Linear Algebra
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Lecture No. 6.2
Rank of a Matrix

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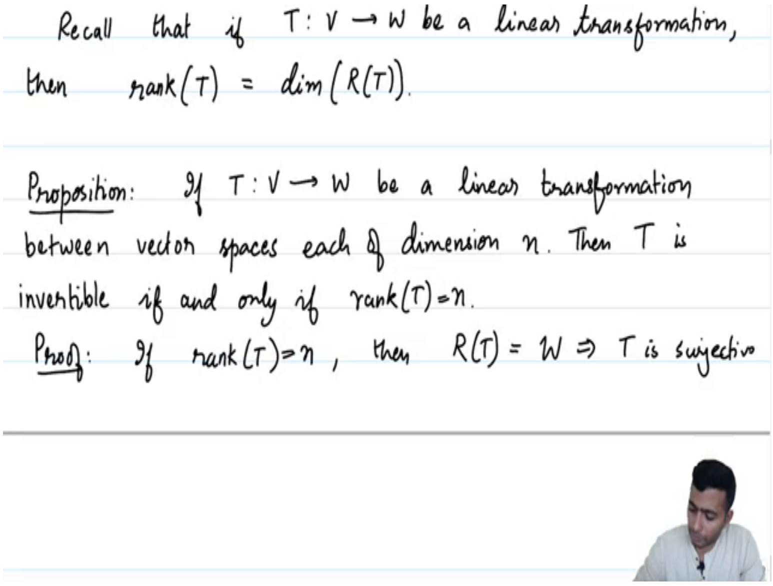

So we are already familiar with the notion of the rank of a linear transformation. So recall that if T is a linear transformation from V to W then the rank of T is the dimension of the range of T, then rank of T is equal to the dimension of the range of T. So recall that this is a very useful notion to have because this is one of the key ingredients of the dimension theorem, which said that look a dimension of V. V is finite dimensional, dimension of V is equal to the dimension of the null space of T plus the dimension of the space of rank space of T, range space of T which is the rank of T.

So the rank of T tells us a lot about the linear transformation. So one example would be, let me just leave it as an exercise or maybe I will just prove it. Proposition, the proposition states that, so if T from V to W be a linear transformation between equidimensional vector spaces, say between vector spaces each of dimension n. Then T is invertible if and only if rank of T is also equal to n.

So one direction is quite straightforward, if T is invertible then in particular T is a surjective map and therefore, R of T will be exactly the same as W and therefore, the dimension of R of T is same as dimension of W which is n. So, I will just leave that as an exercise. The other part is suppose, if rank of T is n, so observe that what happens when rank of T is n.

Then notice that this implies R of T is an n dimensional subspace of an n dimensional vector space, then R of T; I will just write that this directly implies R of T is equal to W because rank of T equal to n implies dimension of R of T is equal to n. R of T is a subspace of W and we know that W is of dimension n. And therefore, any n dimensional subspace of an n dimensional vector space will be the entire space, which means that R of T is W, which implies in particular that T is surjective but that is not all.

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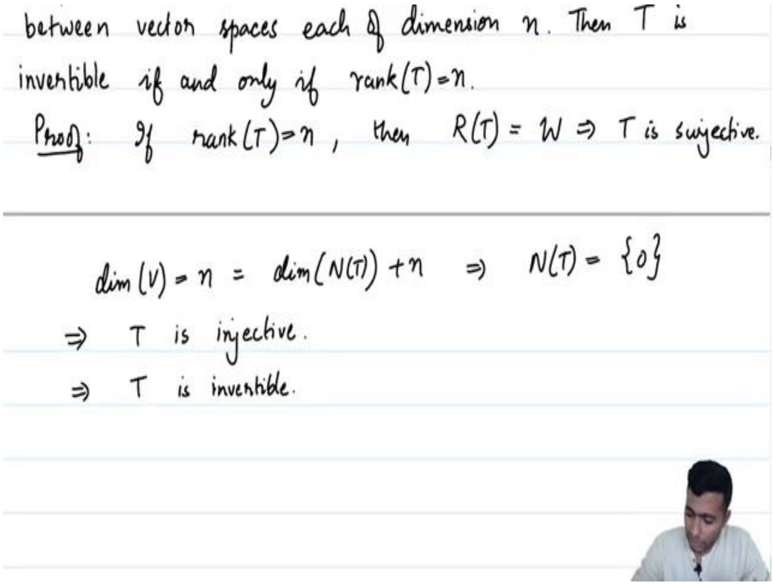

Now, let us invoke dimension theorem which tells us that dimension of V which is equal to N is the dimension of the null space of T plus the dimension of R of T which is dimension of W which is equal to n. So which means that N of T has dimension zero, which means that N of T is the zero subspace. The only subspace which has dimension zero is the zero subspace, which means that T is injective. So what do we know about linear transformation which is bijective? It is an invertible linear, it is invertible. This implies that T is invertible.

So we have used multiple results which we have proved in maybe weeks 4 and 5 and it is a good exercise for you to sit and check at what step which of the theorems were used. Alright, so this was just an example to highlight that the notion of rank is a very useful one. Rank also is a well-behaved object. So for example, if you take linear transformation from say V to W and if you composite by an invertible linear transformation the composition or the product, product of linear transformation and product of composition are just same notion with different words right. So you look at the composition of the given linear transformation with an invertible linear transformation, the rank of the composition as the same as the rank of T.

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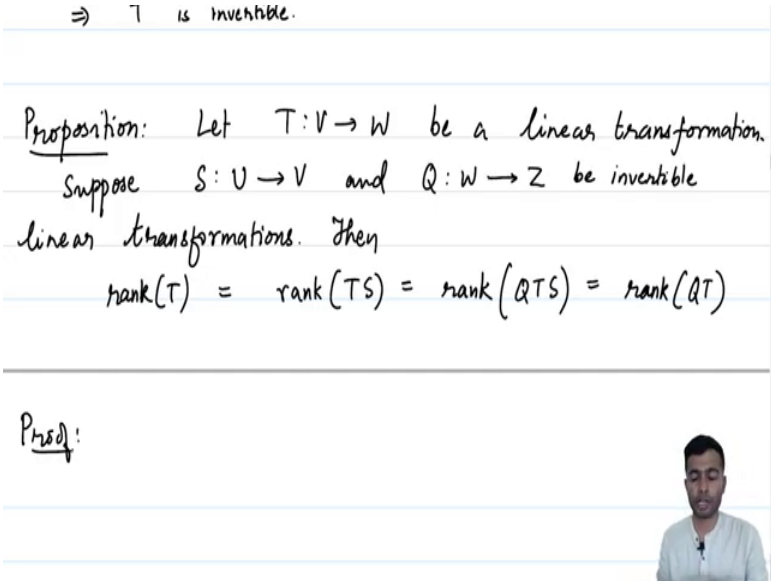

So let me just write down a proposition, which captures what I just said. So let T from V to W be a linear transformation. Suppose U, let us call it S from U to V and Q from W to Z be invertible linear transformations or isomorphisms. Then, rank of T is equal to rank of TS. Observe that S is a linear transformation from U to V and T is a linear transformation from V to W, so this is our linear transformation in particular from U to W. This is the same as rank of QTS, which is the same as the rank of QT.

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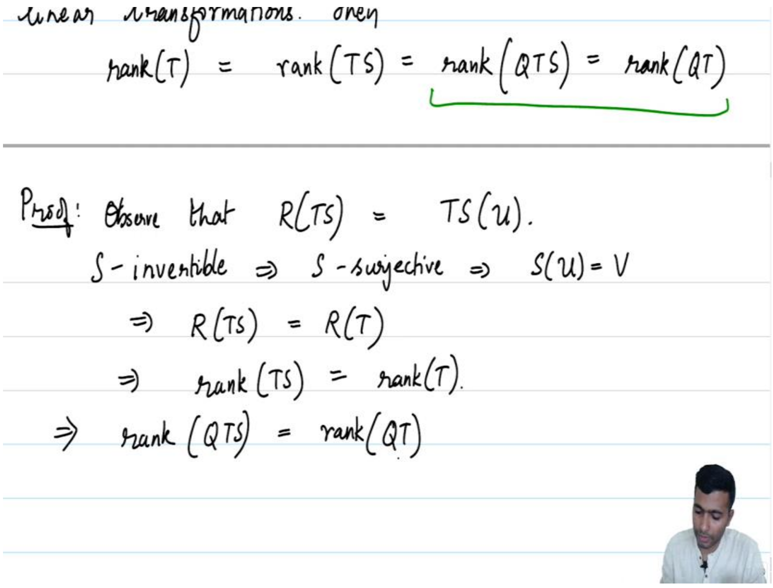

So let us see, let us try to prove this proposition step by step. Let us first prove that rank of T is the same as rank of TS. Okay what does that mean to say that rank of T is the rank of TS? Rank of T is just the range of T right, dimension of the range of T and what is the rank of TS?

That is the dimension of the range of TS. But what is the, observe that R of TS is equal to the range of, this is basically if we write it as T of S of, this is basically T of S of u, right. The set of all elements of the type TS of small u, where u belongs to capital U. But observe that S is an invertible linear transformation. S is invertible and what is the property of, what is a characterization of an invertible linear transformations? So a linear transformation which is both injective and surjective in particular S is surjective. S is surjective and this implies that S of u is equal to V. Every element of V has a pre image under S.

This implies that R of TS is nothing but the R of T, right. This implies that the dimension of R of TS which is the rank of TS is equal to the rank of T, okay. I would say that this also indicates this if P is replaced by QT we have essentially shown that, okay so this also indicates or implies rank of QTS is equal to rank of QT. Instead of, so T was an arbitrary map right, for any T if you compose it with an invertible matrix it is going to preserve the ranks. So in particular rank of QTS is the same as rank of QT. So the only thing that is left to prove would be to check that rank of TS is the, sorry rank of QT is the same as rank of T.

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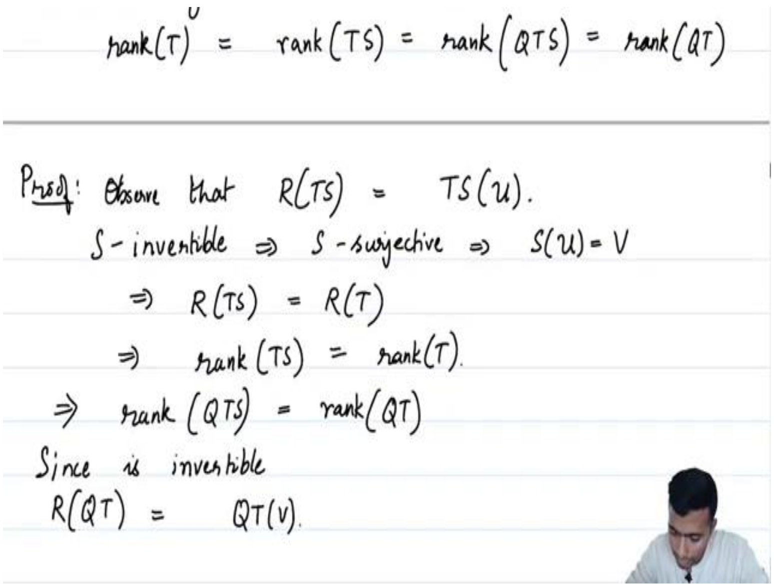

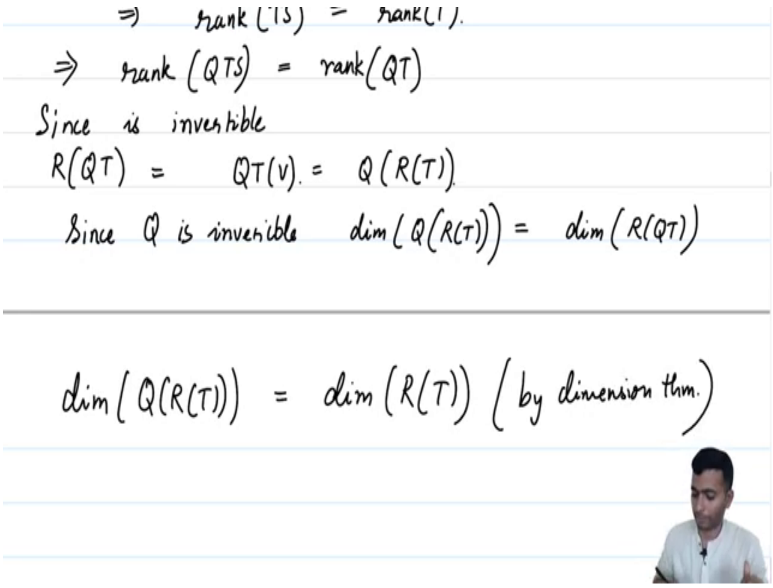

Okay, so what is R of QT? Our next question is to address what R of QT is. Whatever be the range of QT, one thing to note is that Q is an invertible map. Since Q is invertible, the dimension of the range of Q is the same as the dimension of the domain of Q. So this is equal to I will just write it in a crude manner QT of V. Remember that, did I write the preimage here correct? Did I write it as U? Yes, it is U.

Okay, so here T is a map from V to W and Q is a map from W to Z, so P of V is a subspace. So this is basically Q of the range space of T, if you observe. So Q restricted to T of V will map isomorphically onto its image, right, so the range of T. Since Q is invertible, so Q by dimension theorem, dimension of Q of R of T is equal to the dimension of the null space of Q plus dimension of Q of T of V, which is R of QT.

What is our left-hand side? That is nothing but dimension of R of T itself because Q is an invertible map and, or rather let me put it this way. The right-hand side dimension of R of QT, maybe a little more careful. Dimension of Q of RT, yes, this is precisely what we should be looking at. Q of RT is the same as dimension of R of T. This is what the, this is what we can get from the dimension theorem. The previous theorem is just a rewriting of terminologies, this is by the dimension theorem. I will just write it as by dimension theorem here.

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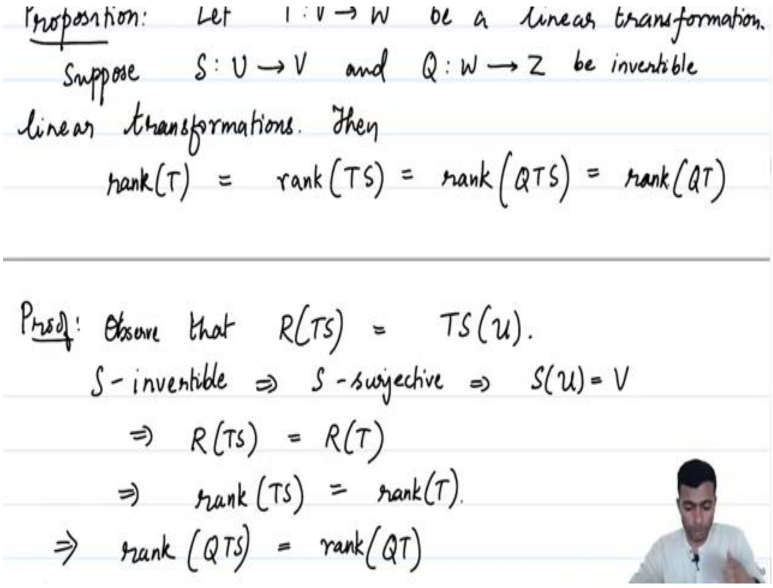

Then, you already know that this is, so I will put a equal to here and tell you that rank of QT is now equal to the rank of T. So we have, what have we shown? We have shown the, where did the equality go? Yes, we have shown the first equality and the last equality was realized as a consequence of the first equality and that rank T is equal to rank QT has been proved and hence, all the ranks are preserved.

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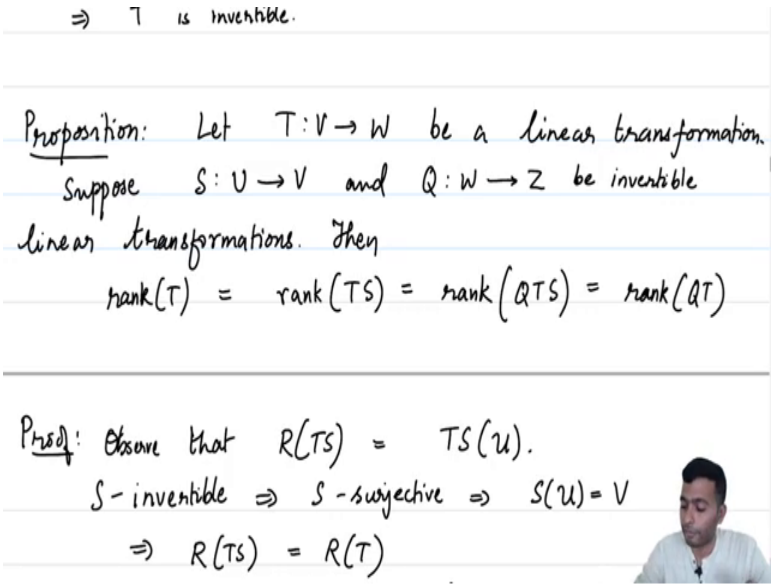

Okay, so let us look at the proposition again, the statement is quite important. What the proposition tells us is that if you consider T to be a linear map from V to W, and suppose S is a invertible map from U to V then if you composite with the linear transformation T the rank of the composition is the same as the rank of T. Same is the case with a matrix cube, not matrix, a linear transformation Q. So the rank behaves well under composition by invertible maps.

Let us now discuss the rank of a matrix. We have discussed the rank of a linear transformation already in quite detail earlier. Our goal here would be to define the rank of a matrix and somehow link it to the notion of the rank of a linear transformation. The rank of a matrix, as was noted earlier, is something which is well studied and well understood. So we will reduce many times the problem of rank of a linear transformation to the rank of a matrix and work with the matrices.

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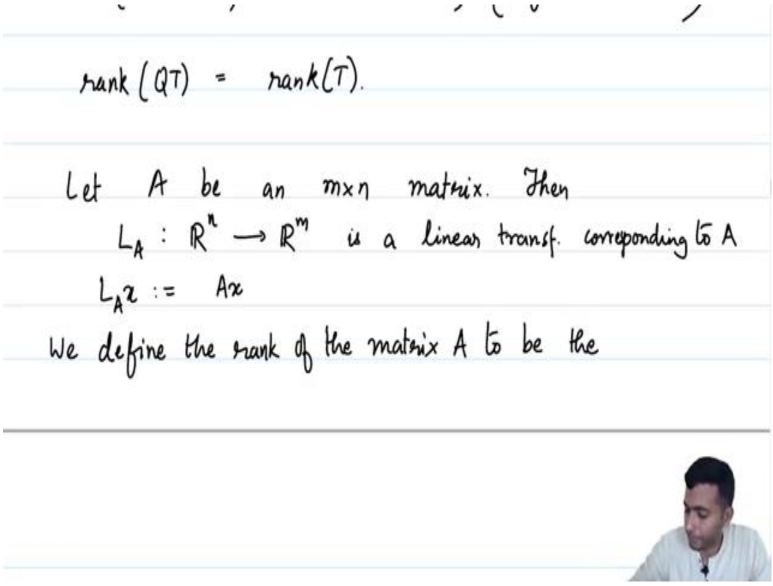

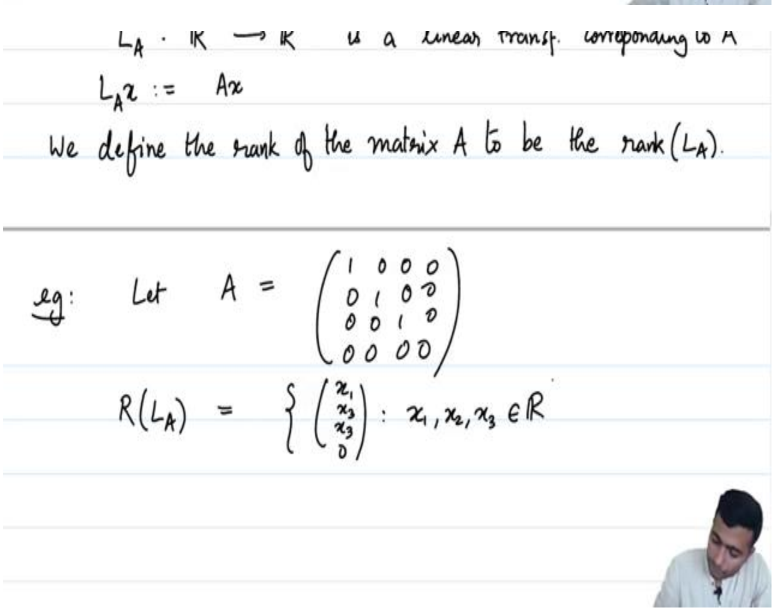

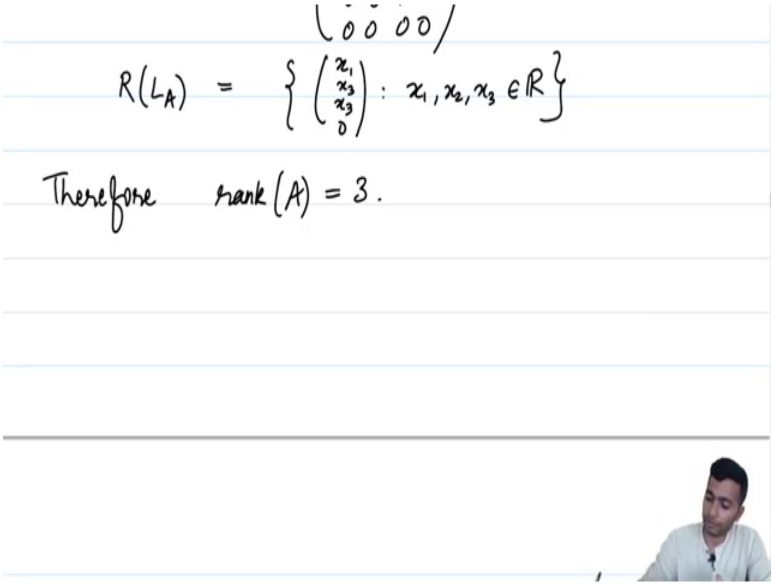

So let us now focus on matrices and the rank of a matrix. Okay, so let A be an m cross n matrix then LA, with respect to the standard basis, then LA, recall that this is just left multiplication by A in the standard basis. Where is this map from? This is from Rn to Rm is a linear transformation corresponding to A. If you recall what LA was?

Let me just remind you what LA of x is, this is just left multiplication of the matrix A with the column representation of x. Remember this was like this and we did look into many properties of LA. For example, LA plus B was LA plus LB. We also checked that LAB, maybe it was not checked it was given as an exercise to prove the LAB is LA times LB and so on. Okay, so given any matrix A, an m cross n matrix A, we can associate a corresponding linear transformation. We are also familiar with the notion of the rank of a linear transformation.

We will define the rank of the matrix A to be the rank of this linear transformation LA. So we define the rank of the matrix A to be the rank of the linear transformation LA. Let us look at an example. So let A be the matrix 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0 and 0, 0, 0, 0. So this is a 4 cross 4 matrix and notice that the rank of A is the dimension of the range space of LA. Notice that, I will just leave it as a check for you to notice that this is nothing but the set of all x1, x2, x3, comma zero where x1, x2, x3 are real numbers.

And therefore, rank of A is equal to 3. Of course in this particular case it was quite easy to consider LA and explicitly calculate the dimension of the range. But that is not generally an easy problem, it can happen that for say big M and big N and more complicated a matrix. It is far more difficult to compute the dimension of the range of LA. So we will spend some amount of time and energy to get hold of methods with which we will be able to simplify this particular calculation.

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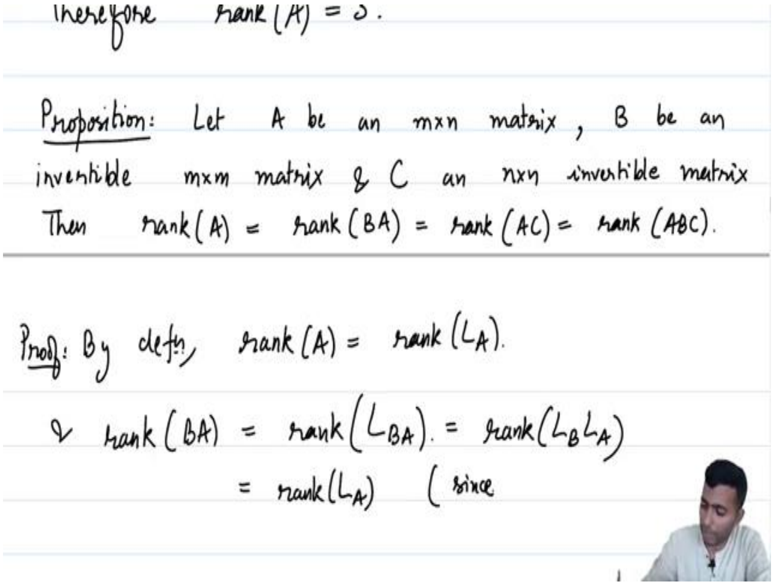

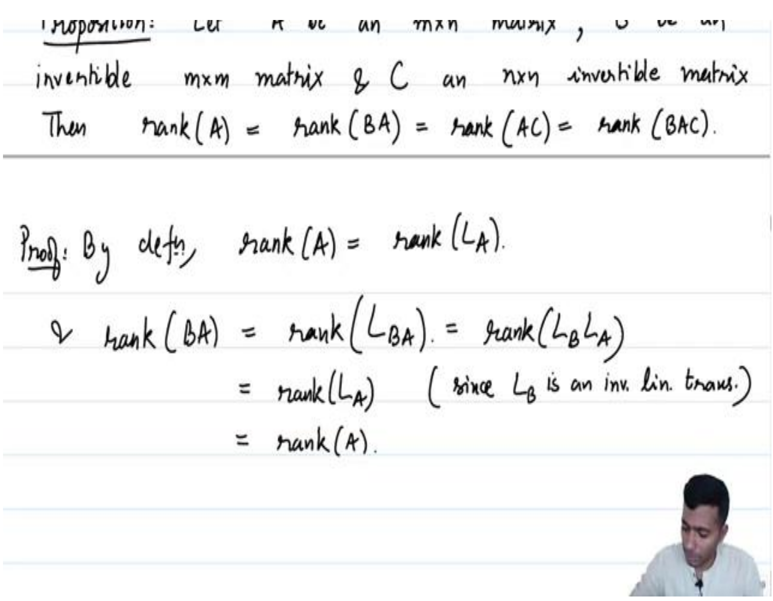

We noticed that if we compose a linear transformation by an invertible linear map to the left or to the right, the rank of the image, sorry the rank of the composition does not change. A similar property holds for matrices. So, let me just write it down as a proposition, this actually will just follow from the case we have proved for linear transformation. So suppose, so let A be an m cross n matrix. B be an invertible m cross m matrix and C, an n cross n invertible matrix. Then rank of A is equal to B is an m cross m matrix, so this is rank of BA, which is the same as the rank of AC, which is the same as the rank of ABC.

Okay, so let us go over the details. What is rank of A? By definition, let us give a proof. By definition, rank of A is the rank of the linear transformation LA, is the rank of LA, and how about rank of BA? Rank of BA is the rank of LBA. But if you go back to the lectures two weeks back, we did discuss this and we noticed that LBA is LB times LA. This is nothing but rank of LB times LA because LBA is nothing but LBLA. So what can we say about LB when B is an invertible matrix? It happens to be the case that, we have proved this as well, B is invertible if and only if LB is invertible. B is an invertible matrix if and only LB is an invertible linear transformation.

And by the proposition we have proved earlier, this is equal to rank of LA since, so let me write down the reason here, LB is an invertible linear transformation. I have written short forms of all of that but it is self-evident, but what is the rank of LA? Rank of LA is nothing but the rank of A. So by a similar argument it is easy to prove that rank of A is the same as rank of AC and rank of equal to rank of BA immediately tells us that; rank of A is equal to rank of AC immediately tells us that rank of BA is the same as rank of, oh, I have written something wrong here.

I am a bit sorry about that, this is BAC, yes. Rank of BA is the same as rank of BAC. Alright, so I will just assume that you will be able to complete the proof here. Idea is extremely similar to what I have just observed. Just notice that LBAC is the same as LBLALC, and that LB and LC are invertible corresponding to invertible matrices B and C.

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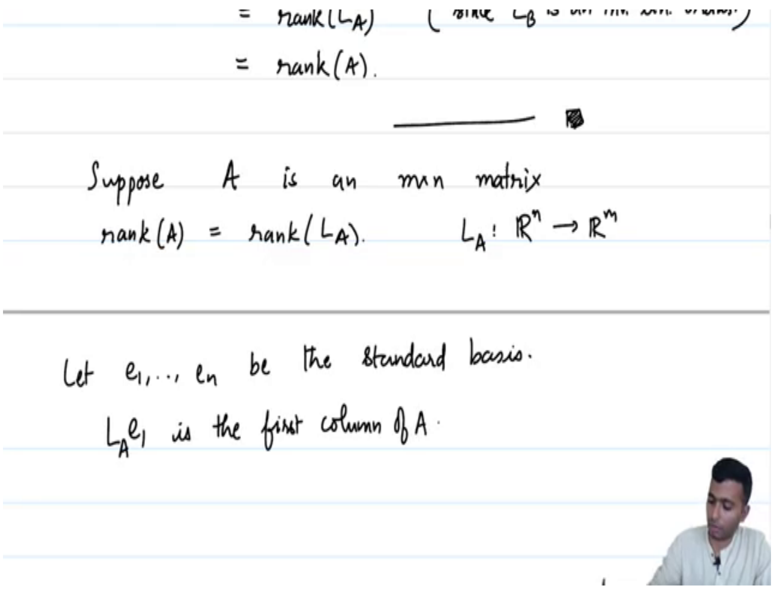

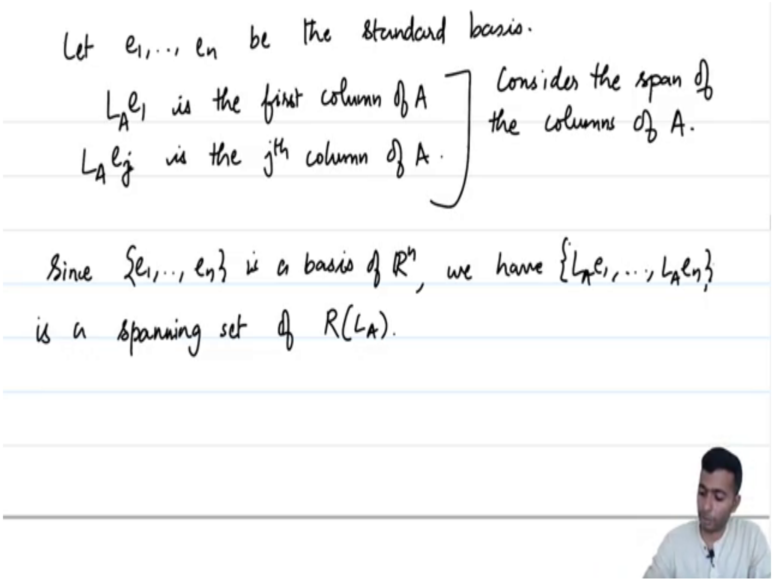

Alright so, this is good, we are in good shape. Okay, so as was noted earlier, it might not be easy to compute explicitly the dimension of the rank of a matrix A, when A is a very complicated matrix. Nevertheless, we can still say something so suppose, so let me just put a stop QED symbol here. So suppose A is an m cross n matrix. Let us define the column space of A to be the subspace of RM, which is obtained by looking at the span or the columns of A. So let me just define it, or before that let us try to see, before even we get into our column spaces let us try to see what the rank of A should be like.

So what is rank of A? Recall that rank of A is nothing but the rank of L subscript A, right? Now what is LA? Remember that LA is a map from Rn to Rm. This is an m cross m matrix, so Rn to Rm is the map. So LA, if you consider the standards basis of Rn, then, so, let e1 to en be the standard basis. Then LAe1 will just turn out to be the first column of A, is the first column. In fact, LAei is the ith column, or LAej is the jth column of A.

But what do we know about LAe1, LAe2 up to LAen. The fact that even e1, e2 up to en is a basis of Rn implies that LAe1, LAe2 up to LAen should be at least a spanning set of the range space of LA. So, since e1 to en is a basis of Rn we have LAe1 to LAen. This set is a spanning set of range of LA. So we will call this the span of LAe1, LAe2 up to LAen as the column space of A because it is the span of the columns of A, right. As noted here, from this, consider the span of the columns of A, what we had just noted is that this space is exactly equal to the range of L of A.

And therefore the dimension of the range space is the dimension of the span of the columns and that is precisely equal to the rank of our matrix A. Again, we have not simplified our problem much, we have just rephrased it into a machinery or into a language, which is obtained from the matrix itself.

So given a matrix A, without going or referring to the linear transformation corresponding to A, we will be able to say that the rank of matrix A is obtained by looking at the column space of this matrix in Rm and looking at its dimension, right. Okay, so let us now get to work to simplify our notion of, we have actually developed all the machinery needed to simplify the notion of rank, not the notion of rank the method of finding rank. So we have just noted that if we multiply a given matrix by an invertible matrix, the rank of the product is preserved or rank of the product is the same as the rank of the matrix A. So next proposition tells us and we also know that you okay, I did not explicitly write it down, it is an exercise for you to check that every matrix can be reduced to row echelon form.

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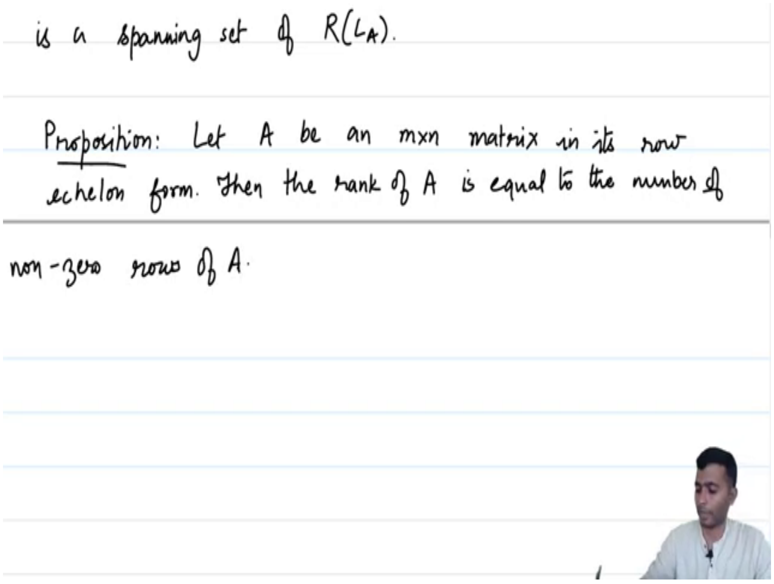

This proposition tells us that if a matrix A is in its row echelon form then the rank of the matrix A is equal to the number of non-zero rows of that matrix A. So let A be an m cross n matrix in its row echelon form. So recall that row echelon form, matrix A is set to be in its row echelon form if every row satisfies the following property. Either the row is the zero vector or the zero, every entry of the row is zero or the first non-zero entry of row is 1 and all entries below in that column is zero. So you take any matrix in the row echelon form, then the rank of A is equal to the number of non-zero rows of A. Okay, so let us give a proof of this.

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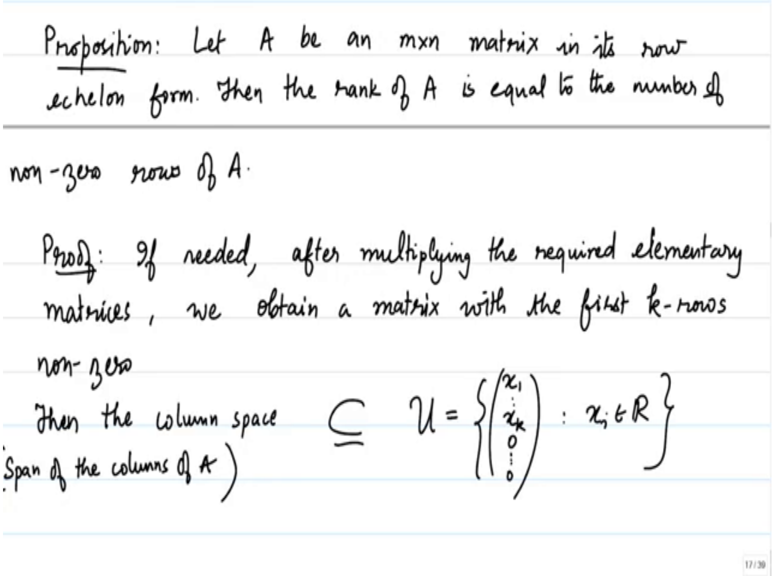

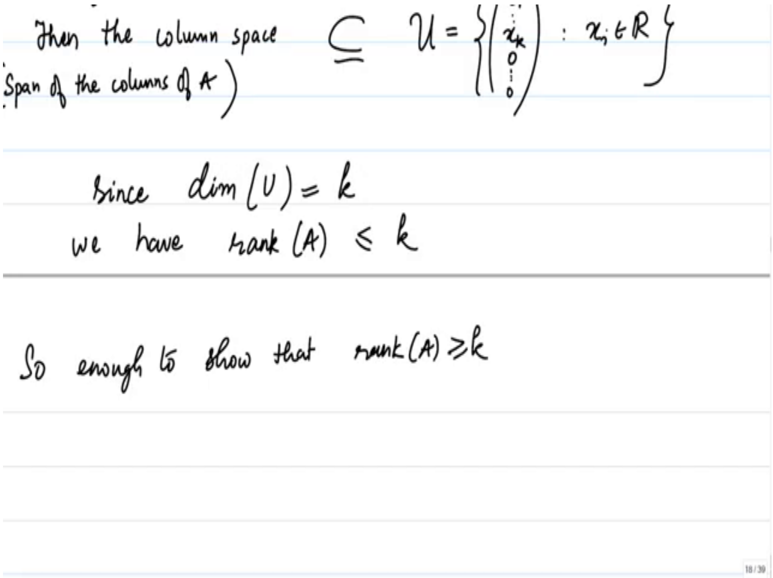

So we noted that the rank of a matrix A is not changed after you multiply by an invertible matrix, so if needed multiply by the elementary matrices which interchange rows after multiplying by the required elementary matrices or multiply by the required elementary matrices, we obtain a matrix with the first say, k-rows, with the first k-rows non-zero. So you get a matrix where A11 up to say Ak1, or rather let me not put it this way. Say if this is the first k-rows and this is say, the n minus k rows, it is a zero matrix here. The n minus k, the last n minus k cross m, not n, m minus k cross n matrix at the bottom is the zero matrix. Okay, that is what this means. If needed multiply by the relevant elementary matrices to get the first k-rows to be non-zero.

So clearly, then the row, the column space which is the span or this span, well, I will just write it down, span of the columns of A. This is contained in, so let me just throw this out I will not get space otherwise. This particular column space, this is contained in the subspace u of Rm, which is given by say x1 to xk and then zero dot, dot, dot, zero where xi is in R. So u is basically the subspace of Rm, which satisfies the property that the last m minus k entries are zero. And we know that dimension of u is equal to k right, since dimension of U is equal to K we have the dimension of the column space which is the rank of A is also less than or equal to K.

Because this is a subspace of u after all, so the dimension of this subspace should be less than or equal to the dimension of u. We will show that the rank of A is, or the column space will have at least k as the dimension. So enough to show that rank of A greater than or equal to k, right? If we do this we have established that our rank is equal to k.

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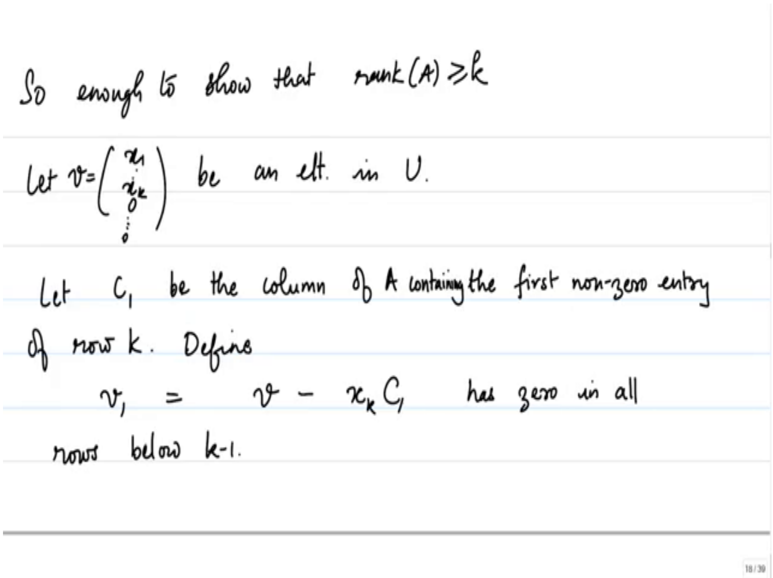

So let us pick an arbitrary element of U. We will show that it is in the span of the column space of A, and therefore U is contained in the column space of A and therefore dimension of U is less than or equal to the dimension of the column space of A, which is the rank of A that means we will get k is less than or equal to the rank of A. Okay so let x1 to xK, zero, dot, dot, dot, dot, zero be an element in U. Now A is in a very special type, it is in the row echelon form with the first k-rows non-zero and the remaining m minus k rows zero.

So let us focus on the kth row. Let Ck, C1, let us call it C1 be the column of A such that the first non-zero entry of row k. Let me put it, let me rephrase the statement, let C1 be the column of A containing the first non-zero entry, the first non-zero entry of row K. The first non-zero since it is in the row echelon form, so the first non-zero entry should necessarily be 1. And the definition of row echelon form tells us that every entry below that 1 should be zero, right.

So now define, so let this be v and let v1 be equal to v minus xk times C1. Now notice that C1 has one in the kth row and zeros below. So xk times C1 will have xk in the kth row and zero below. V has xk in the kth row and zero below. So if you look at v1 this will be a vector as zero in all the rows below k minus 1. Should think about it for a minute and get convinced that this is the case.

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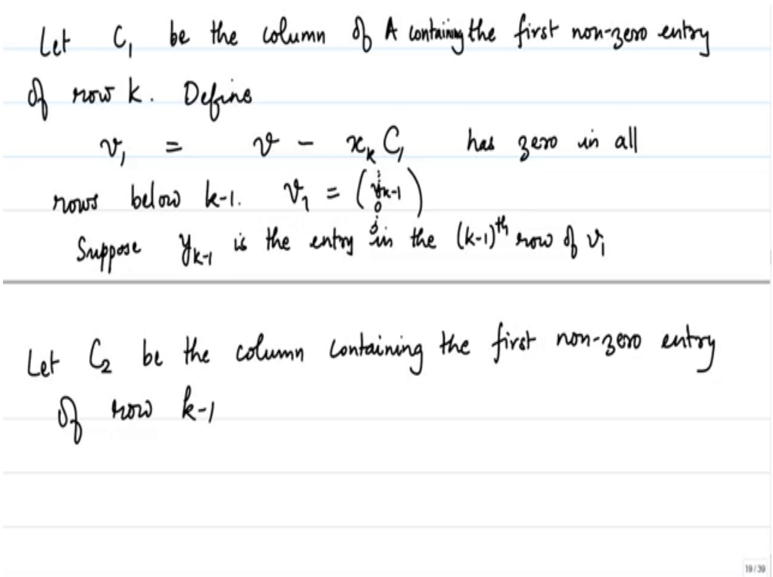

Now suppose, yk minus 1 is the entry in the k minus 1th row of v1. So let me tell you exactly what this means. v1 will be something like say, something here in the k minus 1th column, it is yk minus 1 and then zero below. Here, it could be anything, at the top it could be anything but k minus 1th row will be something let us call it yk minus 1. And let C2, let us follow the same trick as earlier, let C2 be the column containing the first non-zero entry of row k minus 1. Notice that all entries below that 1, the non-zero entry should be 1 and all entries below that should be zero by definition and therefore this C2 has to be a different column from C1, right? It is forced because C1 had a 1 in the kth column, kth row and that cannot happen here.

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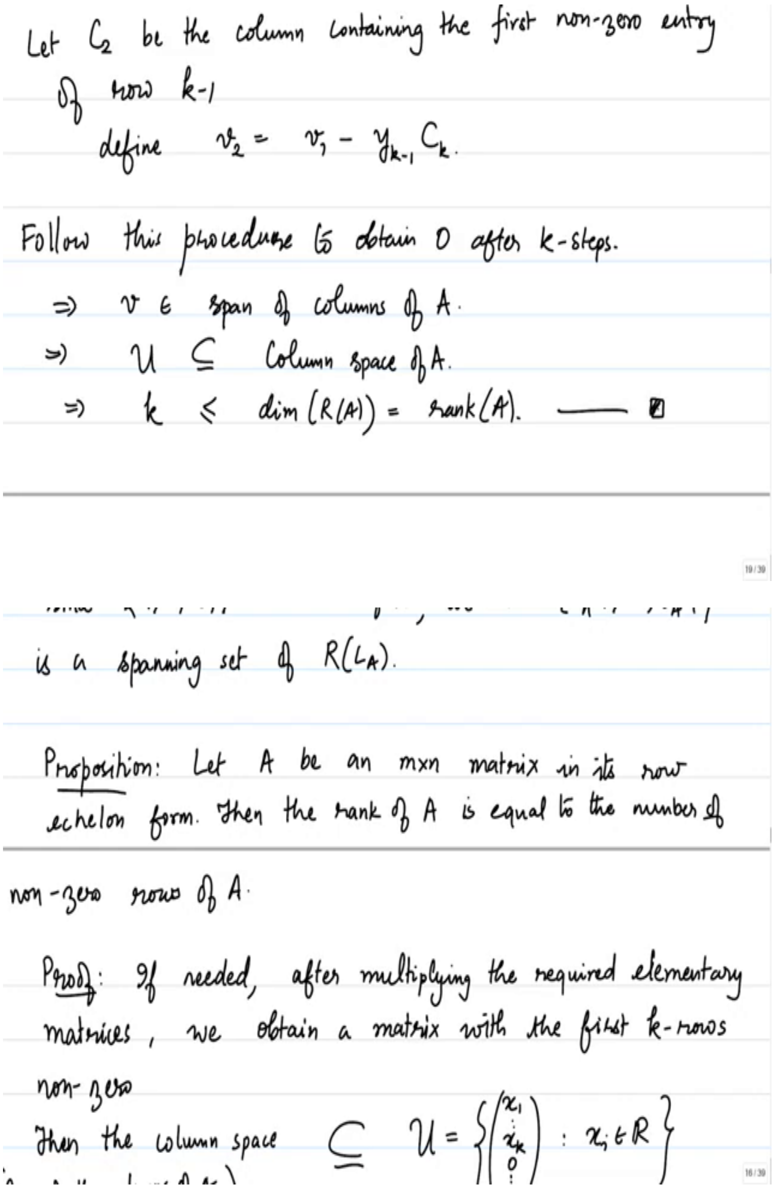

Anyway, that is certainly being used, and also notice that all entries below the k minus 1th row of C2 are zero. Define v2 to be equal to v1 minus yk minus 1 times Ck and follow the same procedure. Follow this procedure. After k steps what do we get? To obtain the zero vector after k steps and if we backtrack what we did, we would have returned v as a linear combination of C1, C2 up to Ck where CIs are columns of our matrix A. This implies that our vector v is in the span of columns of A. This implies that U is contained in the column space of A or the subspace of Rm which is spanned by the columns of A.

And therefore k, which is the dimension of U is less than or equal to the dimension of the column space which is the dimension of the range of A which is equal to the rank of A. And therefore, we have established our theorem. So what we have shown is that, so let me show you the proposition once again. If we have a matrix, which is in its row echelon form then the rank of the matrix A is exactly equal to the dimension, to the number of non-zero rows of A.

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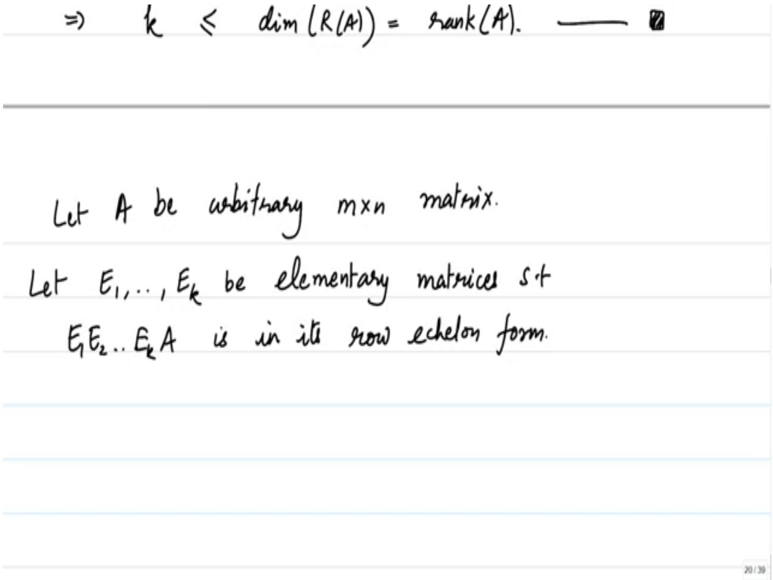

So let us start with an arbitrary matrix. So let A be an arbitrary matrix, m cross n matrix and as I was mentioning earlier it can be reduced to its row echelon form after finitely many row operations. So let E1, E2 up to E, say Ek be elementary matrices such that E1, E2, dot, dot, dot Ek times A is in its row echelon form. Then the rank of E1, E2, Ek times A is the same as the rank of A because each of these E are invertible matrices and we just proved that multiplying by invertible matrices does not alter the rank. So to talk about the rank of A, we just need to reduce it to its row echelon matrix and look at the number of non-zero rows that will give you the rank of our matrix.

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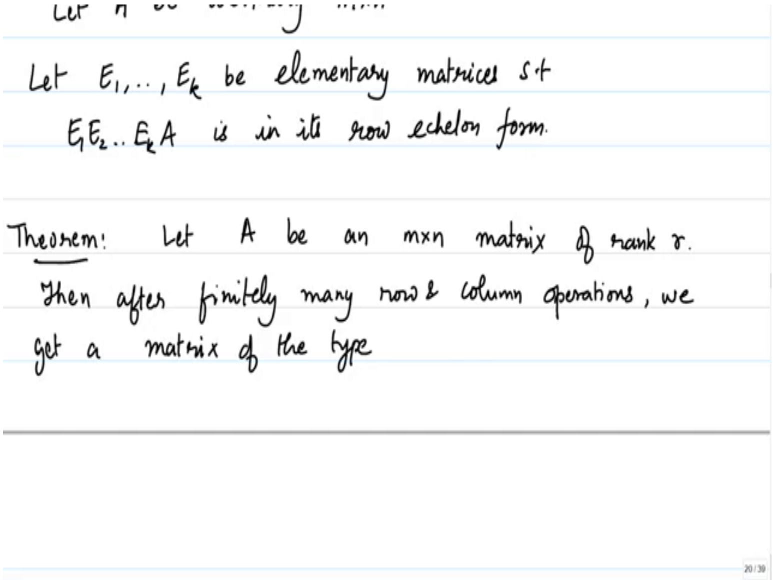

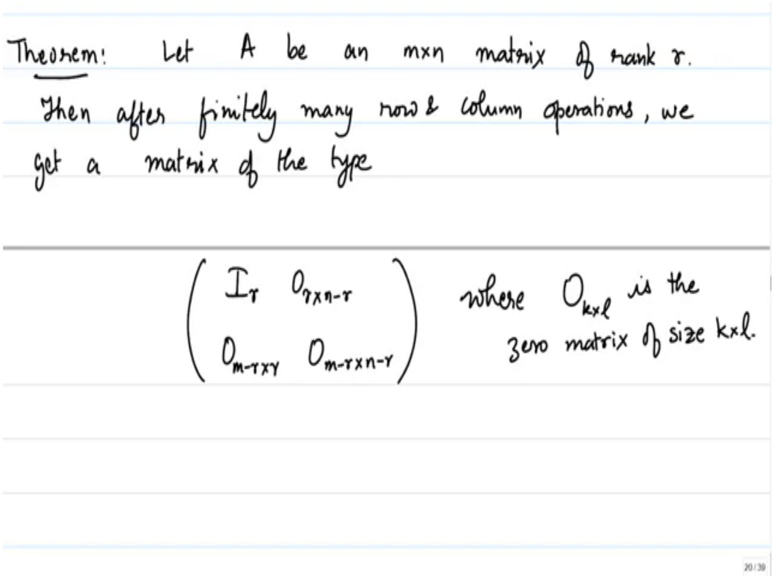

But then the row echelon form need not be quite elegant. It might be still bad. We can do even better. Why are we just considering row operations? Column operations will also be multiplication by invertible matrices just from the right, that is all and therefore the rank is still preserved if you multiply by matrices from the right. So let us now observe that we can do much better. By considering column operations as well, we can exactly find out a very, very nice form or we will be able to derive to a very nice form which will give us the rank or based on the rank the matrix will be in a very nice shape. So let us just have a look at that particular statement.

All right, so that is captured in this theorem. So let A be an m cross n matrix of say, rank r then we can prove that after finitely many column operations and row operations A can be reduced then we can say that, okay let me say that again. Then we can say that after finitely many row operations and column operations A can be reduced to a particularly nice form.

Let me just write it down, then after finitely many row and column operations we get a matrix of the type, I will write it down here, Ir which is the identity matrix of size r this will be an n minus sorry, r cross n minus r matrix. This will be an m minus r cross r matrix and this will be an m minus r cross n minus r matrix, where zero, well k cross l is the zero matrix of size k cross l. Zero matrix meaning all entries are zero. So the theorem is quite powerful in the sense that it tells us that after any row operations and column operations if r is the rank of this matrix you can reduce it to this particularly nice form.

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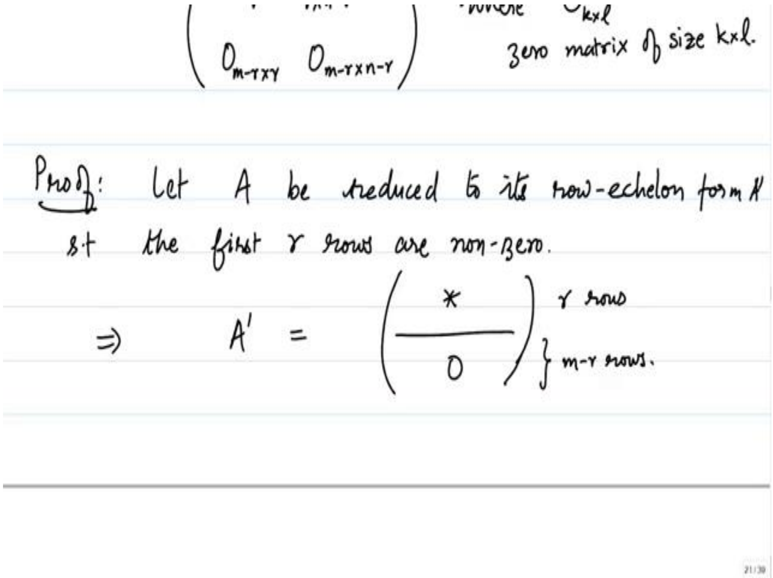

So let us give a proof of this. So after finitely many row operations, let us assume that our matrix is in the row echelon form with the first r rows being non-zero. So let A be reduced to its row echelon form, and assume that after a few more if needed, a few more elementary rows operations of type 1 the first r rows are non-zero. So notice that the previous theorem or the previous proposition tells us that the rank of the matrix A will be precisely the number of non-zero rows in the row echelon form, right. The row echelon form will have the same rank. So yes, it will be the exactly the number of non-zero rows and because r is the rank of our matrix A, that will be r non-zero rows in its row echelon form.

So now let us exchange or interchange rows to obtain first r of them being non-zero. So now that means let us call the matrix A, A prime reduce to its row echelon form A prime, let me just call it A prime. What is our A prime going to look like? A prime will have something here in the first m minus r rows and zero in the final. The first r rows it is going to be something and in the final m minus r rows it is already zero, okay.

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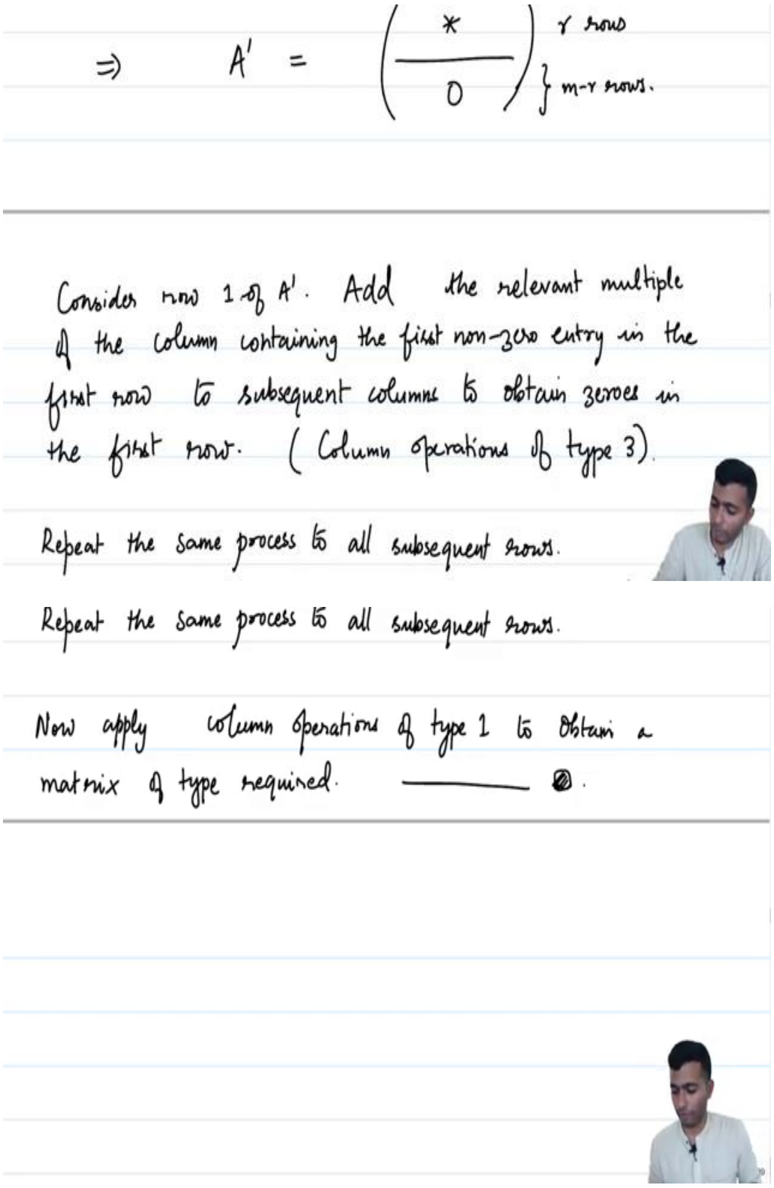

Now let us work more on this, on the matrix A prime. Okay, what do we know about the row echelon form? The row echelon form has a very nice property. After the, till the first non-zero entry is 1 and every entry below it is zero. So consider row 1 of A prime. Then let us now do some column operations. What are the column operations? Subtract these relevant multiple. We are doing a column operation of type 3 now to this matrix A prime and recall what is the column operation of type 3. We subtract a multiple of one column or add multiple of one column to the other, right.

So addition and subtraction or we can just add. Let me just use the word add the relevant multiple of the column containing the first non-zero element, first non-zero entry, which is 1 here in the first row to = subsequent columns. So if say for example, to obtain zeros in the first row. So if for example, the first row has 1 in the 5th column, 6th to remaining up to n, you subtract the A1k for k greater than 5 times the 5th column, and then we will get all the entries after the 5th column to be zero.

Notice that this does not do anything to subsequent entries in the column, only the first column is affected. Why? Because in the row echelon form every entry below 1 is zero, and therefore this does not do anything to other columns. So what happens after this particular column operation is the first entry, the first row will have 1 in one entry and zero elsewhere. Now repeat the same process to the second row, the same process to the second row, and notice that there will be only one entry which is non-zero, which is 1 in one of the columns, and every other entry now will be zero.

And this particular 1 will meet in a different column to the 1 in the first row to all subsequent rows, in fact. Not just the second row, all subsequent rows. And what do we finally get? We obtain matrix with every row containing a non-zero element 1 in one of the columns and zero elsewhere. And these 1s are all distributed in different columns that is what we will end up with. Now apply column operation of type 1 where you interchange, apply column operations many actually, many might be needed operations or none might be needed depends on the requirement. Apply column operations of type 1, so this was all type, remember that these are all column operations of type 3, okay.

Right, so now apply column operations of type 1 to obtain a matrix. So the first r rows have 1 in one of the columns and zero elsewhere. So if you swap or interchange the columns accordingly, we will get a matrix of the type required. Okay so if R is the rank we have a particularly nice way of representing reduced form of that matrix. Let me just note what we did.

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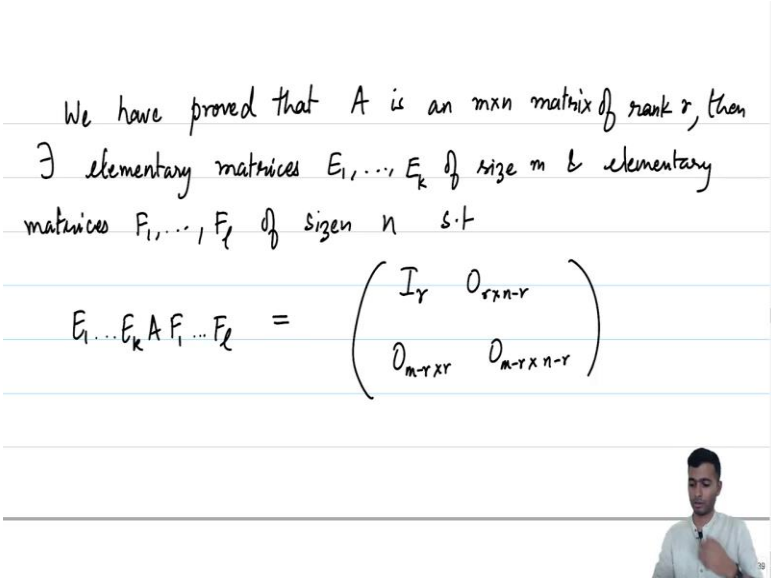

So essentially, we have proved that if A is an m cross n matrix of rank r, then excess elementary matrices E1, E2 up to Ek of size m and elementary matrices F1, F2 up to Fl of size n such that E1, dot, dot, dot multiplied to A times F1 dot, dot, dot multiply it to up to Fl. This is of the type Ir zero n minus r cross n minus r zero m minus r cross r zero m minus r cross n minus r.

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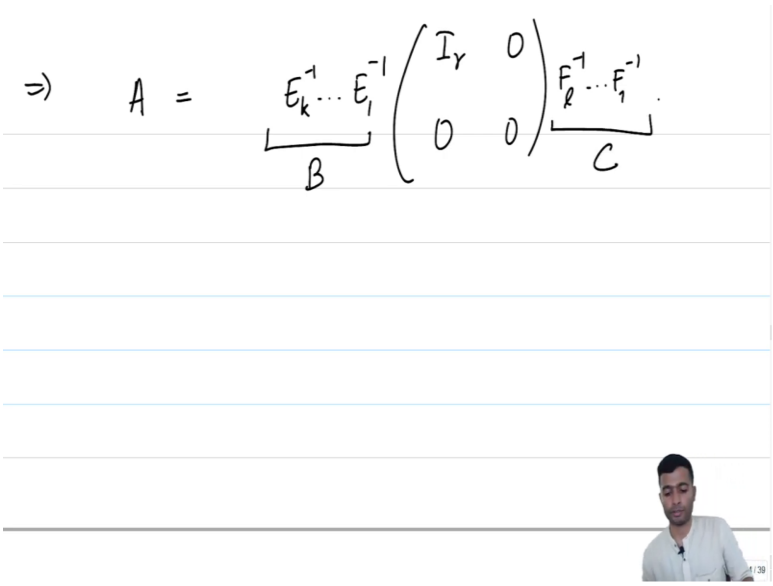

But notice that this E1, E2 up to Ek are all invertible, so is F1, F2 up to Fl. This means that A can be written as Ek inverse dot, dot, dot E1 inverse times our matrix Ir is zero, I will not bother writing the size, it is the same times Fl inverse, dot, dot, dot F1 inverse. Because these are all invertible matrices. So let us call this something, let us call it B and let us call this something C and this is precisely what we have proved in the last two propositions.

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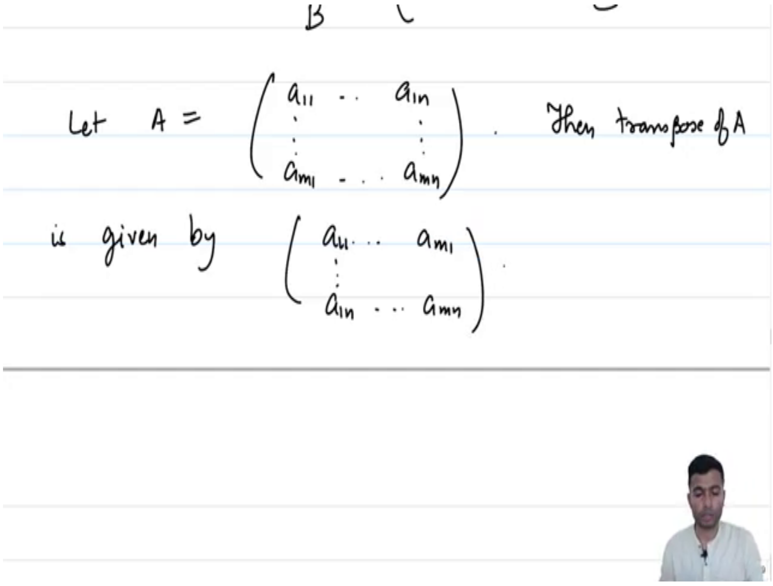

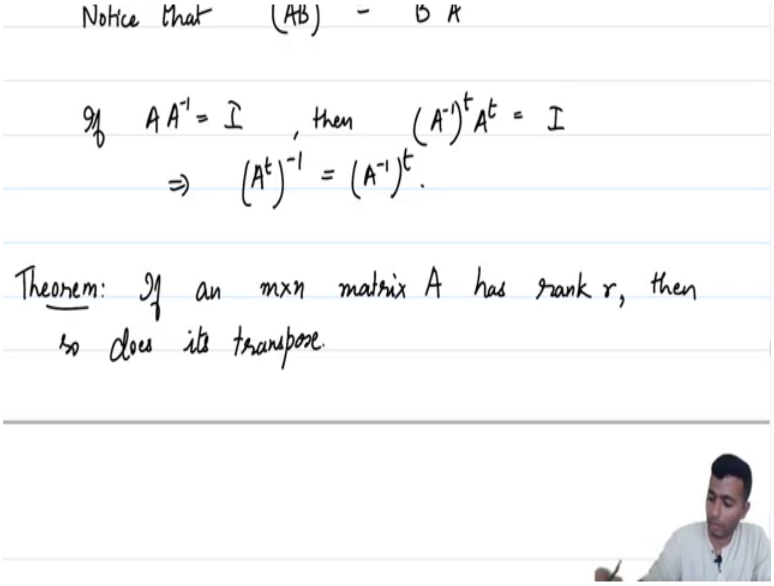

Okay next let us try to address what would happen if you look at the transpose. We will show that the transpose of a matrix will also have the same rank as A. So recall that a matrix A, the transpose of a matrix is obtained by reflecting along the diagonal, so let A be equal to say a11 up to a1n, am1 up to amn. Then the transpose of A, what is transpose of A? This will be given by a11 to am1, a1n to amn. Notice that this is a n cross m matrix. A few properties of transpose are important to be noted here. Notice that, check that I will leave it to you again, notice that you would have seen it already, AB transpose is the same as B transpose, A transpose.

And further, if A is invertible, so is A transpose. So if A inverses identity, let us look at the transpose of this, then A inverse transpose, A transpose is equal to identity transpose which is the same as identity. And that implies that A transpose inverse is equal to A inverse transpose. So the transpose of a matrix is also invertible. So now, let me give you a theorem, which tells us that if a, if an m cross n matrix A has rank r, then so does its transpose. The transpose also has the same rank as A. and how do we go about proving this? It is quite straightforward.

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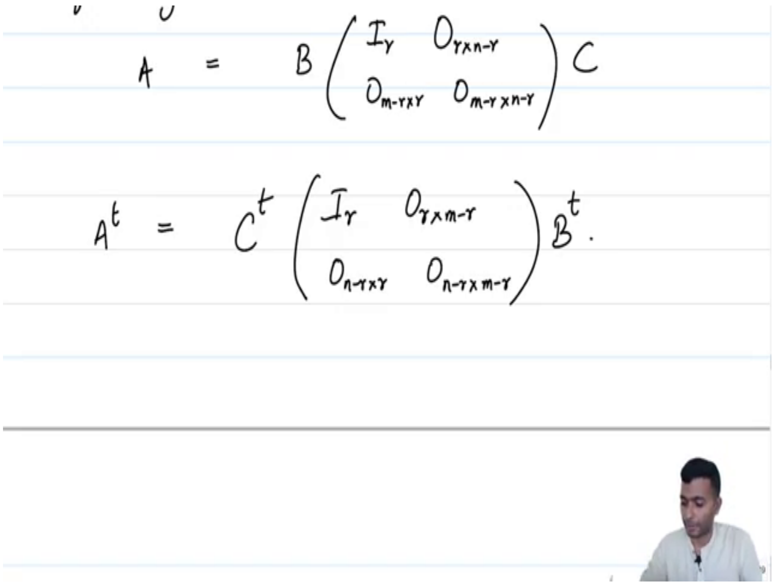

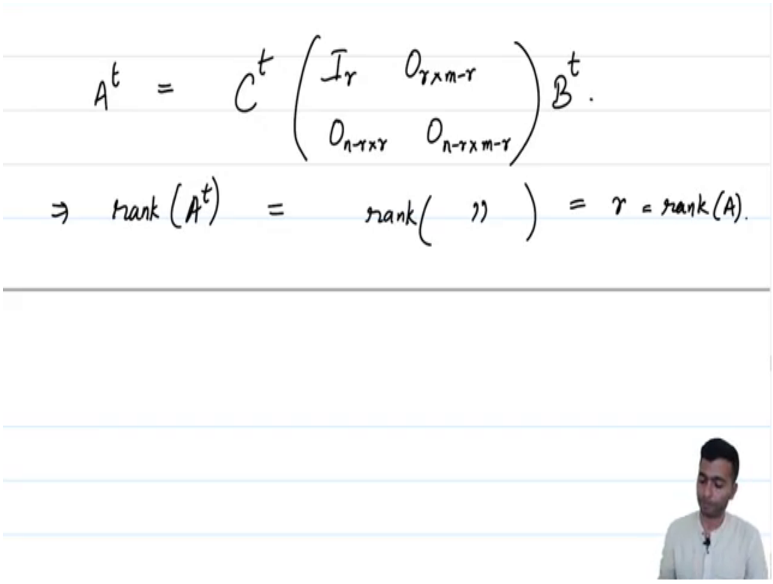

So let us write A, by the above theorem, let us write A in the special form we have. By the above theorem, we have A is some B times Ir zero r cross n minus r, zero m minus r cross r, zero m minus r cross n minus r times C, where B and C are invertible matrices. Now let us take the transpose of A. Transpose of A will give you, by the transpose it inverts the order in which we do it, we should check that this is the same as Ir zero r cross m minus r, zero n minus r cross r, zero n minus r cross m minus r. This is something which you should check yourself times B transpose, this is precisely what the transpose will look like.

But then, if B and C are invertible so are B transpose and C transpose, and if you multiply by invertible matrices the rank will not change. So rank of, this gives rank of A transpose is equal to the rank of this particular matrix. Let me just put it like this, I do not want to write it down entirely. But that is precisely equal to R which is the number of non-zero rows, which is equal to the rank of A. So it is not something arbitrary we have shown, if you just think about it the rank of A transpose will be the dimension of the column space of A transpose, but that is exactly the row space of A.

So the row space of A, if you think about it is in RN and the column space is in RM. So if you take a matrix, look at its column space in RM, suppose it has dimension R, then you look at the column, the row space of the matrix A in RN that will also necessarily have dimension R. So these matrices are, they are the relation is being captured well in this particular theorem. Okay, so we have spent some considerable amount of time developing simpler ways to look at the rank of a matrix. How do we relate it to the rank of linear transformation? So that is going to be our next quote.

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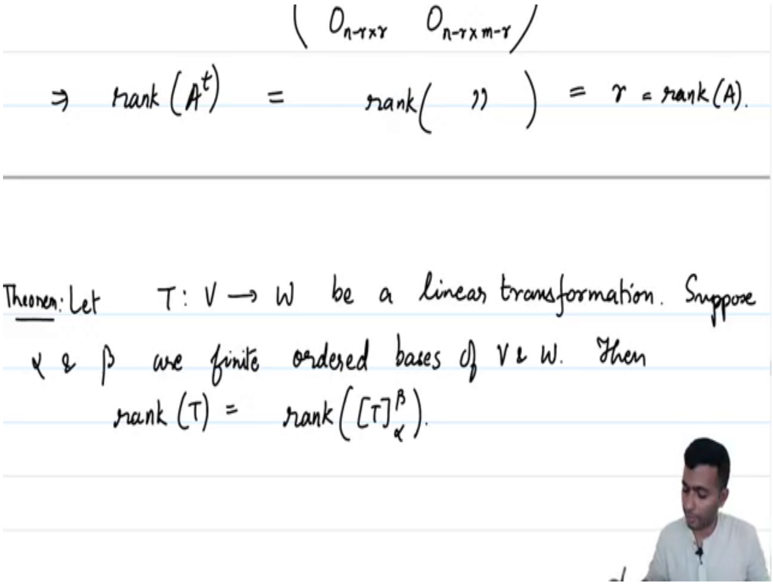

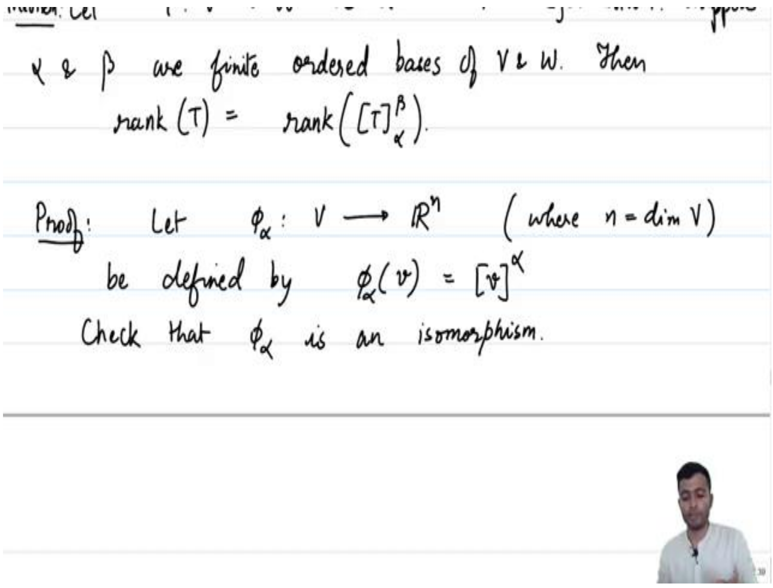

So let theorem, it is a theorem. So let T from V to W be a linear transformation. Suppose alpha and beta are finite ordered basis, so when I say finite its cardinality is finite and therefore, V and W are finite dimensional. Suppose the alpha and beta are finite ordered basis of B and W, then rank of T is equal, as a linear transformation we can talk about rank, and this coincides with the rank of the matrix of T with respect to alpha and beta.

Irrespective of what ordered basis alpha and beta you take and look at its matrix rank of T is equal to the rank of this matrix. So let us give a quick proof of this, it is an elegant proof. We will first define, let phi alpha be a map from V into Rn, where n is equal to the dimension of V. We defined by, we have not defined this map, let us define this map to be phi alpha of a vector V is the column representation with respect to alpha. So notice that this is going to be a n column which can be thought of as an element in Rn.

And I will leave it to you as an exercise to check that phi alpha. We know that it is linear transformation, if you look at phi alpha V plus W, we have checked that this will be V alpha plus W alpha. V1 plus V2 if you look at it, it is going to be V1 alpha plus V2 alpha. Same with scalar multiples. Again, if you have not seen it, it is a good exercise to check that phi alpha is not just a linear map but a isomorphism, it is an exercise.

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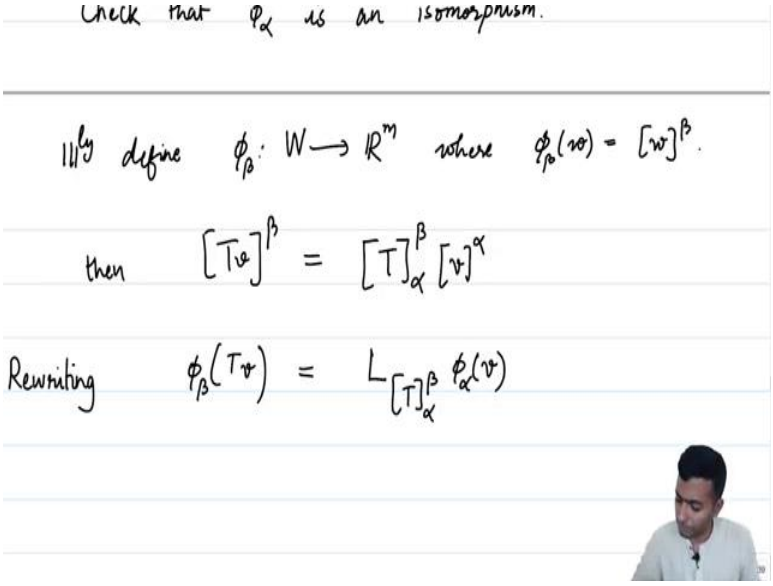

Similarly, we can also define phi beta. Similarly, define phi beta, which is from V to in this case Rm not V, W to Rm from the image, where phi beta of say a W is the column representation of W with respect to beta. Now the matrix representation of T with respect to alpha beta tells us that, then Tv with respected to beta is equal to T alpha beta v alpha right, this is the impact of going down to basis alpha beta. So if I had to write it slightly differently, this is nothing but phi beta of Tv is equal to, let us look at LT alpha beta. And then this is going to be phi alpha of V right. If we are to look at Tv as a vector in Rn and, sorry Rm and if we were to look at v alpha as a vector in Rn, then this is exactly what, this is rewriting we get this our matrix multiplication is captured by L subscript A, here A is T alpha beta.

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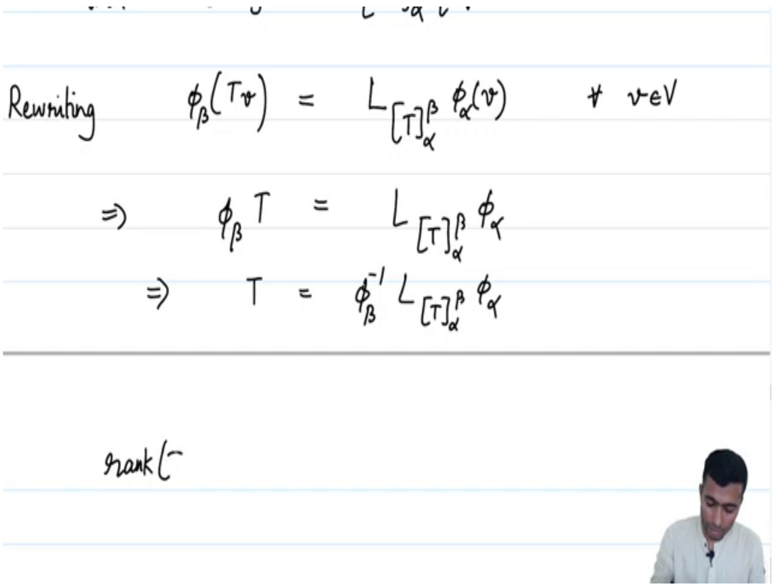

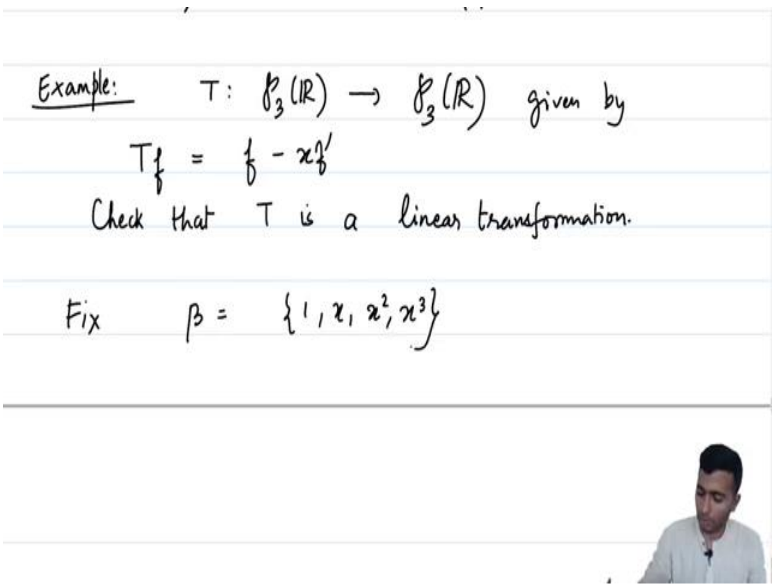

Let us carefully observe what has happened, this is for all V in capital V right. This implies phi beta composed with T is equal to LE alpha beta composed with phi alpha because it is getting satisfied for all V, and we know that phi beta as an isomorphism. This is all, I am not writing down the reasons, phi beta being an isomorphism I am just orally telling it, this is equal to LT alpha beta phi alpha. But then what is the rank of T now?

Rank of T is the rank of the thing in the right-hand side here, this part. But phi beta and alpha beta both are isomorphisms and hence, they are invertible. And therefore this is the same as the rank of linear transformation LT alpha beta because if you compose with invertible linear transformations the rank is not changed, but that is precisely by a definition equal to the rank of T alpha beta.

Okay, so we have now linked the notion of the rank of a linear transformation to the rank of its matrix. And we have spent a considerable amount of time trying to develop all the tricks needed or all the techniques needed to reduce our matrix A into a good form which will give us the rank. So let me stop this video with an example. Suppose T is a map from P3 of R to itself given by Tf is equal to f minus x times f prime. Should check that this is a linear transformation, check that T is a linear transformation. Now if you fix the basis 1, x, x square, x cube. 1, x, x square and x cube, let us try to calculate what the matrix of T with the expected to beta is.

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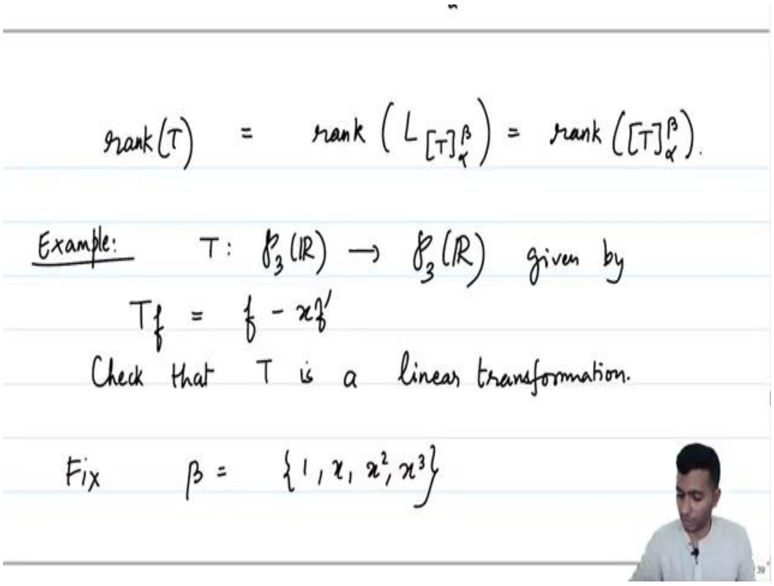

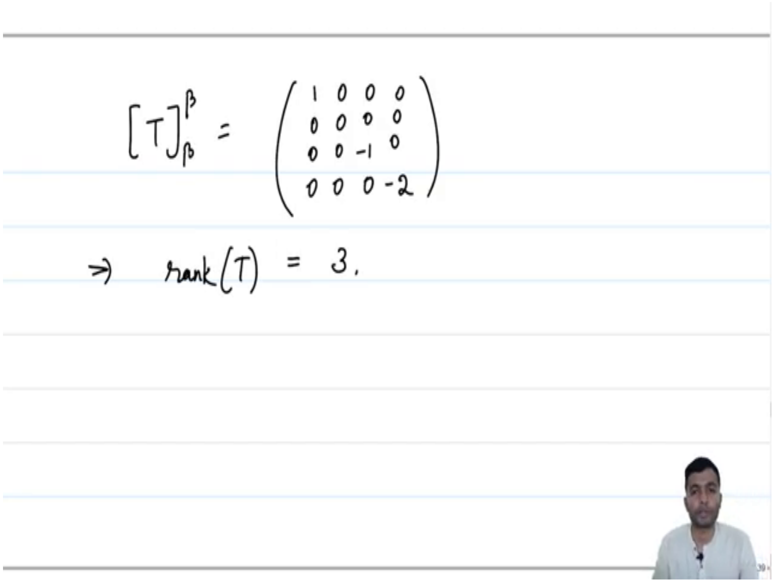

The first column is going to be T of 1, T of 1 will be 1 minus x times 1 prime is zero, so it will be 1, so it is going to be again, 1 and this is going to be 1, 0, 0 ,0 because 1 is just one times 1, zero times zero remaining things, okay. What is T of x? T of x will be x minus x times 1 which is zero, so the zero vectors. How about T of x square, it is going to be x square minus 2x square which is minus x square right. So this will be 0, 0 minus 1, 0 and x cube will be x cube minus 3x square which is minus 2x square, sorry minus 2x cube. So this is going to be 0, 0, 0 minus 2. And this implies that rank of T is equal to 3 because rank of T is the rank of its metrics with respected to beta, alright.