Linear Algebra
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Lecture 6.3 Inverting Matrices

So we have already seen how elementary row operations and column operations are powerful tools in helping us determine the rank of a matrix. So next, let us try to explore how these elementary operations can be used to determine the inverse of an invertible matrix.

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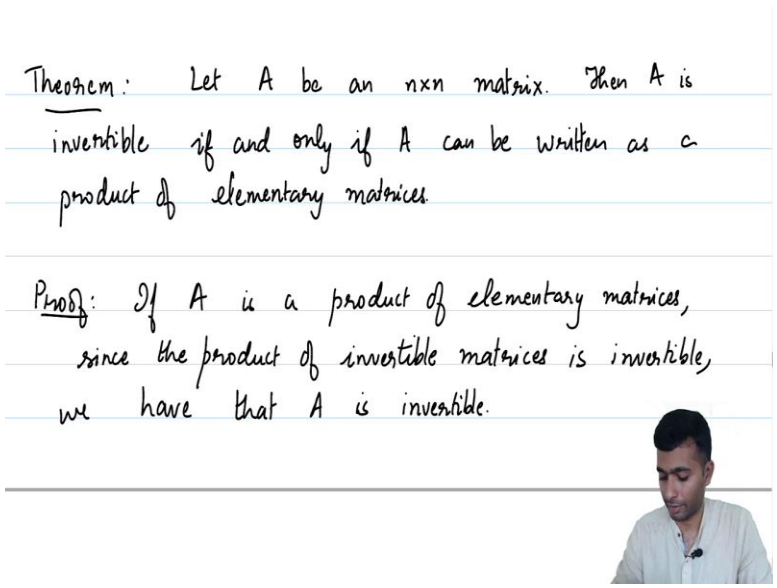

Let us begin with a theorem directly. Theorem, so let A be an n cross n matrix. Then A is invertible if and only if A can be written as a product of a elementary matrices, if A can be written as a product of invertible matrices. If you notice that A can be written as a product of not invertible matrices, elementary matrices product because if you can write A as a product of invertible matrices it will be invertible which we will use, but we will be writing a more specialized statement and only if A can be written as a product of elementary matrices.

Okay, let us give a proof of this. So we are telling something substantial, take any invertible matrix, you might be able to write it as E1, E2, E3 up to Ek where E is our elementary matrices. Let us see how we prove this theorem. One side is straightforward. If A is a product of elementary matrices then notice that each of the elementary matrix is an invertible matrix. Then, product of invertible matrices is also an invertible matrix. Since, the product of invertible matrices is invertible we have A is invertible, we have that A is invertible. Okay, so let us try to prove the other direction.

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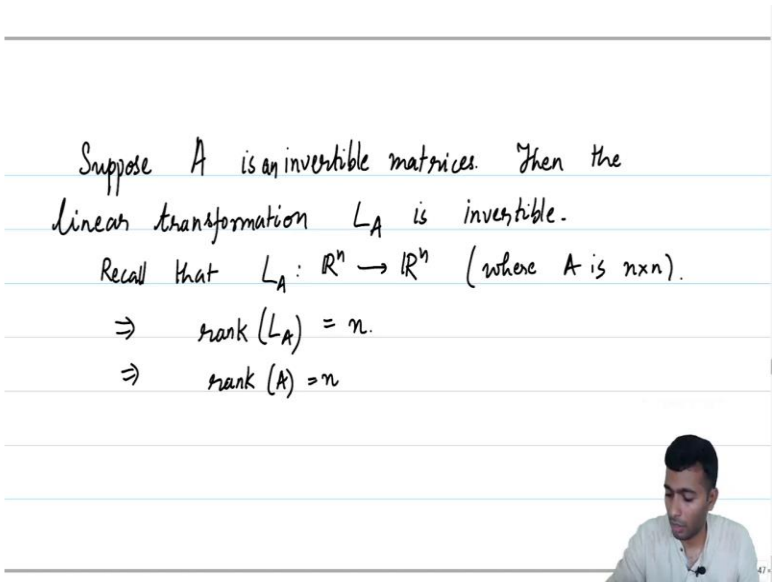

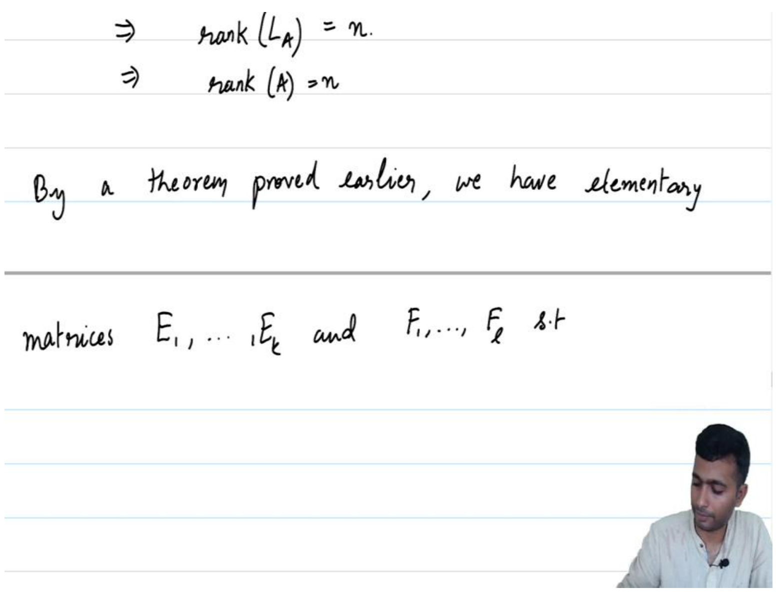

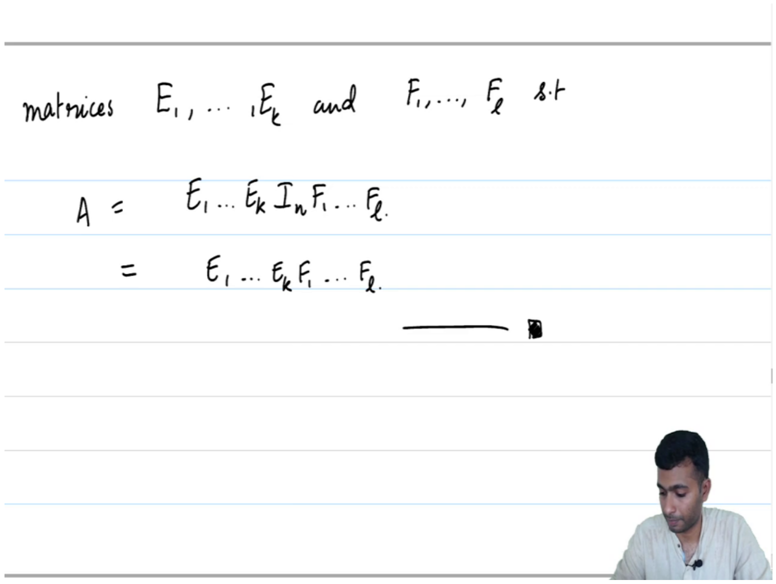

Let us start with an invertible matrix, suppose A is invertible. Let us try to write A as a product of invertible matrices, sorry, elementary matrices. Okay, how do we go about doing that? So suppose A is an invertible matrix, what do we know about the associated linear transformation LA, if A is invertible then the linear transformation LA we know that this is also invertible linear transformation is invertible, but recall that LA is a linear transformation from Rn to itself.

Recall that LA is a map linear transformation from Rn to itself because where A is n cross n, where n is the size of A. And what do we know about invertible linear transformations from vector space to itself? We actually did explicitly prove that this implies rank of LA is equal to the dimension of Rn, which is equal to n, which implies the rank of the matrix by definition is equal to rank of LA this is equal to n.

Now let us invoke one of the theorems which we proved in the previous video, where we showed that if we start with a matrix A which has rank r, then there exist elementary matrices A1, A2 up to, E1, E2 up to Ek and F1, F2 up to Fl such that our matrix A is E1, E2 up to Ek times a block matrix where the first block has identity in the r cross r identity in the first block and zero elsewhere times F1, F2 up to Fl, right.

So let me just write it by a theorem proved earlier we have elementary matrices say E1, E2 up to Ek and F1, F2 up to Fl such that our matrix A is equal to E1 dot dot product of

E1 to Ek times Ir, in this case is In, and let me just draw your attention to what would be the remaining blocks. There are no remaining blocks is the point because r here is n, so n cross n matrix, n minus R is n minus n which is zero so, it is just going to be In here. The only block is the identity block followed by F1 dot dot Fl. But then identity is the identity map, so Ek times the identity is just Ek which means that this is E1 to Ek, F1 to Fl and since A is an n cross n matrix each of the Ei's and Fi’s are n cross n elementary matrices and hence, we have proved our result.

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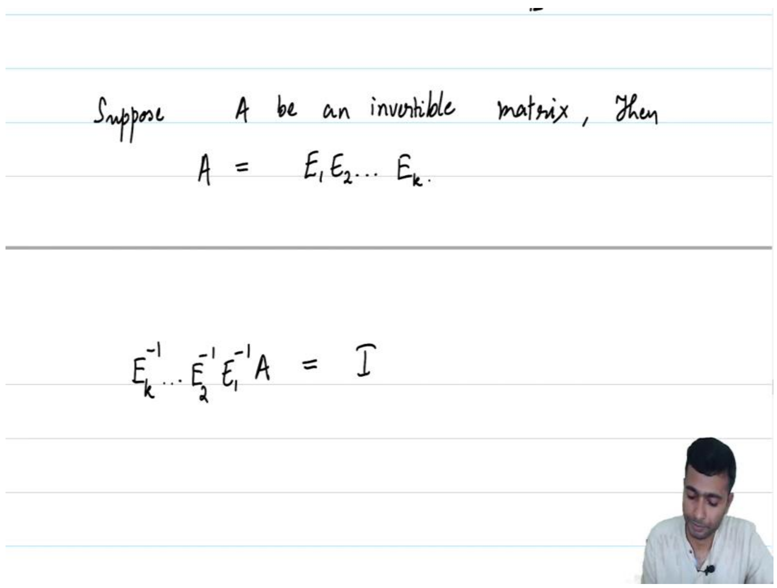

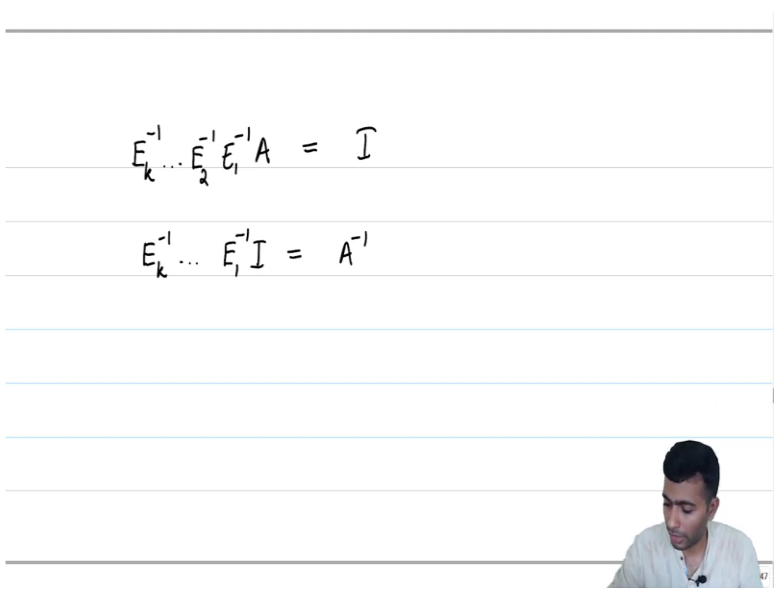

So how is this useful? Of course, we have done, we have made an important observation that every invertible matrix can be written as a product of elementary matrices but can we do more with this and let us see what we can do with this. So suppose, A be an invertible n cross n matrix then by what we just proved, A can be written as something like E1, E2 up to say Ek, right. It is a product of invertible matrices, sorry product of elementary matrices. I keep making this mistake, but elementary matrices is what I mean.

Now, if you take an elementary matrix and look at its inverse, what we get back is also an elementary matrix. So if we multiply to the left of A by the following E1 inverse E2 inverse Ek inverse times A, this will just give us the identity and n cross n identity matrix. If you carefully observe what is this E1 inverse, E2 inverse, Ek inverse? They are all multiplication by elementary matrices from the left, multiplication of E1 inverse with A which is a row operation. So what it essentially tells us is that these row operations will give us the identity matrix. Or if you multiply A inverse to the right, Ek inverse up to E1 inverse is equal to A inverse.

So these elementary matrices which after the relevant row operations give us the identity if you capture them, the product of it turns out to be or rather let me put it this way, the same elementary row operations applied to the identity matrix will give us the inverse of the matrix. This is a strategy, which we can use to compute the inverse of a invertible matrix.

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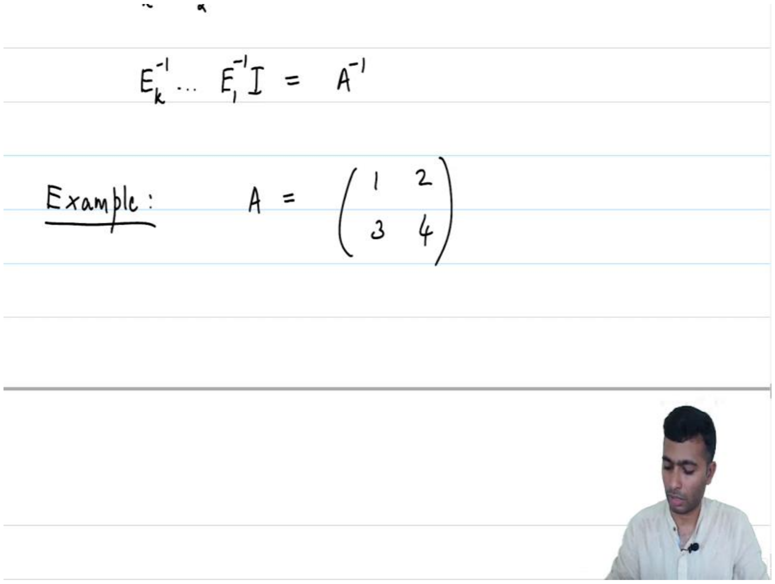

So, let us look at one example. Let us take a simple straightforward matrix that we can think of. Let us take say 1, 2, 3, 4 and let us see whether we can use this algorithm to calculate the inverse of A.

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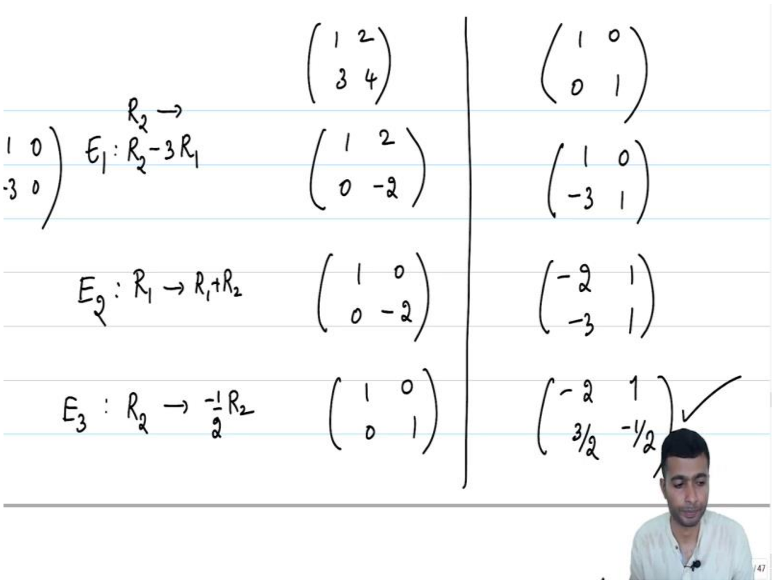

So let me divide the next page into two, and let us capture the first matrix here and the identity matrix here. Let us apply the same elementary row operations to both these matrices. Let us try to reduce the first one to identity and let us see what we get in the second one. So the first operation E1 would be to make the second entrance zero, so this is basically three times row one is reduced from row two. This give 1, 2 nothing changes, this will be zero and 4 minus 6 is minus 2. Let me just write it like this row 2 is minus 3 times row 1, this is what we are doing.

So if you want the corresponding matrix will be identity and the second row is minus of 3 comma 0. This is what the corresponding E1 will be. Okay, so what will be the effect on the identity matrix? 1, 0 remains unchanged, so our minus 3 comes here and we have already calculated what that is. So first row operation that we did. And how about the second one, E2? E2 will try to make the entry about two to be, above minus 2 to be 0, we just add so this is R2. So R2 is changed to this one now the first row is changed to R1 plus R2 just adding.

Again, type three, this is a type 3 row operation, this is a type three row operation. We will get 1, 0, 0 minus 2 here, and if we add above this will be minus 2, 1 minus 3, 1. So I am not going to write what E2 is, a matrix you should calculate yourself. I have however captured what E2 times E1 is in the right.

Now E3 will be a type 2 row reduction where we will multiply the second column by minus of 1 by 2. So R2 we will replace it by minus of 1 by 2 times R2 to get the identity matrix and here it will be minus 2, 1. The second row is multiplied by minus of 1 by 2 to get 3 by 2, and here that will be minus 1 by 2. So yes, this is what is the candidate for the inverse of 1, 2, 3, 4 and it is a simple check from you to see that this is indeed the inverse.