Linear Algebra
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Lecture 6.4: Determinants

Next, let us recall the notion of a determinant of a square matrix of a given size. Many of you might be already familiar with the notion of a determinant. Determinant is a number which is defined in a complicated manner using the entries of the given matrix. However, it has some extremely nice properties, especially with respect to product of matrices and with respect to the row operations or column operations for that matter. So let us recall the definition of a determinant, we will define determinant recursively.

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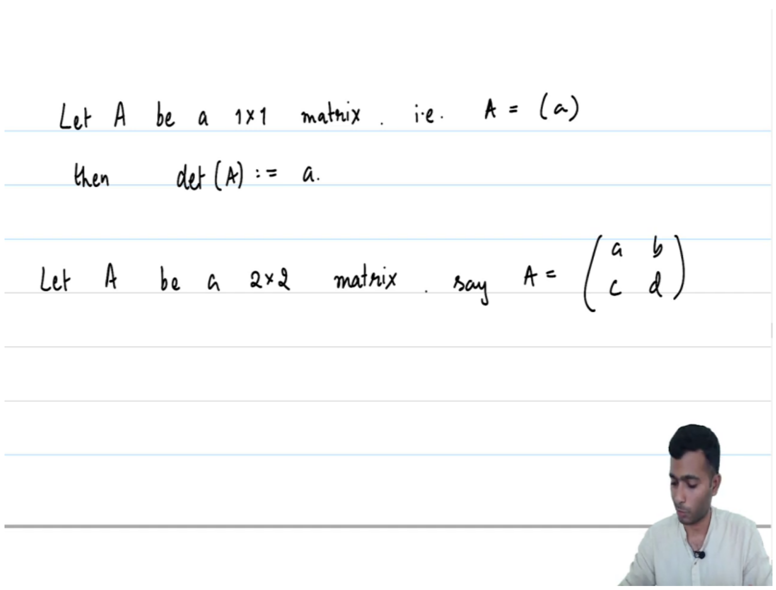

So let A be an n cross, sorry, be a 1 cross 1 matrix, so let us define the determinant for our 1 cross 1 metrics, i.e., A be just one entry a, then the determinant of A is just defined to be the entry a. Now, let A be a 2 cross 2 matrix. Let us say A is something like a, b, c, d. So this is the definition, so let me put two dots.

So if A is a matrix of this type, then define determinant of A to be equal to ad minus bc. So we will be giving the definition of determinant in a recursive manner. So for an n dimensional, sorry, for an n cross n matrix, the determinant will be using the knowledge of what the determinant is often n minus 1 cross n minus 1 matrix. So in order to do that, we should first develop the notion of what is called as a minor.

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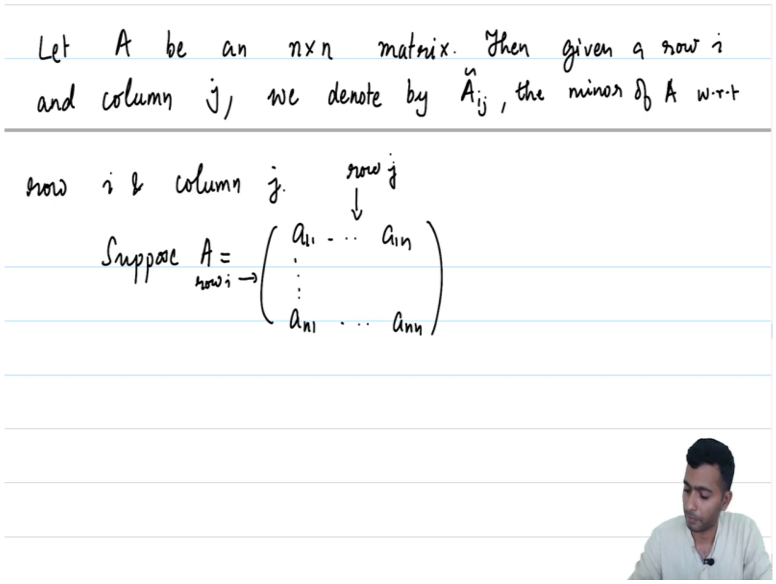

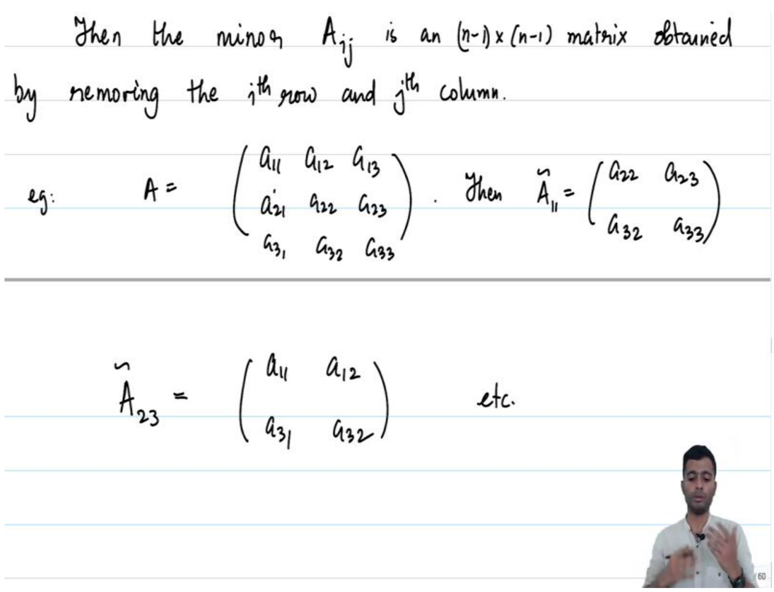

So definition, so let A be n cross n matrix. So, we have given the definition for n equal to 1 and n equal to 2 of a determinant. For more general n we have to develop certain notions which let us focus on, so let A be n cross n matrix. Then given a row i and column j, we call or we denote by Aij tilde, the minor. So I will define what this is, minor of A with respect to row i and column j. So we will see what the minor is.

So suppose A is an n cross n matrix, which is of this type a11 up to a1n, an1 up to ann. Suppose, this is our matrix A and suppose this is our row i and this is row j. Then the minor of A with respect to row i and row j is the n minus 1 cross n minus 1 matrix which is obtained by removing the ith row and the jth column.

Then the minor Aij tilde is obtained by removing. So it is an n minus 1, let me just specify that, this is an n minus 1 cross n minus 1 matrix obtained by removing the ith row and jth column. So let us look at the case when n is equal to 3 so suppose, so for example, suppose A is a11, a12, a13, a21, a22, a23 and a31, a32, a33. Let us look at some of the minors here. Then, what will be the first minor? Let us say a11 tilde that is just going to be equal to a22, a23, a32, a33. As you can notice, the first row and the first column has been removed.

What is say, for example, a23 tilde? a23 tilde will just be the matrix obtained by removing the second row and the third column, so this is going to be a11, a12 there is no third column and there is no second row. So this is a31, a32 and so on. So this is how we obtain the minors of a given matrix. Once we have the minors of a given matrix, we can define the definition or sorry, we can define the determinant of a matrix by using these minors, so let us see how it is done.

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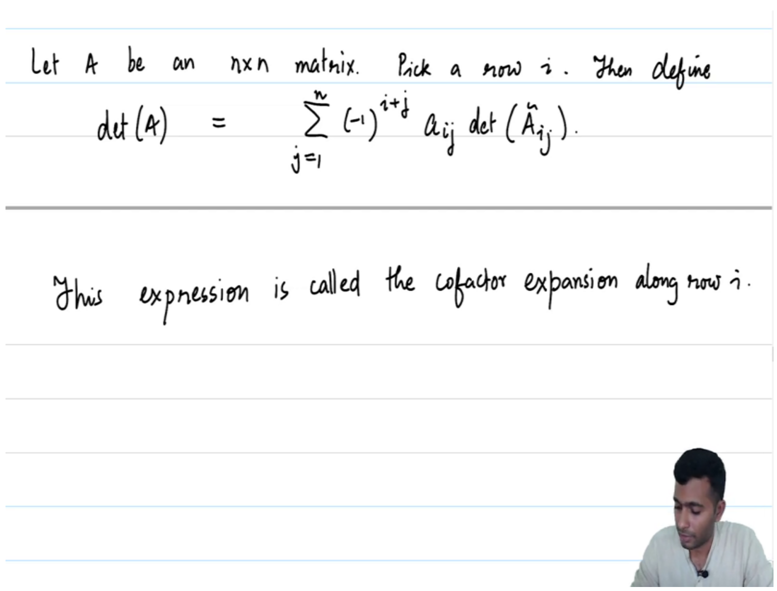

So definition of a determinant. So let A be an n cross n matrix, any n cross n matrix. Of course, all through our matrices have real entries are all n cross n matrices over r. Pick a row i, then define determinant of A to be equal to the sum minus 1 to the power i plus j, where the sum is from j equal to 1 to n of aij times the determinant of Aij tilde.

So you will look at, you pick a row i and then look at row i and column 1 removed. Look at the minor, take the determinant we know it is a n minus 1 cross n minus 1 matrix so assume that it has been defined up to order n and determinant of Aij tilde has already been defined and it makes sense and then you look at this expression. This is called the determinant of A.

So this expression is called the cofactor expansion along row i. The determinant of the minor Aij tilde is sometimes called the ijth cofactor. Anyway, let me not write that down. So this expression is called the cofactor expansion and this is exactly what the determinant of A is.

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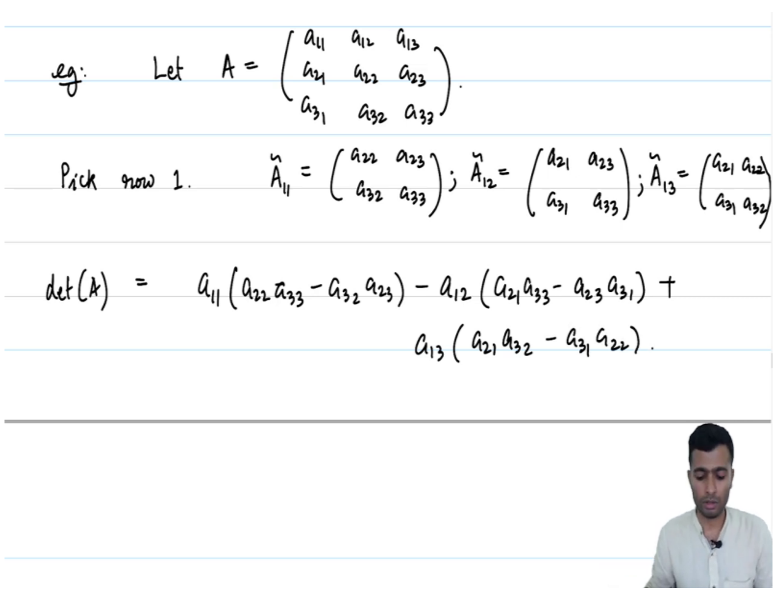

So let us look at the example of a determinant of a 3 cross 3 matrix. So example, let A be as above a11, a12, a13, a21, a22, a23, a31, a32, a33. So let us pick the first row, pick row 1 and then we can see that A11 tilde is a22, a23, a32, a33; A12 tilde is a21, a23, a31, a33 and A13 tilde is a21, a22, a31, a32. Let me be careful, there is a bracket which is there, yes.

So determinant of A by our very definition, if we look at the cofactor expansion along row 1, this is going to be equal to minus 1 to the power 2 is 1, a11 times determinant of A11 tilde, which is a22 minus, sorry, a22 times a33 minus a32 times a23. So plus minus 1 to the power 1 plus 2, which is minus 1, so minus a12 times, let me quickly write down the expression and I hope I would not make any mistake. Plus a13 times a21, a32 minus a31, a22.

So as you can see, the expression for the determinant of a 3 cross 3 matrix is already becoming cumbersome. But many times it is the properties of the determinants, which are very useful. We will come to that in a matter of time, but before that I would like to make an observation when we defined the determinant of a matrix, the definition of the determinant, yes.

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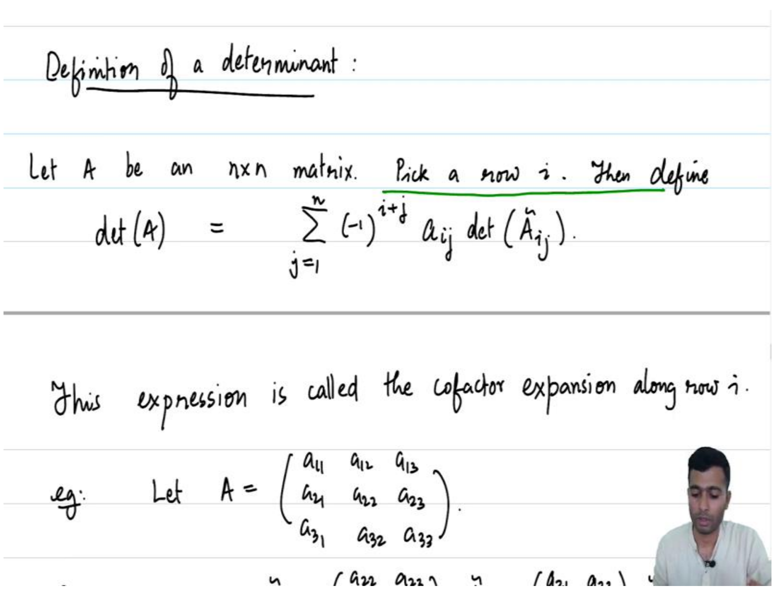

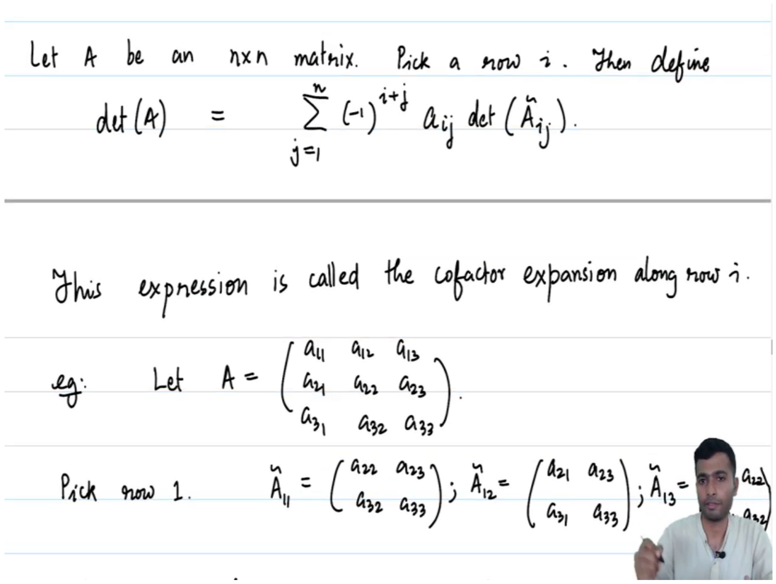

As you can see, we have made a choice of the row along which we are looking at the cofactor expansion to define the determinant of A. It is true that this choice does not matter. So what if we pick say, row 1 and get a number and row 3 gets another number, then determinant is not a well defined quantity, right?

But as it happens, the different expansions around different rows will give us the same number. It is a remarkable property, which I will not be proving for lack of machinery, let me put it that way, but this is something which you should keep in mind maybe at a later date when you look at more advanced mathematics a simpler proof can be given than what can be given here.

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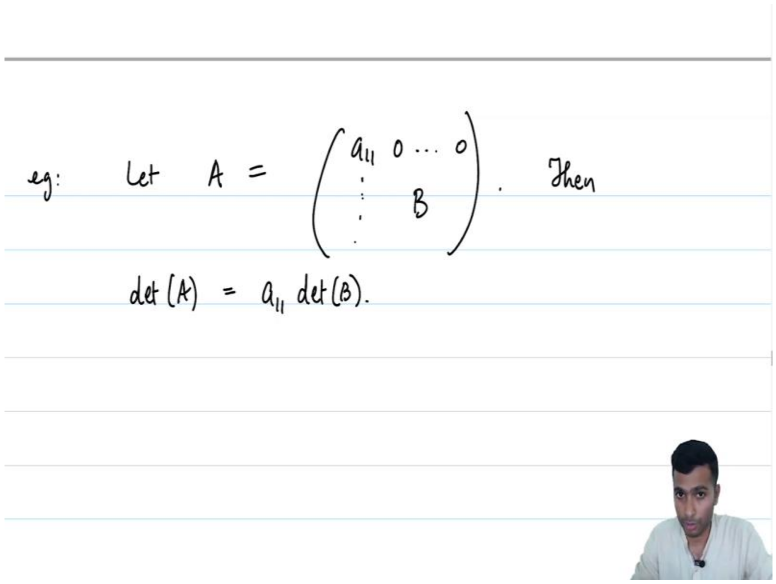

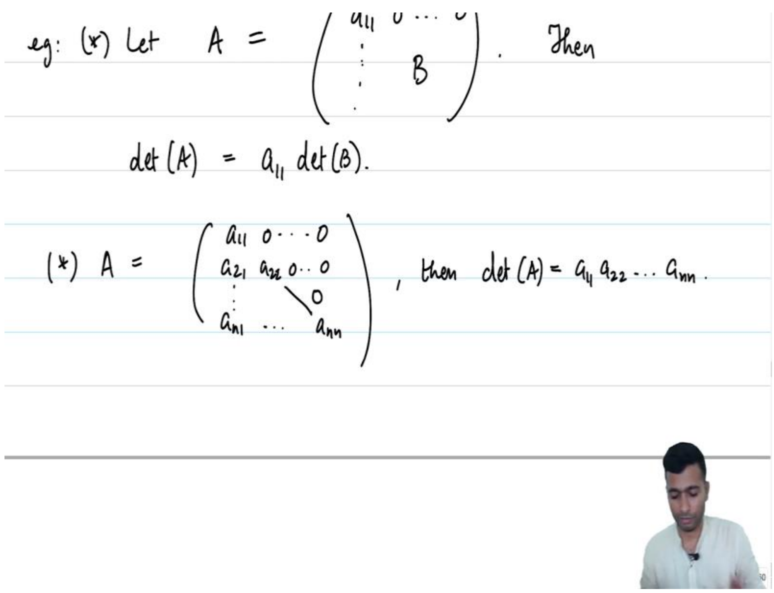

Let us now look at the determinant of certain special types of matrices. So suppose for example, as an example let us consider A to be of this type, that is a11 here and then there are zeros in the remaining entries. Does not matter what it is here but suppose the minor of a11 of the row 1 and column 1 is B, then determinant A is equal to a11 times the determinant of B. Because the remaining expressions in the cofactor expansion are zero.

Now by induction, if we therefore have A is equal to say, a11, 0, 0, a21, a22, 0, 0, an1 up to ann so all zeros across the diagonal and lower diagonal, then determinant of A will just turn out to be equal to a11 times, a22 times up to ann. Should check that this is indeed true. And we can say, well, let us not say more yet in particular, if you take a diagonal matrix, then the determinant of the diagonal matrix is just the product of the diagonal entries.

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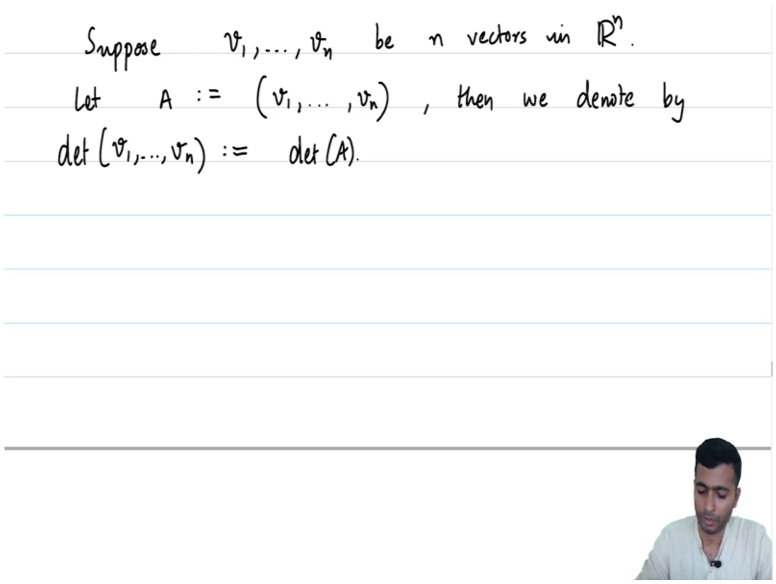

Alright, now let us explore some properties of the determinant. Before we explore some properties of the determinant let us also look at the determinant of a matrix obtained by some columns of say, Rn. Suppose v1, v2 up to vn be n vectors in Rn, then let A be the matrix which is obtained by the column vector representation. So I have just not put the brackets, it is an abuse of notation, v1 here represents the column vector, v1 and so on up to vn, so v1 up to vn are column vectors, and suppose A is the matrix.

Then we denote by det or determinant of v1, v2 up to vn to be equal, so this is by definition equal to the determinant of the matrix A. So the order in which v1 to vn are being taken does matter, we will come to that in some time. Okay, let us now look at some properties of the determinant.

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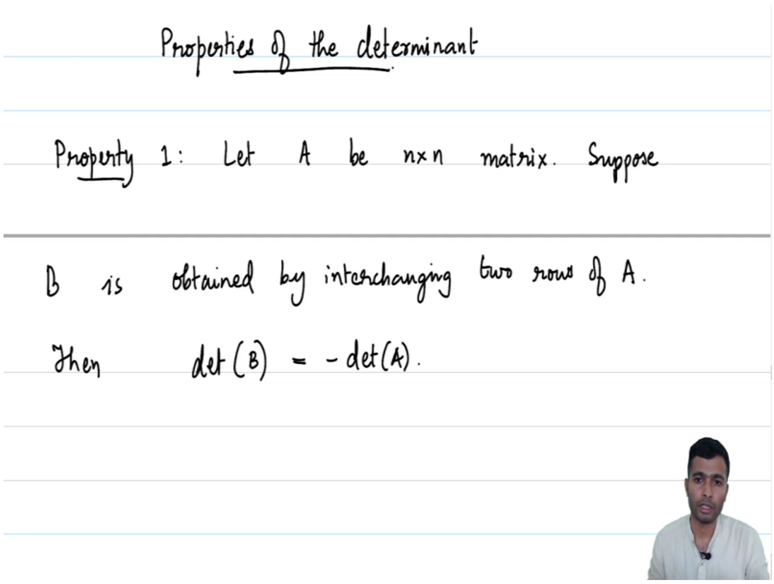

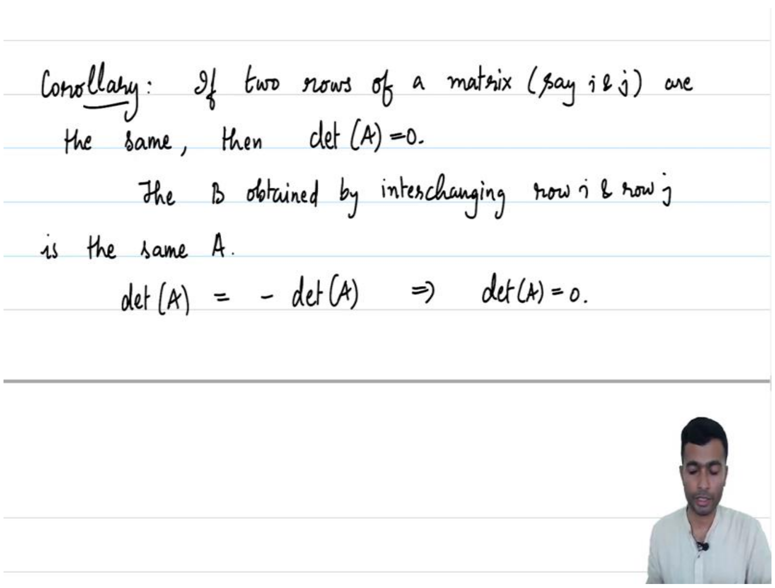

The first property that I would like to discuss is the following. So let A be n cross n matrix. Suppose, B is obtained by interchanging two rows of A. Then determinant of A is equal to sorry, determinant of B is equal to minus of the determinant of A. So the property of interchanging rows is antisymmetric, it inverts the sign of the determinant.

So in particular, if you look at the row operation of type one, what it does is to invert the sign of the determinant. We will not give a proof of this, however let us look at a corollary to this particular statement. As a corollary, we can conclude that if two rows of a matrix are the same then the determinant should be zero.

So if let me write it down, two rows of matrix say i and j are the same, then determinant of A is equal to zero because do the elementary row operation of type one where we invert row i and row j, the matrix B that we get is actually the same as matrix A and then the B, so let me just note that the B obtained by interchanging row i and row j is the same as A as the matrix does not change.

And the property tells us that determinant of B, which is equal to the determinant of A is just minus of determinant of A, and therefore two times determinant of A zero, which implies determinant of A is equal to zero. So notice that if we multiply the determinant of sorry, if we multiply A from the left by an elementary matrix of type one, then the determinant, the sign of the determinant changes.

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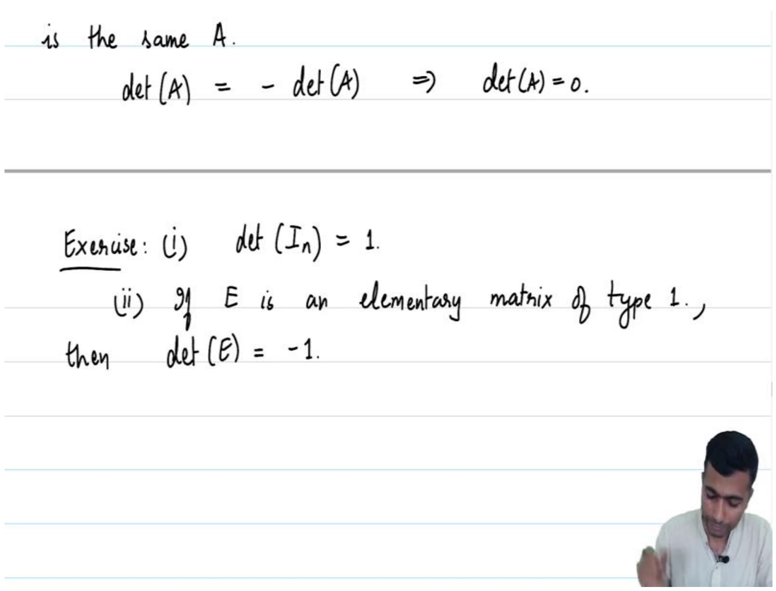

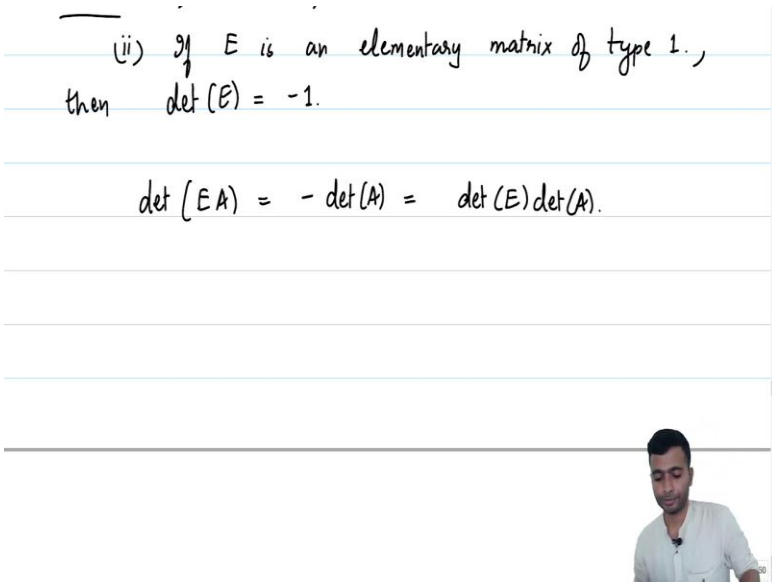

Also note that it is an exercise for you to check that the determinant of an elementary matrix of type one is equal to minus 1. In fact, first exercise before even we go further determinant of In is equal to 1. The identity matrix we have already seen that the diagonal matrices will have determinant equal to the product of the entries and we know that the identity has all the diagonal entries once. So this exercise is quite straightforward.

This exercise, however, is a little more work. Maybe you should spend some time to do it. If E is an elementary matrix of type one, then determinant of E is equal to minus 1. And therefore what this tells, what this property tells us is that determinant of E times A is equal to minus of determinant of A, which is the same as determinant of E times determinant of A.

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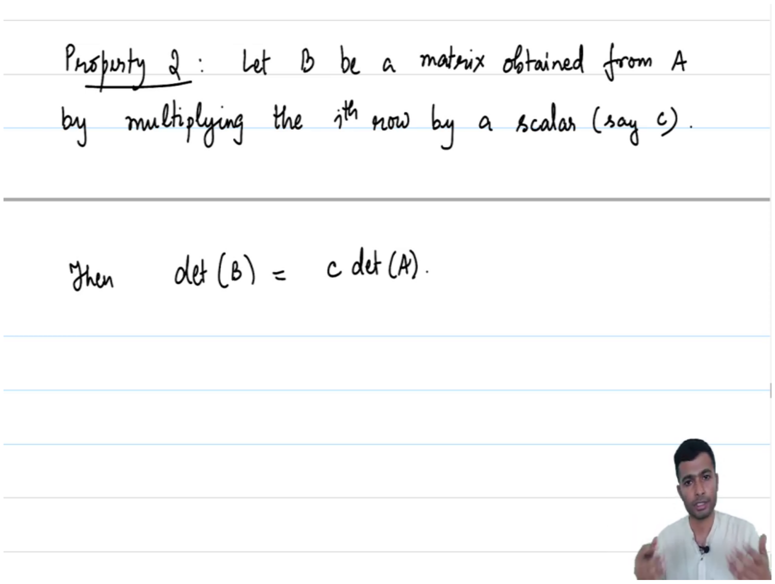

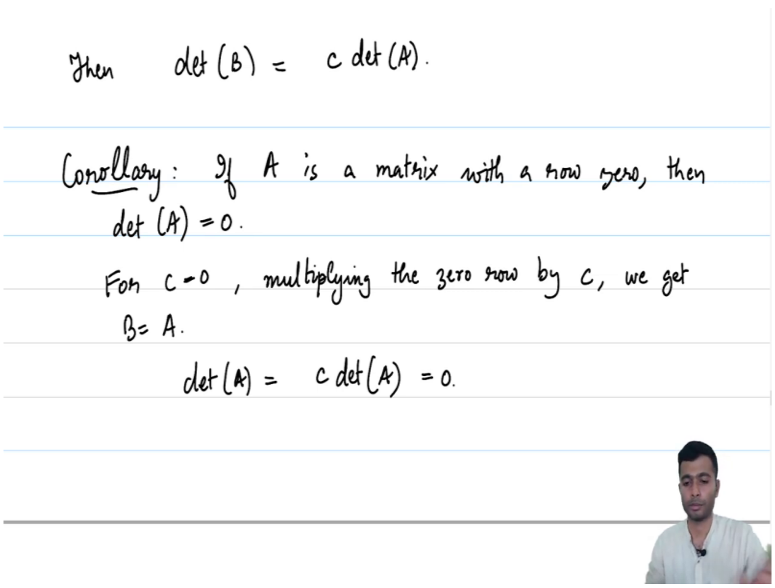

Alright, so now let us look at the next property. Let us call it property 2. The next property is keeping in mind the elementary row operation of type two. If you recall, it is just multiplying a particular row by a constant. So let B be a matrix obtained from A by multiplying the ith row by a constant, a scalar say c. Then determinant of B is equal to c times the determinant of A.

I will again, not prove this property. It is quite straightforward however to check this property by considering the cofactor expansion along i itself and notice that the Aij that features in the expression will now be replaced by c times Aij, the c comes out common c times whatever remaining is the determinant of A and therefore, this property follows.

As a corollary to this property however, we do have that if there is a matrix, which has all entries zero in a row. So if A is a matrix with a row identically equal to zero, then determinant of A is equal to zero. So let B be the matrix obtained by, so notice that even though in the elementary row operation of, say type two, we did demand that we always multiply by nonzero scalars. Here, in this property, we do not need to bother about that even for c equal to zero this property holds. And that is precisely what we will be doing here.

So for c equal to zero, multiply the zero row by c and this gives us, multiplying the zero row by c, we get back, we get B is equal to A. Therefore, determinant of B, which is now determinant of A is equal to c times the determinant of A. But c is zero, which is equal to zero and therefore if you have a matrix which has one row zero, then the determinant is zero.

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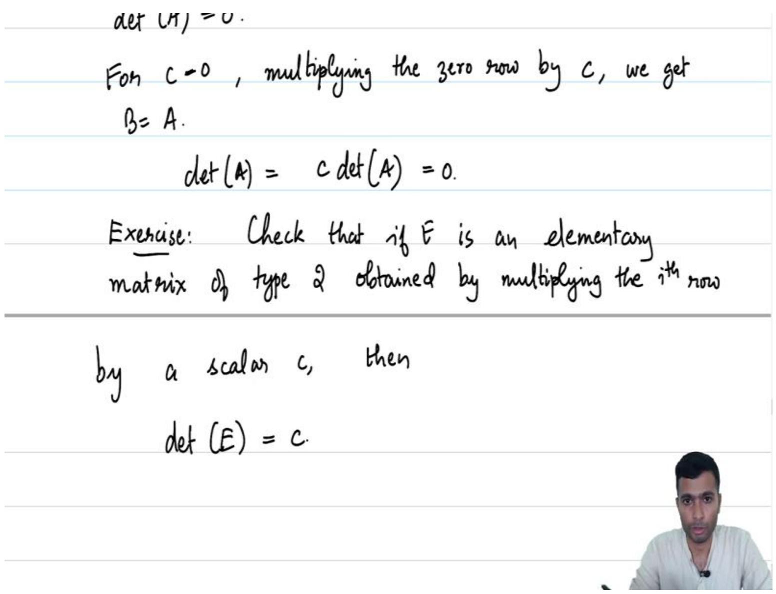

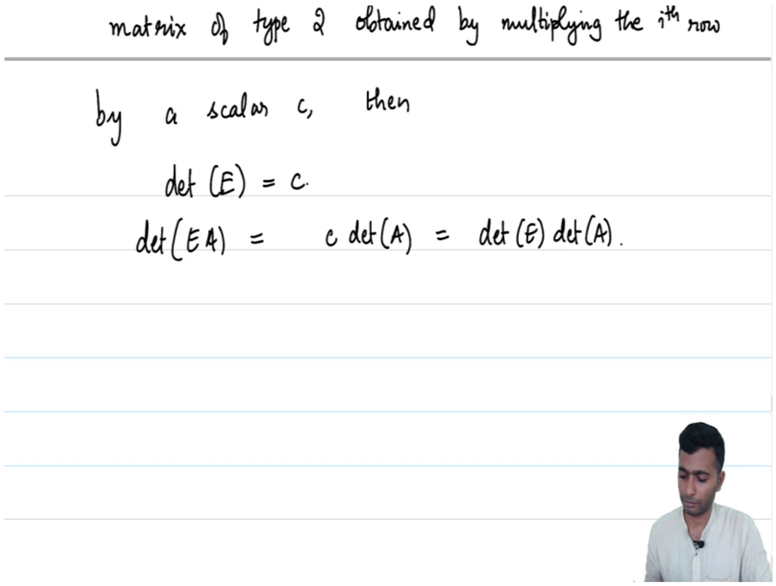

So now, an exercise is for you to check by brute force that, check that if E is an elementary matrix of type two obtained by multiplying it straightforward, multiplying the ith row by a scalar c, then determinant of E is equal to c. Actually this exercise is quite straightforward because after multiplying one of the rows in the identity matrix by a scalar c we end up with a matrix which has one in all the diagonal entries except that ith one where it will be c.

And therefore the product will just be equal to c and therefore, what we have is determinant of E times A is again equal to determinant of E, c times determinant of A because the ith row is getting multiplied by c and we just saw by property two that manifests in this manner, but this is nothing but determinant of E times determinant of A.

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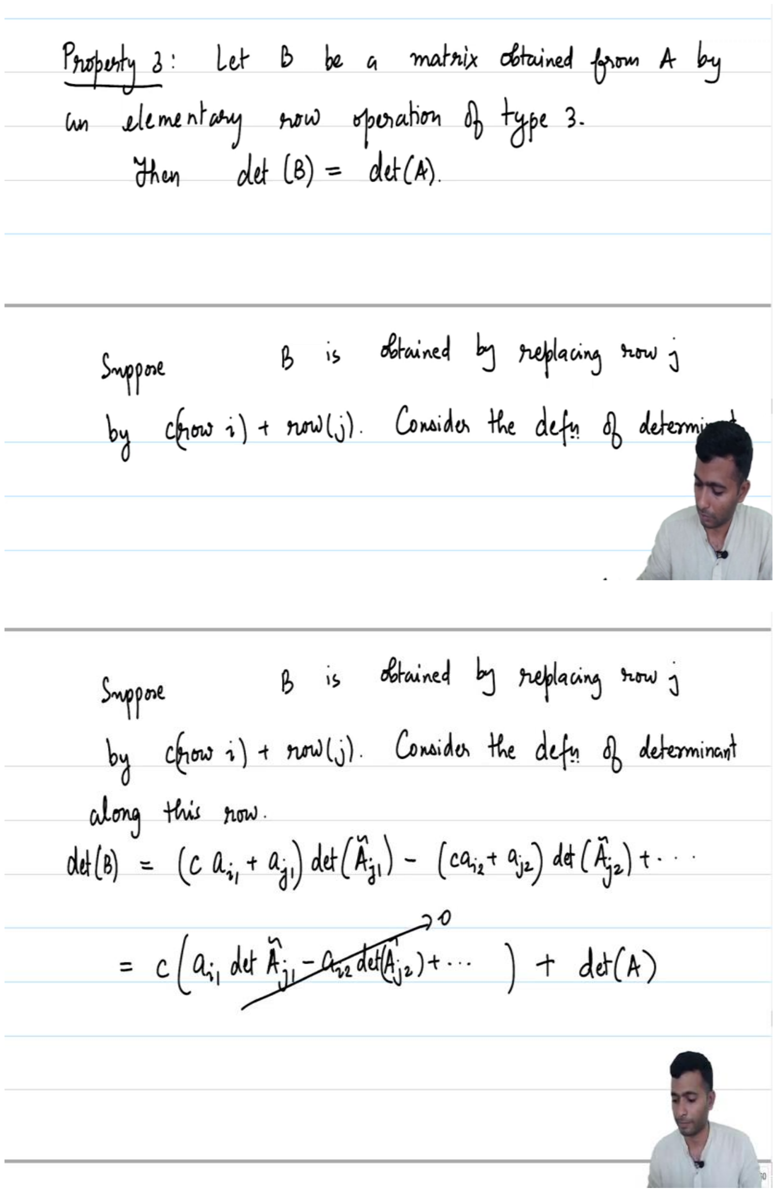

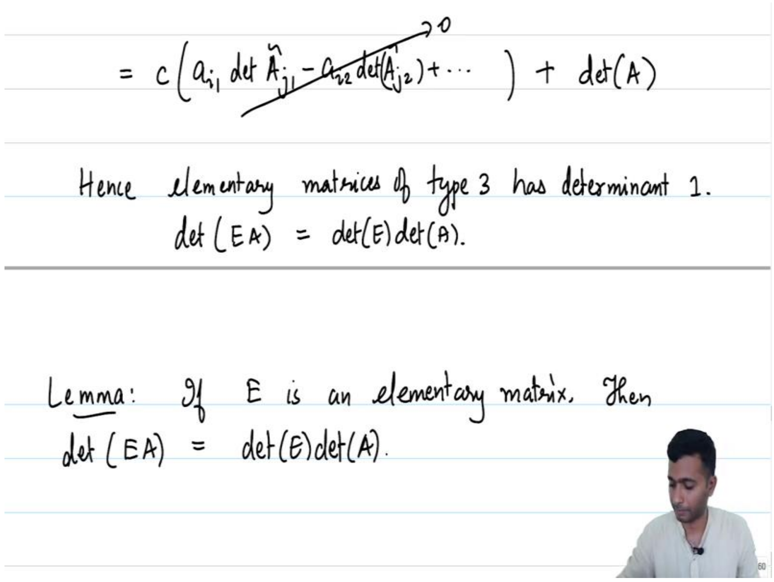

So let us now look at property 3. This should concern, this should certainly concern the elementary row operation of type three. So let B, let me just write it like this, let B be a matrix obtained from A by an elementary row operation of type three. So let me not write down the details, there is an i and j and there is a scalar c. Row i is, or row j is replaced by row j plus c times row i.

Then determinant of B is equal to the determinant of A. Again, I will not prove this, the only thing to note is that the determinant of B should be evaluated along row j and there each of the expression will have a row i, c times row i and row j component which remains, then let us extract out the, maybe I should write it down.

So suppose, B is obtained by replacing row j by c times row i plus row j. Consider the definition of determinant along this row. It can be shown that it splits, it is a linear property. The linearity will become evident by considering the expansion here the typical, let me just write down the expression for you.

Then the determinant of B will just be c times ai1 plus aj1 times determinant of A. This is along row j, so j1 tilde minus c times ai2 plus aj2 determinant of Aj2 tilde plus so on. We will be able to extract out c times ai1 determinant of Aj1 tilde minus ai2 determinant of Aj2 tilde and so on. And then whatever remains is determinant of A.

But if you look at what is inside the bracket A it is the determinant of the matrix A, the determinant of a matrix say B prime, where B prime is obtained by B by replacing the jth column by the ith column. And we have just seen a few minutes back that if you have a matrix which has two rows identical, then the determinant is zero. So you get that this quantity will be zero whatever remains is determinant of A.

So this has a good implication or this has a deep implication on elementary matrices of type three, so hence, elementary matrices of type three determinant has determinant 1. Again, we have determinant of B is E times A which is B is equal to determinant of A itself which is one times the determinant of A which is determinant of E times determinant of A. So we have just proved the following lemma if E is an elementary matrix irrespective of which type then determinant of EA is equal to determinant of E times determinant of A.

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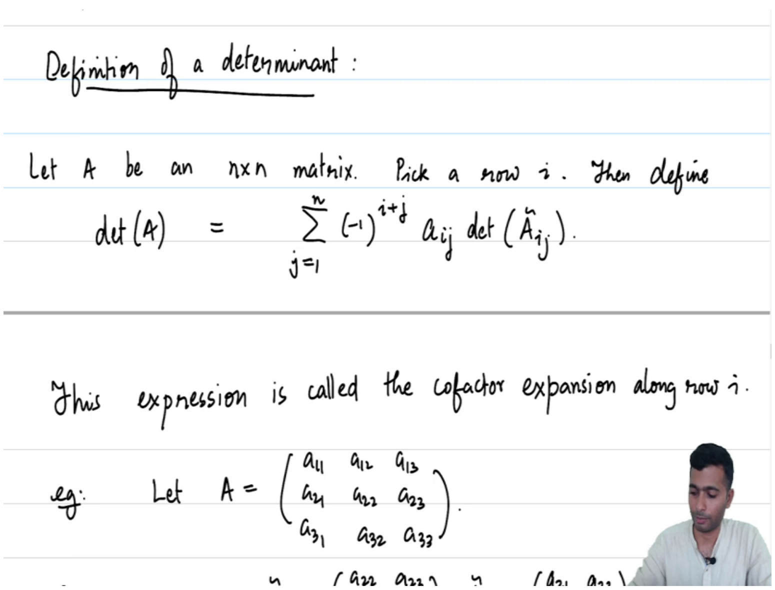

So at this point I would like to again draw your attention to the definition of the determinant that we gave. If you recall that the determinant was defined by looking at the cofactor expansion along a row. A similar definition could have been defined by looking at the cofactor expansion along a column. And just as how it does not, the definition does not depend on the row, the definition using the cofactor expansion along a column will also give us back the same number.

All these beautiful properties do exist for the determinant A. However, we will not be proving this statement, it falls in the same category. So if you look at the expansion, cofactor expansion along the column say, column j then the fact that the number that we get will be the same as this number is also something which we will assume right now. The reason I would like to note this particular point is that all the statements that we have mentioned with respect to rows do hold for columns as well.

And these properties also hold for column operations. So in other words, if A is an n cross n matrix if E is a elementary matrix and if you look at A times E then determinant of A is equal, determinant of A times E is equal to determinant of A times determinant of E. So all these expressions go through for column operations as well. Alright, so these are the basic properties, which I would like to note. Now, let us look at some of the more remarkable properties that we will be able to say from this.

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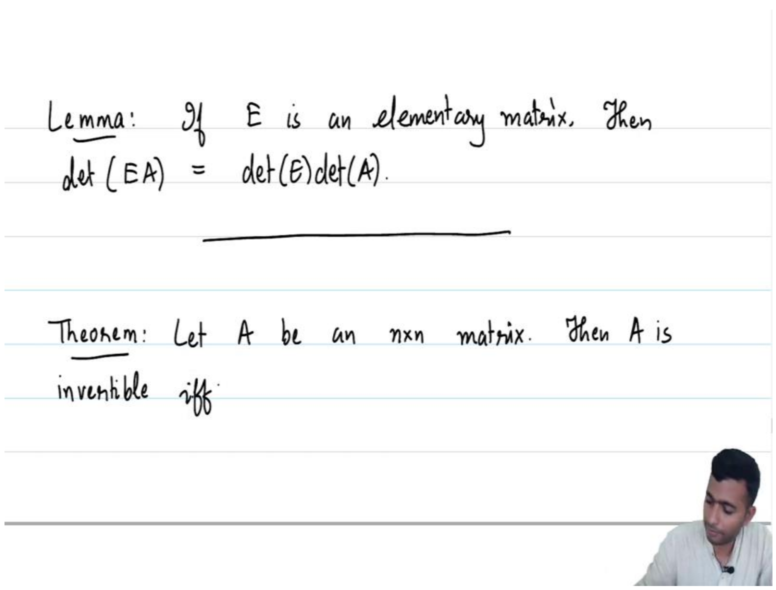

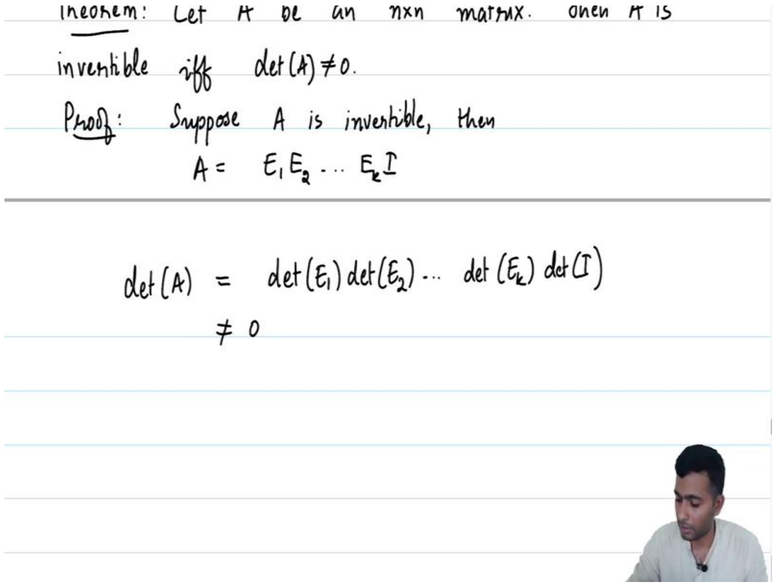

The first one being a theorem. This theorem tells us that the determinant measures how far our matrix is from being an invertible matrix. So let A be an n cross n matrix, then A is invertible if and only if determinant of A is not equal to zero. So let us give a proof of this. It is quite remarkable that invertibility is being captured in whether the determinant of the given matrix is non-zero.

So suppose A is invertible then a result from the previous video tells us that A can be written down as a product of elementary matrices. Then A is equal to E1, E2 to some Ek times the identity. But we just noted that determinant of A then will be equal to after repeatedly using the theorem or lemma which we proved earlier, this is equal to determinant of E1 times determinant of E2 dot dot determinant of Ek times determinant of I which is equal to 1.

Now, notice that each of the EI's are elementary matrices, which have nonzero determinants as we had just noted, and hence this is not equal to zero. Product of nonzero real numbers will not be zero. So yes invertibility immediately tells us that the determinant cannot be zero. Suppose determinant is nonzero that is where we.

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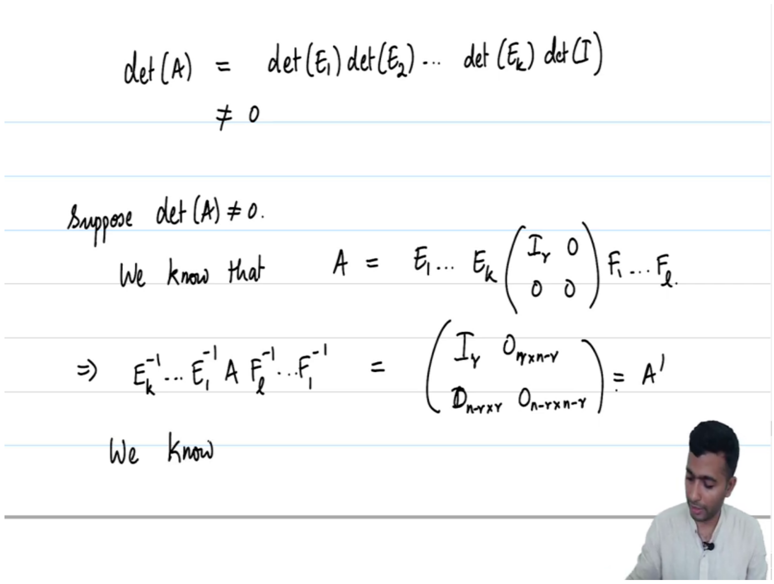

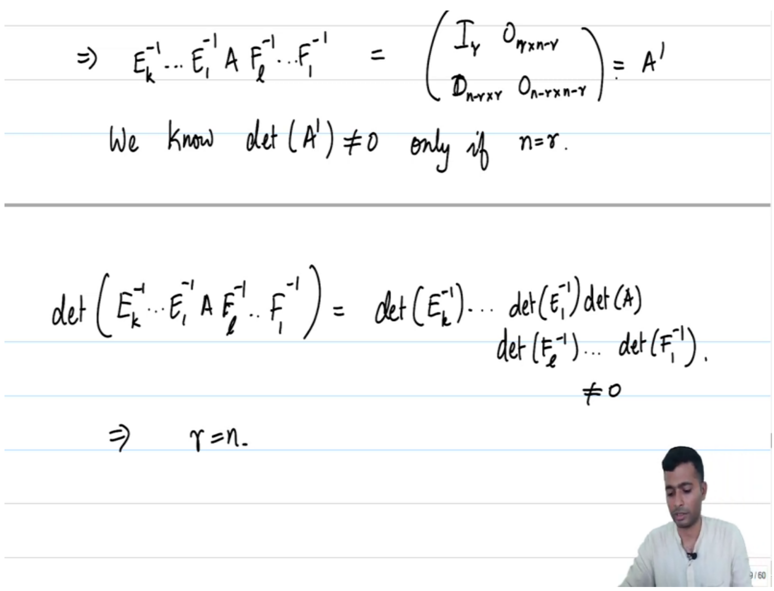

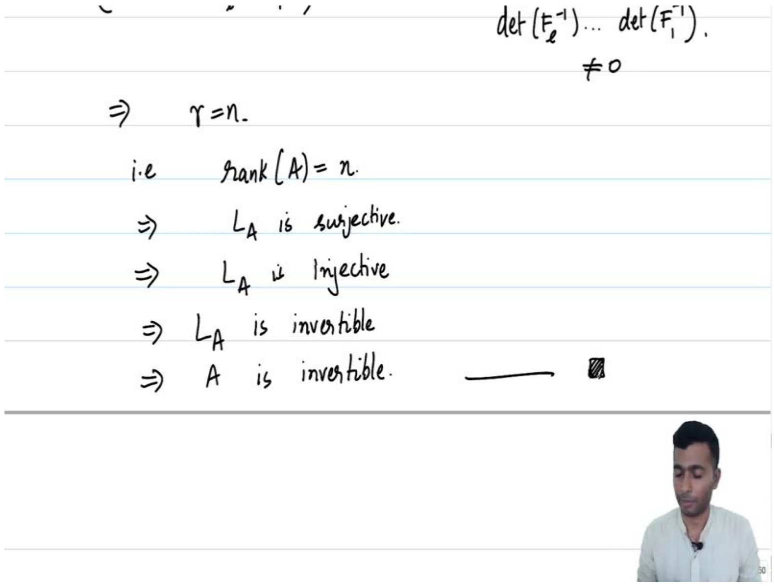

So suppose, determinant of A is not equal to zero. And we would like to now argue that A is an invertible matrix, but we know that again, from the previous or two videos back we have that, we know that A is equal to E1 dot dot dot Ek times Ir then zero, zero, zero in block matrices, and then F1 to Fl.

This implies that Ek inverse dot dot E1 inverse times A times Fl inverse up to F1 inverse is equal to Ir zero n minus r cross n minus r, let me write it down. Zero n minus r cross r and that is zero n minus r cross n minus r. And we know exactly what the determinant of the matrix on the right hand side is.

So let us call this matrix something, let us call it A prime. So we know that determinant of A prime is not equal to zero only if n is equal to r, and what about determinant of the LHS? Ek inverse dot dot E1 inverse A, Fl inverse up to F1 inverse. What do we know about the inverse of an elementary matrix? It will again be an elementary matrix and therefore this is just going to be equal to, we just checked that for elementary matrices the determinant of E times A for example is determinant of E times determinant of A.

By repeatedly applying this we get this is equal to determinant of A times determinant of Fl inverse dot dot dot determinant of F1 inverse and again, because determinant of A is not equal to zero, the right hand side here is not equal to zero. And this forces, this is not equal to zero and this forces n, sorry r to be equal to n. But what does it mean to say that r is equal to n?

What was r to begin with? r if you recall is the rank of our matrix A, i.e. rank of our matrix A is equal to n or to state it differently, LA is surjective. But what do we know about a surjective linear transformation from rn to itself or for that matter from a vector space to itself in the finite dimensional case. It should necessarily be an injective map as well by the dimension theorem.

LA is injective, so I have slowly stopped writing the reasons, it is for you to fill it up I have orally mentioned it. This will tell us that it is an injective surjective linear transformation and therefore LA is invertible. But we know that LA is invertible if and only if the matrix A is invertible. And that is precisely what we were trying to prove. So yes, if determinant of A is not equal to zero matrix is necessarily invertible.

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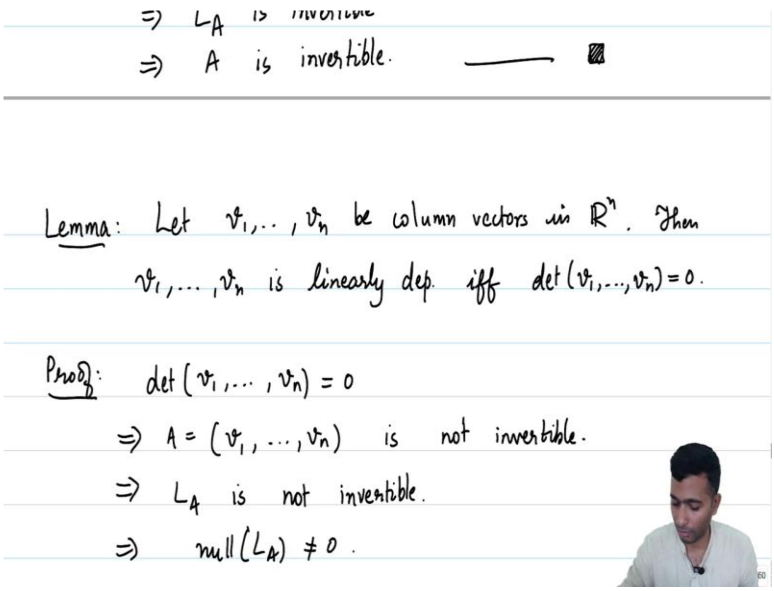

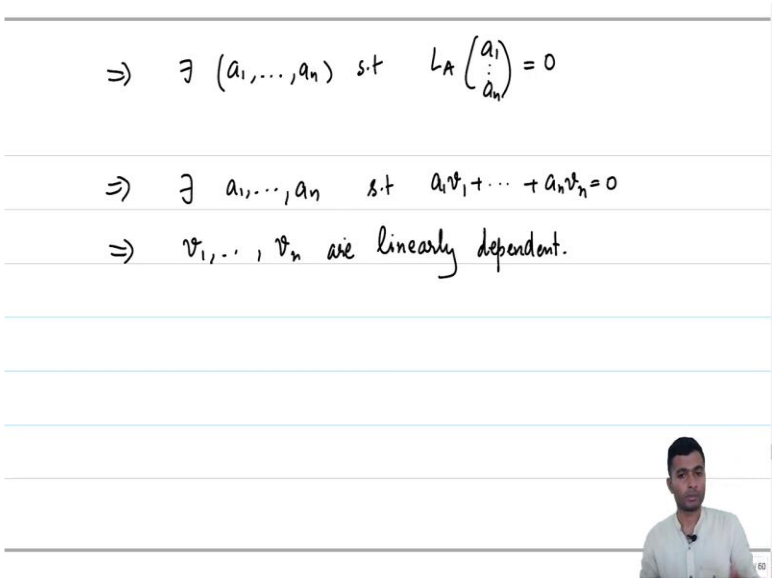

Let us look at some more properties. As a corollary to this statement, let us prove that a collection of vectors v1, v2 up to vn is linearly dependent if and only if the determinant of the corresponding matrix is zero. So we have already developed or associated a notation. So if, so let v1 to vn be vectors, column vectors in Rn, then v1 to vn is linearly dependent if and only if determinant of v1 to vn is equal to zero. So the determinant is also capturing linear dependence of its columns.

So let us give a proof of this. So in particular, if we have a matrix as determinant equal to zero, then the columns of that matrix should necessarily be linearly dependent, and therefore it cannot have full rank. That is what it says. Let us give a proof of this. So, determinant of v1 up to vn is equal to zero. What does this mean? This means that the matrix v1 to vn is not invertible. Let me write it down.

So this implies that the matrix A which is given by v1 to vn, the columns are v1 to vn is not invertible. So this is the case by the previous theorem, as a corollary to the previous theorem, we have that A is not invertible. What does that mean? This means that A, the null space of A, null space of LA, this means that LA is, okay I will write it down.

LA is not invertible because LA is invertible if and only if A is invertible and here A is not invertible and this implies LA is after all a map Rn to itself and it is not invertible implies that it is not injective. This implies that the null space of LA is not equal to zero. There is some vector in the null space.

So this implies that there exist a1 to an such that LA of a1 dot dot dot an is equal to zero, which implies there exist a1 to an. What is LA of a1 up to an? It is just a times this, and what is eight a times this? a1 v1 plus up to an vn is equal to zero, which implies that v1 to vn is linearly dependent. We have proved one side of the theorem. We proved that if the determinant is zero then v1 to vn are linearly dependent.

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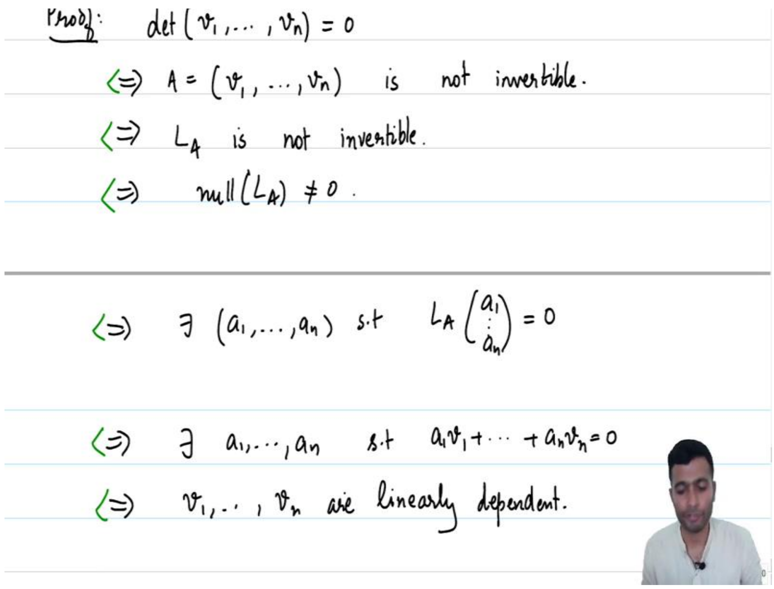

Let us assume that v1, v2 up to vn are linearly dependent. And let us just look that each of the green arrows will also get satisfied. That is an exercise for you to check and therefore, determinant of v1, v2 up to vn. So the determinant of a matrix also tells us whether its column vectors are linearly dependent or not. Determinant also behaves very well with product of matrices.

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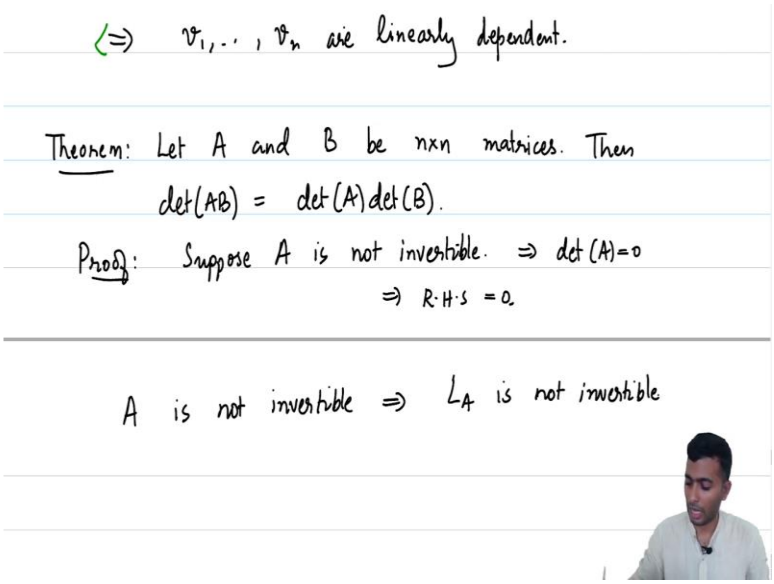

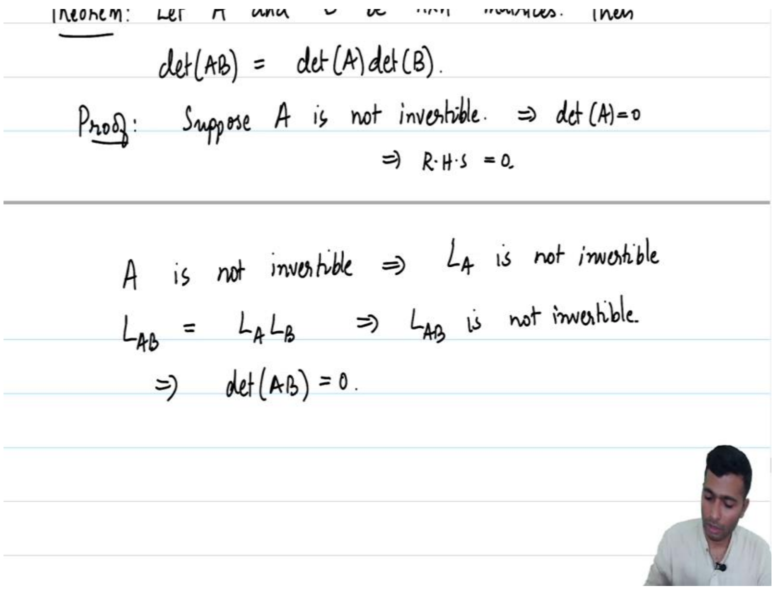

So let us solve, let us prove another theorem. Let A and B be n cross n matrices then determinant of AB is equal to the determinant of A times the determinant of B. If you notice, we have not imposed any restriction on A and B, it could happen that A is in not invertible or A is invertible. B is invertible or B is not invertible. In any of these cases, this property holds. So let us give a proof. We will divide the proof into cases. Suppose A is not invertible. When do we have that A is not invertible? When this is if and only if LA is not an invertible linear transformation.

But we already know that A is not invertible implies that determinant of A is equal to zero, which implies that RHS is equal to zero. Now let us prove whether, let us prove that LHS is also equal to zero. Well, A is not invertible implies that LA is not invertible and we will look at, so these are all corollaries to the theorem just proved, the theorem prior to the previous one.

Now if you look at LAB that is equal to LA LB and the fact that LA is not invertible implies that LA is not surjective and therefore, LA LB can also be not be surjective. This implies that LAB is not invertible and this implies that AB is not invertible and therefore, let me skip a step and directly write determinant of AB is equal to zero because AB is not invertible. If determinant were nonzero then it will be invertible. So if A is not invertible then we have established that both LHS and RHS are zero.

If B is not invertible a similar argument tells us that RHS is equal to zero because determinant of B is equal to zero, but B not invertible implies that LB then, this is one conclusion. B not invertible implies that LB is not invertible. As a linear, it is not an invertible linear transformation. That means that LAB which is equal to LA LB is not invertible.

Why is that the case? Because if LB is not invertible, then LB will not be injective. There will be something in the null space of LB and the vector will continue to be in the null space of LA LB and therefore, it will be in the null space of LAB. Therefore, LAB does not have the zero subspace as the null space. Therefore, LAB cannot be injective therefore, it cannot be invertible. So this implies that determinant of AB is equal to zero, it is not nonzero and therefore, when A or B is not invertible then we are done.

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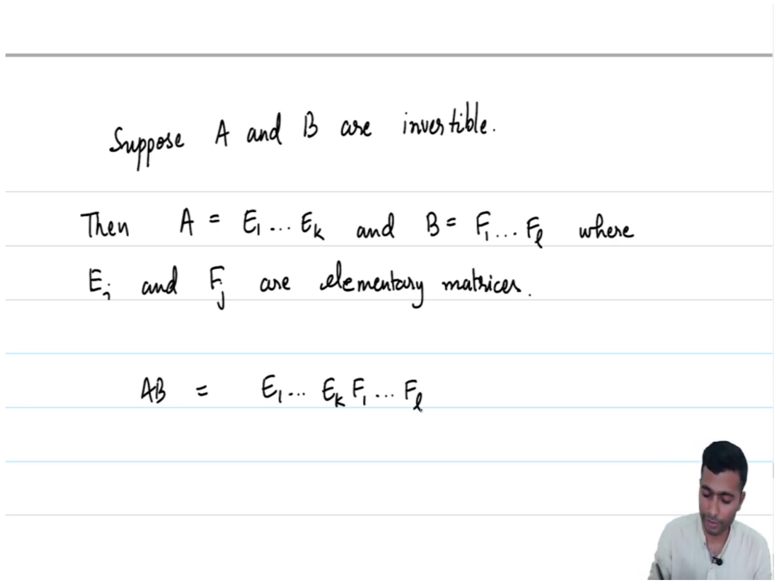

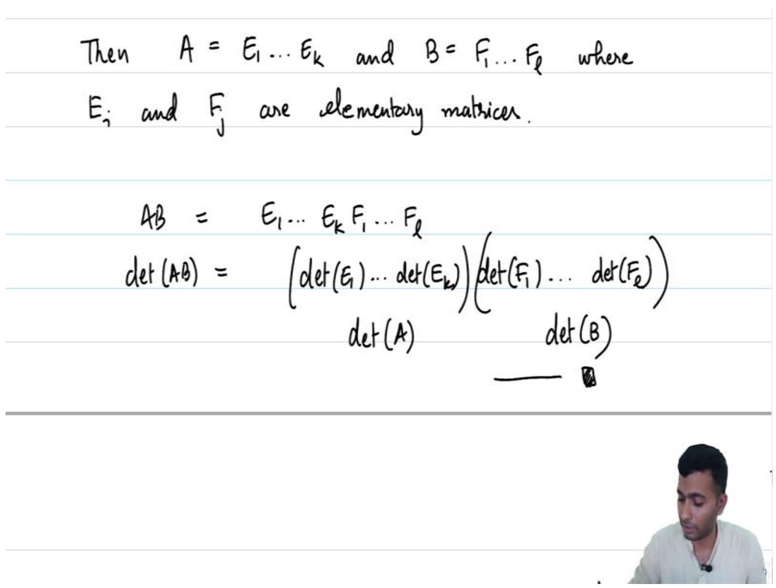

Suppose A and B are both invertible, that is the only case left. We have already proved that if a matrix is invertible then it can be written as a product of elementary matrices. So then A is equal to E1 dot dot Ek and B is equal to F1 dot dot Fl where Ei and Fj are elementary matrices.

This tells us that determinant of, and this tells us that AB will just be equal to E1 dot dot Ek, F1 dot dot Fl for some k and l. But these are all elementary matrices and we know that if you look at determinant of AB this is equal to the determinant of the RHS which splits because it is elementary matrices. We already checked at it splits for the case when it is an elementary matrix.

But if you notice carefully that we put a big bracket here, if I group it like this, the things inside the bracket is just equal to determinant of A, and this is just equal to the determinant of B. And this is precisely what we were trying to prove. So determinant behaves extremely well, when it comes to matrix multiplication.

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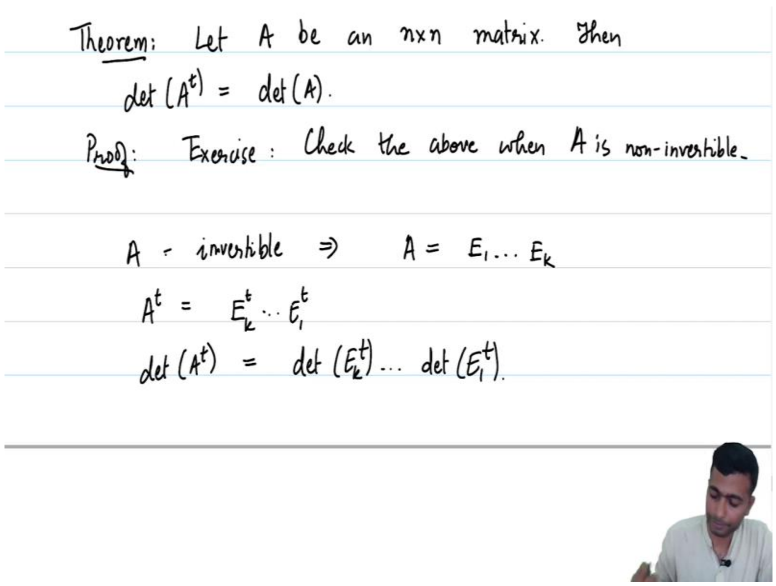

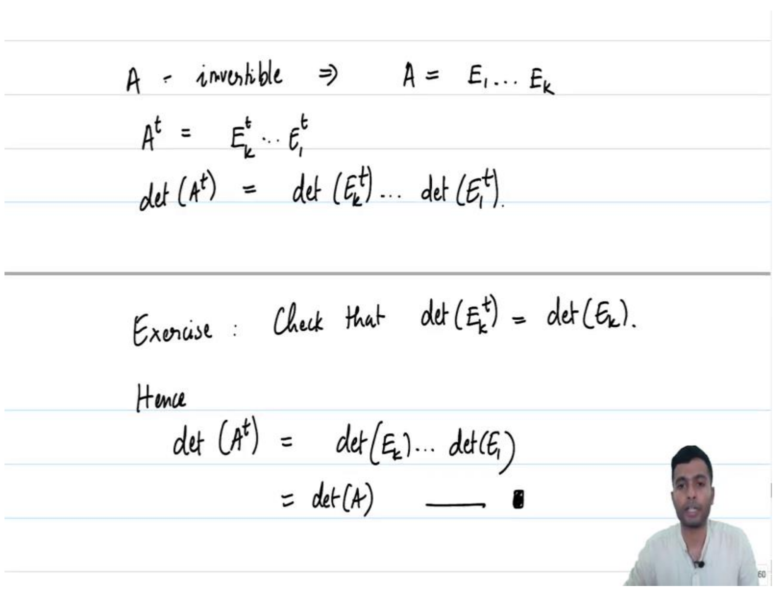

We will now prove that the transpose of a matrix, the determinant is again preserved, so let us do that. So theorem, so let A be an n cross n matrix. Then determinant of A transpose is equal to the determinant of A. So again, we will split this into two parts, the proof. The first is when A is not invertible. I will leave that as an exercise, just make the following observation.

If A is not invertible then LA is also not invertible, LA transpose will also be not invertible and therefore determinant of A and determinant of A transpose both will be equal to zero. Let us hence assume so, the case so, let me leave it as an exercise. Check the above when A is non invertible. Okay, let us now assume that A is invertible.

A is invertible implies that A can be written as E1 dot up to Ek where even E1, E2 up to Ek are all elementary matrices, and therefore A transpose will just turn out to be equal to Ek transpose up to E1 transpose and therefore, determinant of A transpose will be equal to determinant of Ek transpose dot dot determinant of E1 transpose.

E1, E2 up to Ek are elementary matrices and for elementary matrices it can be checked explicitly that the transpose does not alter the determinant. Check that, exercise, check that, these are fairly simple matrices, is equal to the determinant Ek. And with this, we will get hence, determinant of A transpose is equal to the determinant of Ek times up to determinant of E1, which is nothing but the determinant of A. We are done.

So this is an alternate way to convince yourself that if you look at the cofactor expansion along a column, it should give you the same number. Of course, there might be some circularity in the argument here but this is a good indicator to believe that, yes, even if we look at the cofactor expansion around a column or any column for matter, it should be equal to the cofactor expansion along any row.