricted to U, it is an isomorphism onto the range of T.

So in particular, T is a surjective map, then what this problem essentially says is that there exists a subspace U of V such that when T is restricted to U as a map from U to W, it is an isomorphism. That comes later, let us first prove the problem to the existence of such a subspace U. So in order to prove the existence of such a subspace, let us focus instead on the range space of T. So, since V is finite dimensional, so notice that in the problem, there is no assumption on the finite dimensional W, we do not need W to be finite dimensional to establish something of the sort. So since V is finite dimensional however, this forces by the dimension theorem, dimension of R of T which is dimension of V minus the dimension of null space of T is finite.

This is a finite dimensional vector space, subspace of W. So, let us say n is the dimension of R of T and what does it mean to say that n is a dimension? There is a basis, there are basis of R of T of size n and beta be the one such basis, W 1 to W n be a basis of R of T. But what do we know about R of T? R of T is precisely the range of T. So, in other words, every W i is the image of some vector V i. So, let V j rather be vectors in capital V such that T V j is equal to W j. Now, let alpha be equal to or rather let U be equal to the span of all these V 1, V 2 up to V n. My claim is U is the required subspace.

So, when I say the required subspace what does it mean? Let me just take you back to the statement of the problem. The problem says that, underline it in green, there exist a subspace U such that U intersected with null space of T 0 and R of T is exactly equal to this So, let us give a proof of this claim and establish it.

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$Tv_j = w_j$.

Let $U = \text{span}(v_1, \ldots, v_n)$

Claim: U is the required subspace.

Let $v \in U \cap \text{null}(T)$.

$\Rightarrow v = a_1 v_1 + \cdots + a_n v_n$ & $T(v) = 0$

Then $T(v) = a_1 Tv_1 + \cdots + a_n Tv_n$

$= a_1 w_1 + \cdots + a_n w_n = 0$

$\Rightarrow a_i = 0 \;\forall\, i$

$$\Rightarrow \quad a_j = 0 \quad \forall j$$

$$\Rightarrow \quad v = 0.$$

Hence $U \cap \text{null}(T) = \{0\}$.

Let $w \in R(T)$

$$\begin{aligned} w &= b_1 w_1 + \cdots + b_n w_n \\ &= b_1 T v_1 + \cdots + b_n T v_n \\ &= T(b_1 v_1 + \cdots + b_n v_n) \\ &= Tu \quad \text{for } u \in U. \end{aligned}$$

So, the first part is to show that U intersected with the null space of T is 0, this is what the question mark is. Let v be in the intersection of U and the null space of T. So, V is an element of capital U implies this implies that v is equal to some a 1 v 1 plus up to a n v n because recall that capital U is the span of v 1 v 2 up to v n. Then what is T of v? T of v is 0 and T of v is 0 because it is in the null space. But what is T of v? T of v is a 1 T v 1 plus up to a n T v n, the standard trick which we have been using so many times, this is equal to a 1 w 1 plus up to a n w n. And what do we know now? This is equal to 0. And we had started off with w 1, w 2,… w n, which are basis elements of R of T.

And therefore, they are linearly independent. I am not writing any of these things down. I am just writing that a j is equal to 0 for all j which implies v is equal to 0, so this implies that see, we started off with an arbitrary vector v in U intersected with the null space of T and you have proved that it is supposed to be 0, 0 is obviously in the intersection of both because both are subspaces. So, hence the intersection is equal to the 0 subspace and what else is left to be seen? R of T. So let w be some, let…to show the second part wherein we have to, what was the second part? It was to show that R of T is exactly T u for u in capital U.

I am just underlining that in green right now, this is what we are going to do. So, let us take some arbitrary vector w in R of T, we know that w is equal to a 1, we have used a 1 already, b 1 w 1 plus up to say b n w n for some scalars v 1, v 2 up to v n, which is equal to b 1 of T v 1 plus up to b n of T v n which is equal to T of b 1 v 1 plus up to b n v n, which is an element of capital U, which is equal to T u for u in capital U. That is precisely what we have set up. So yes, we have established this now, it is a good task for you to sit down and check that T restricted to U, we have done all the work already, you have to just check that T restricted to

U is an isomorphism and it is an invertible linear transformation from u to R of T, it is not a part of this problem, but it is a worthwhile exercise for you to sit and think about.

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$= Tu$ for $u \in U$.

Problem 6: Let V_1, V_2, V_3 & V_4 be vector spaces s.t. $\dim(V_1) = 8$, $\dim(V_2) = 5$, $\dim(V_3) = 7$ and $\dim(V_4) = 6$. Let $T_1 : V_1 \to V_2$, $T_2 : V_2 \to V_3$ and $T_3 : V_3 \to V_4$ be linear transformation & let $T = T_3 T_2 T_1$. Then prove that T is not surjective

subspace

Enough to show that $\dim(R(T)) < \dim(V_4) = 6$

Recall $T = T_3 T_2 T_1$

Claim: $R(T) \subset R(T_3 T_2)$

If $w \in R(T) \Rightarrow \exists\, v \in V_1$ s.t $Tv = w$

Recall $T = T_3 T_2 T_1$

Claim: $R(T) \subset R(T_3 T_2)$

If $w \in R(T) \Rightarrow \exists\, v \in V_1$ s.t. $Tv = w$

$\Rightarrow T_3 T_2 (T_1 v) = w$

Let $T_1 v = u$ then $w = T_3 T_2 u$. $\Rightarrow w \in R(T_3 T_2)$

$$T_3 T_2 : V_2 \to V_4$$

$$\dim(V_2) = \dim(N(T_3 T_2)) + \dim(R(T_3 T_2))$$

Problem 6: Let V_1, V_2, V_3 & V_4 be vector spaces s.t. $\dim(V_1) = 8$, $\boxed{\dim(V_2) = 5}$, $\dim(V_3) = 7$ and $\dim(V_4) = 6$. Let $T_1 : V_1 \to V_2$, $T_2 : V_2 \to V_3$ and $T_3 : V_3 \to V_4$ be linear transformation & let $T = T_3 T_2 T_1$. Then prove that T is not surjective.

Proof: T is not surjective if $R(T)$ is a proper subspace

Enough to show that $\dim(R(T)) < \dim(V_4) = 6$

$$\dim(R(T_3 T_2)) = \dim(V_2) - \dim(Null(T_3 T_2))$$
$$\leq \dim(V_2)$$

$\Rightarrow \dim(R(T_3 T_2)) \leq 5$

$\Rightarrow \dim(R(T)) \leq \dim(R(T_3 T_2)) \leq 5 < 6.$

■

Let us now look at one more application of the dimension theorem. So this is problem 6, I guess. It is problem 6. So problem 6, so let V 1, V 2, V 3 and V 4 be vector spaces such that dimension of V 1 is equal to 8, dimension of V 2 is equal to 5, dimension of V 3 is equal to 7 and dimension of V 4 is equal to 6. So, we are given four vector spaces, V 1, V 2, V 3, and V 4, dimensions of these are 8, 5, 7, 6. Now, let us consider linear transformation. So, let T 1 from V 1 to V 2, T 2 from V 2 to V 3 and T 3 from V 3 to V 4 linear transformations and let us call the composition and let T be equal to, be careful with the T 1, T 2, T 3 in the linear transformation, T 1, T 2, T 3.

So, we have seen that composition of linear transformations is again a linear transformation so, in particular T is a linear transformation. The problem is to prove then prove that T is not surjective so, if you notice we have absolutely no information about these specifics of T 1, T 2 or T 3. They are some linear transformations, in fact any linear transformation from V 1 to V 2, you consider call it T 1 consider some other linear transformation T 2 from V 2 to V 3 and so on. And you look at the composition. Whatever be those linear transformations, this problem tells us that it cannot be a subjective linear transformation. So, let us give a proof of this statement. So, what does it mean for some map to be surjective? It means that the range space is equal to the domain.

So, T is not surjective if R of T is a proper subspace but if it is a proper subspace, the dimension will be smaller. And only if the dimension is smaller, will it be proper subspace? Enough to show, hence enough to show that R of T, the dimension of R of T is strictly less than the dimension of what is the image of V 4, is the image of T. What is the image of T? V 4 and this is having some dimension, what is the dimension of V 4? That is being given here to be 6 as you can see here.

So, if we show that this is less than 6, if the dimension of R of T is less than 6, and we are done because it cannot be a proper subspace which has dimension equal to 6. Yeah, so, let us not try to prove that the dimension of R of T is less than 6. But what is T? Recall that T is nothing but T 3, T 2, T 1 and so, we make a claim R of T is contained in R of T 3, T 2. So that is not a difficult claim to prove if say for example, w belongs to R of T, this means that there exist v in Capital V 1 such that T v is equal to w. But that means, T 3, T 2, T 1 v is equal to w. Let us call T 1 v something that, let T 1 v be equal to say u, then w is equal to T 3, T 2 u, which implies that w is in R of T 2, T 3. So, yes, certainly R of T is contained in R of T 3, T 2.

Now, let us focus on T 3, T 2 now, what is T 3, T 2? T 3, T 2 is a map from V 2 to V 4 and the dimension theorem tells us that dimension of v 2 is equal to the dimension of the null space of T 3, T 2 plus the dimension of the range space of T 3, T 2. Or, to put it in another way, dimension of the range space of T 3, T 2 is equal to the dimension of V 2 minus the dimension of the null space of T 3, T 2. In other words, the dimension of the range space of T 3, T 2 has to be strictly, not strictly, less than or equal to the dimension of V 2, what is dimension of V 2?

Let me go back and show what the dimension of V 2 for u is, let me box it in green, it is equal to 5. So, what we have established here is that dimension of the range space of T 3, T 2 is less than or equal to 5, but we just showed that R of T is contained in R of T 3, T 2, this implies dimension of R of T is less than or equal to the dimension of R of T 3, T 2, which is less than or equal to 5, which is less than 6 and therefore it cannot be subjected, this is exactly what we had set up for. In the setup which we had just described, there cannot be linear transformations such that composite, it will turn out to be a surjective.

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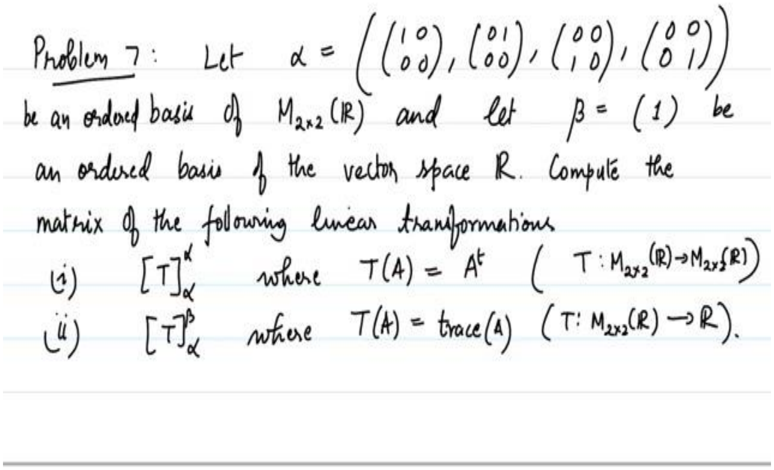

Solution: $T\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$

Recall that in this case $[T]_{\alpha}^{\gamma}$ is a 4×4 matrix.

The next problem is a problem on computing the matrix of a linear transformation corresponding to a given basis, I will just do a very straightforward example this is not a problem which is complicated at all. So let me not spend too much time on it. So, let us fix the basis so let alpha being equal to say the following basis of say 2 cross 2 matrices one, so this be a basis that alpha, the basis of M 2 cross 2 of R, basically 2 cross 2 matrices over R and let beta be equal to the set or rather ordered basis consisting of the constant 1 be an ordered basis both are ordered basis, let me not confuse you by writing it, 1 be an ordered basis here as well, an ordered basis of the vector space R.

So, here R is being considered as a vector space over itself. So, any non-zero element will turn out to be a basis, it is a dimension 1 vector space. So, what is this problem? This problem asks us to compute the matrix of the following linear transformations. What is the first one? First one is T of A, so let me say this is what we have to compute, alpha alpha matrix of T with respect to alpha alpha, where T of A, T and just let me just write it.

This is the transpose of this matrix. So, notice that T is the map from M 2 cross 2 of R to M 2 cross 2 of R. And what about the second one? The second one is T alpha beta where T of A is equal to the trace of A and where is this map from? So this is from 2 cross 2 matrices over R into R. So both, I will not check that these are linear transformations. I will just leave that to you to check whether these are linear transformations.

Let us jump into computing the matrix of these linear transformations. So let us give a solution. So the first one is the matrix of T with respect to alpha, right? So let us write down what T is. T of a 1 1, a 1 2, a 2 1, a 2 2, this is going to be equal to a 1 1, a 2 1, a 1 2, a 2 2, this is precisely the transpose of the matrix, which is just written to the left. And to compute the matrix of linear transformation with respect to a basis or to basis beta and alpha and beta, we compute T of the basis elements in alpha and write it down in terms of beta.

That will turn out to be the coefficients of what we have in terms of beta, will turn out to be the column corresponding to the vector which we have picked. So, in other words, so recall, so I am just recalling, or maybe I will just prove this or find the matrix here and recall the relevant aspects. So, in order to compute the matrix, so here what will be the matrix? So recall that in this case T alpha alpha is a 4 cross 4 matrix. Right, what will be the first column? So, let us see what the first column is.

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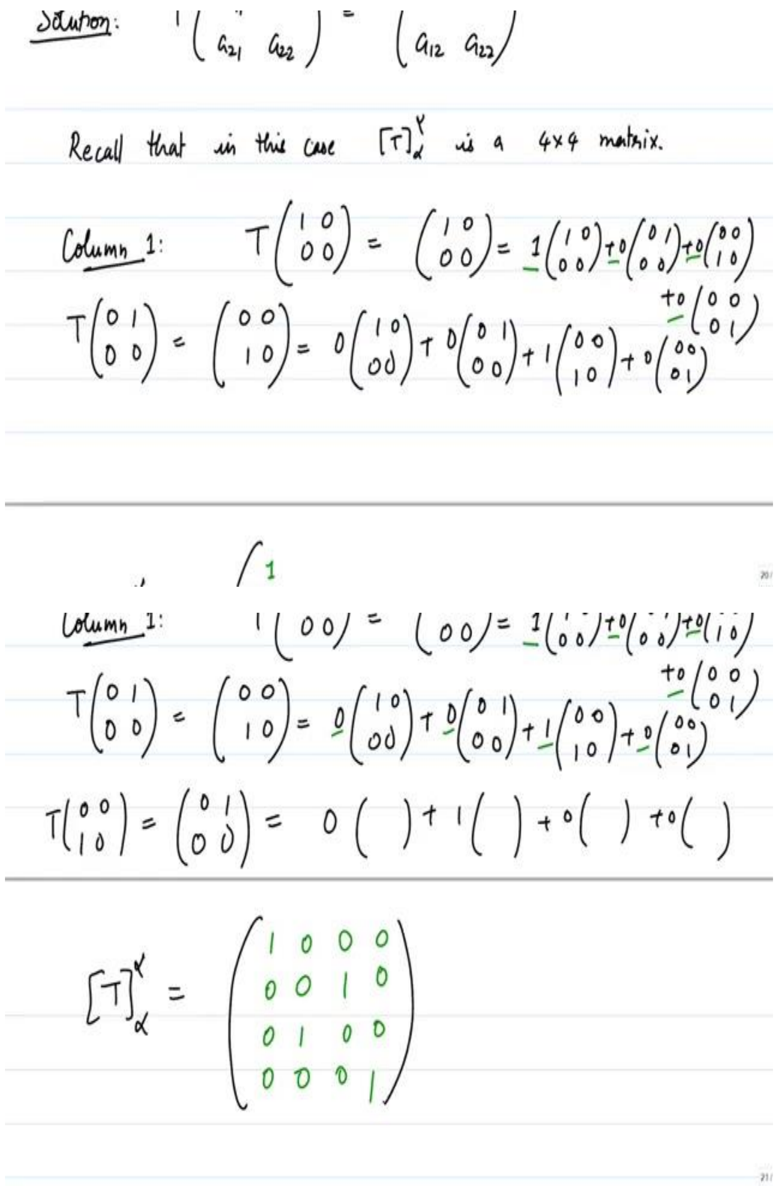

Column 1, column 1 is look at T of 1, 0, 0, 0. And see what it is in terms of the other matrices. The basis alpha, this is just going to be 1, 0, 0, 0 by definition, which is equal to one times 1, 0, 0, 0 plus zero times 0, 1, 0, 0 plus zero times 0, 0, 1, 0 plus zero times 0, 0, 0, 1. What should I do below, that zero times 0, 0, 0, 1 and hence the first, so let me just capture the matrix for you here.

T alpha alpha, the first column will just be these coefficients, which I am now lining in green and therefore, this will just turn out to be equal to 1, 0, 0, 0. How about the second column? Second column will be of corresponding to the second basis vector, which is 0, 1, 0, 0. And if you compute, this is just going to be equal to 0, 0, 1, 0 will be the image of 0, 1, 0, 0, which is equal to 1, which is not equal to 1. This is equal to zero times 1, 0, 0, 0 plus zero times 0, 1, 0, 0 plus one times 0, 0, 1, 0 plus zero times 0, 0, 0, 1. Left, what will be the second column here? That is just going to be 0, 0, 1, 0; 0, 0, 1, 0.

How about T of 0, 0, 1, 0? That is just going to be equal to 0, 1, 0, 0. That is just going to be zero times this plus one times this plus zero times this plus zero times this, whatever is that about. And from there I will take you to write this as 0, 1, 0, 0. And the final, let me not write down, computation is extremely similar. And this is exactly what the matrix of linear transformation T with respect to alpha is going to look like. So that is the first one, how about the second one?

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$$[\ ]_\alpha \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) $\quad T(A) = \text{trace}(A)$

$$T\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} + a_{22}$$

$$T\begin{pmatrix}1&0\\0&0\end{pmatrix}=1,\quad T\begin{pmatrix}0&1\\0&0\end{pmatrix}=0=T\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$T\begin{pmatrix}0&0\\0&1\end{pmatrix}=1$$

$$=\begin{pmatrix}1, & 0, & 0, & 1\end{pmatrix}$$

The second one is the trace to T of A is equal to the trace of A. So let us write it down explicitly and see what it is. This is T of a 1 1, a 1 2, a 2 1, a 2 2. And what is T of a 1 this, what is this? Is equal to a 1 1 plus a 2 2 I will check, I will not check that this is a linear transformation. I will leave that you, let us however, find out what the matrix of T will be here. Yet again, we are going to do the same type of calculations, we will compute T on the basis vectors, write it down in terms of the basis vectors in the image. And hence get hold of the columns of the matrix of the linear transformation.

So here, notice that a is coming from M 2 cross 2 of R, which is a 2 cross 2 matrix with the dimension 4 vector space and the image is a real number in the vector space R, which has dimension 1. So this matrix is going to be a 1 cross 4 matrix. So let us compute what the matrix will be. So what is P of 1, 0, 0, 0? This is just 1. Similarly, what is T of 0, 1, 0, 0? Which is equal to 0 which is the same as T of 0, 0, 1, 0 and what about T of 0, 0, 0, 1? That is also equal to 1.

So, this matrix is just going to be equal to, from here the first column comes, from here the second and the third column comes and from here even I have not write it down, it is 1. Fourth column. So, the 1 cross 4 matrix will look like this so, I did not take more complicated basis, because the calculation will just become slightly more complicated, it will not reveal anything more, you have to take the basis vectors from V, compute what T of those basis vectors is in terms of the basis vectors in w and that will give you the columns of the linear transformation, matrix of the linear transformation.

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Problem 8: Prove that for any vector space V having finite dimension, V and $\mathcal{L}(\mathbb{R}, V)$ are isomorphic.

Proof: Recall that

$$\dim(\mathcal{L}(V,W)) = \dim(V)\dim(W)$$

Proof: Recall that

$$\dim(\mathcal{L}(V,W)) = \dim(V)\dim(W)$$

$$\dim(\mathcal{L}(\mathbb{R}, V)) = \dim(V).$$

By a theorem, two finite dimensional vector spaces are isomorphic iff they have the same theorem.

Problem: Let $T: \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^{n+1}$ be the map $T(p) = (p(0), p(1), \ldots, p(n))$. Prove that T is a linear transformation & that T is an isomorphism.

So, the next problem demands that we prove 2 vector spaces as being isomorphic so, problem 8. So, prove that for any vector space V having finite dimension the vector space is V and V comma F are isomorphic. V and L of or rather R comma V are isomorphic. So, recall that L of V comma w was the space of all linear transformation from V to w, we had proved that that is a vector space. We had also proved that it is isomorphic to the vector space of m cross n matrices over R, where m is the dimension of V, dimension of w and n is the dimension of V.

So, in this case, this is actually a very straightforward proof, recall that L of R comma V which is a vector space of all linear transformations from R to V, this is isomorphic. In fact, let me not write all this. We had shown that it is isomorphic to the matrices of m cross the vector space of m cross n matrices and therefore, L of V comma w is having dimension equal to dimension of V times dimension of w. So recall that dimension of L of V comma w is equal to the dimension of V times the dimension of w.

But in our case, V is just, the role of V is w is now V in our case and V is R in our case, so therefore, by this argument dimension of L of R comma V is equal to the dimension of R times dimension of V, which is equal to the dimension of V. So, we know that now the dimension of L of R comma V is the same as dimension of V.

But what do we know about vector spaces which have the same dimension? By a theorem we proved in the 4th week two finite dimensional vector spaces are isomorphic if and only if they have this m dimension and by invoking this particular dimension, both these vector spaces are isomorphic. So, if you notice, we got hold of a proof of this particular problem, without doing any effort. We did not get hold of an explicit linear transformation from V to L of R comma V. And we did not bother checking whether it is a linear transformation.

In other words, our problem was cut short by the theory that we had developed and using that we get this particular problem in a jiffy.

Excellent, so, let me conclude by finally getting hold of the property of being invertible using the dimension theorem and they are all interconnected to be expressed to be expected. So, this problem asks us to prove that this particular map let T from p n of R to R n plus 1 be the map. So let me not write that, it is a linear transformation that you and I will not check. Let me just leave it to you. P of 1 up to p of n. So, notice that this is an element in R n plus 1. So, T of P is this. So, prove that of course, you have to prove that T is a linear transformation, prove that T is a linear transformation and it is an isomorphism. T is an isomorphism.

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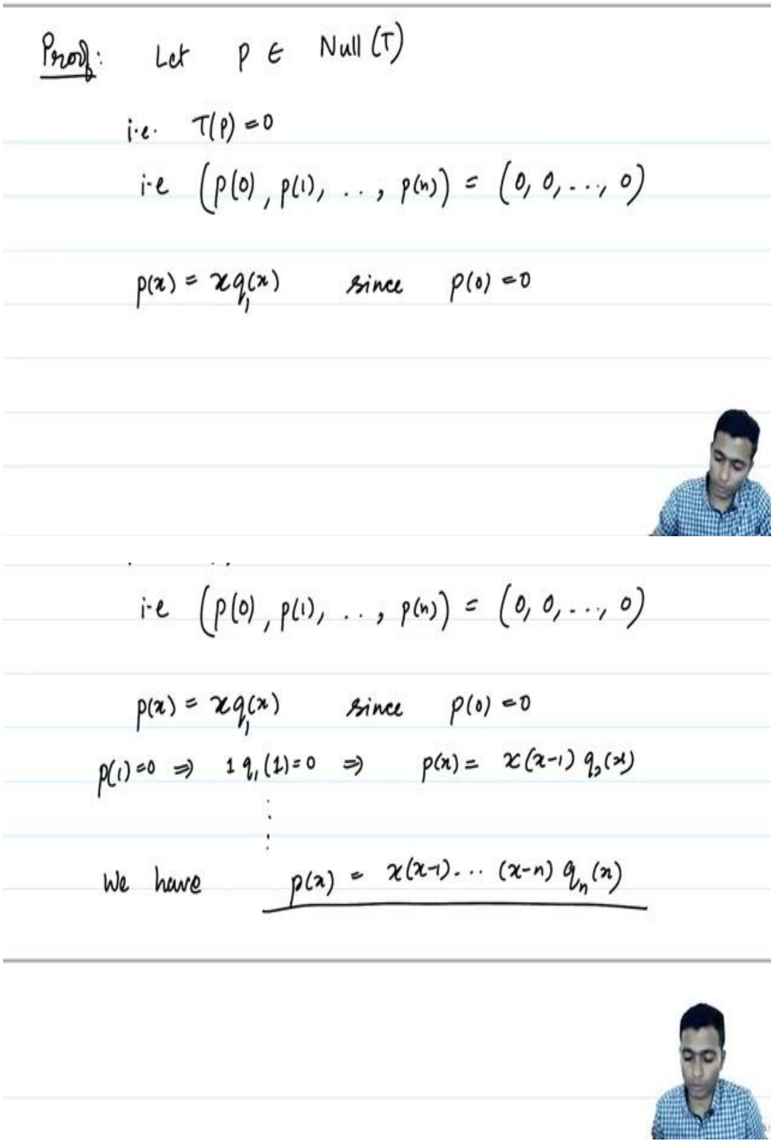

So one of the most essential things to establish that it is indeed an isomorphism is to check that it is injective. Now injective maps, to check that something is injective, what do we need to do? It is injective if and only if the null space is the 0 subspace, right. So, let us pick some vector let P be in the null space of T. What does it mean to say that T is in the null space of T? It means T of P is 0. But what does it mean to say that this is 0? i. e P of 0 is equal to all this is 0, P of 0, P of 1 up to P of n all these are equal to 0s.

That is precisely what it means for P to be in the null space of T but T is a polynomial. And by the factorization property, we know that because P of 0 is 0, P of x is equal to x times q of x or other let me write it as q 1 of x and we know that similarly, x times q1 of x does not satisfy, so observe that this implies q since… let me not rush, since p of 0 is equal to 0. Next p of 1 equal to 0 implies 1 times q 1 of 1 is equal to 0, this implies p of x is equal to x times x minus 1 times q 2 of x.

And continuing like this, we have p of x is equal to x into x minus 1, into up to x minus n times q n of x. But notice that this polynomial will have degree greater than or equal to n, if q n of x is non-zero and that will not be an element of P n of R. So we need this element to be in P n of R.

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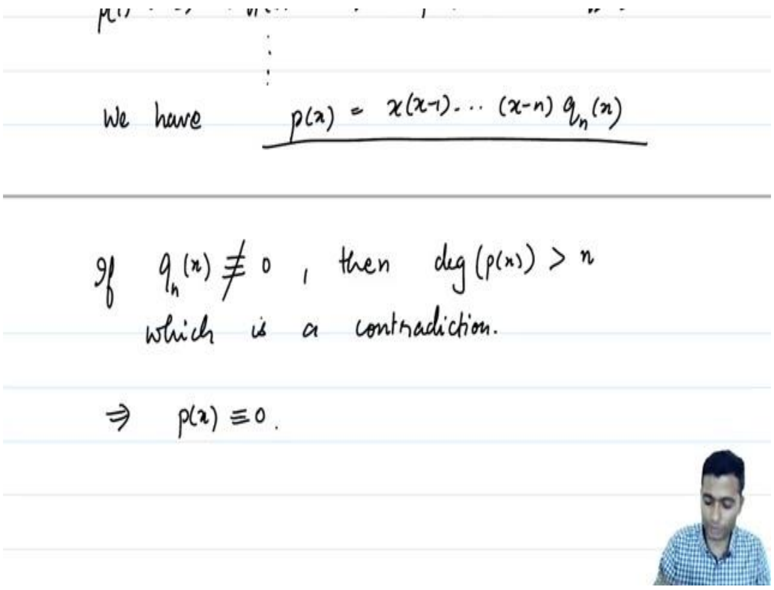

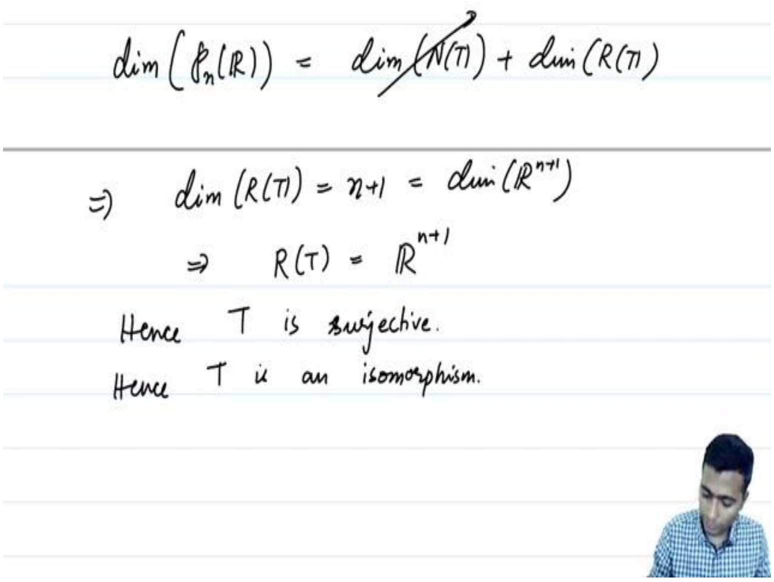

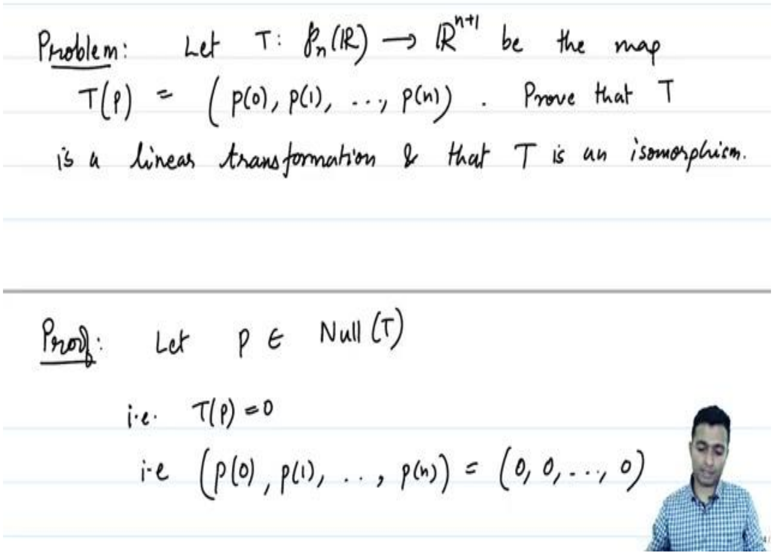

If q n of x is not identically equal to 0, then degree of p of x will be greater than n plus 1 which is a contradiction, since p of x to begin with was an element in p n of x, n plus 1 which will be greater than or equal to n plus 1, it will be greater than n for sure, which is a contradiction and therefore, q n of x is identically equal to 0. This implies that p of x is identically equal to 0. So, yes, we have established that the null space of T is just the 0 vector.

But then now we will invoke the dimension theorem to say that dimension of P n of R is equal to the dimension of the null space of T plus the dimension of R of T and this we know is 0 because null space is the 0 vector space which gives dimension R of T is equal to n plus 1, which is the dimension of R n plus 1.

So we do not need to explicitly check for surjectivity in this particular case, which might have been a complicated thing to do. The dimension theorem directly tells us that because our map is into R n plus 1, our dimension is known to be n plus 1 and this implies that R of T is equal to R n plus 1. Hence T is surjective. What can we say about injective and surjective linear transformation, should necessarily be an isomorphism. That is the definition because it will be invertible, right? Hence T is an isomorphism. So if you notice this is an isomorphism, which we have defined without going down to a basis.