**Linear Algebra**
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**Lecture No. 6.1**
**Row Operations**

So till now, we have discussed quite extensively what linear transformations are, how they are related to matrices and how this interaction plays out and in this week we will continue with that study. So we will focus in this week, we will start with the study of row operations of a matrix and also discuss what the row echelon form of a matrix is. Many of you might be already familiar with this notion but nevertheless, it is quite useful to recall it and put it in a formal setting. So, let us recall from the beginning, what row operations of a matrix are. So an elementary row operation can be thought of as a map, which takes an m cross n matrix and gives you back another m cross n matrix.

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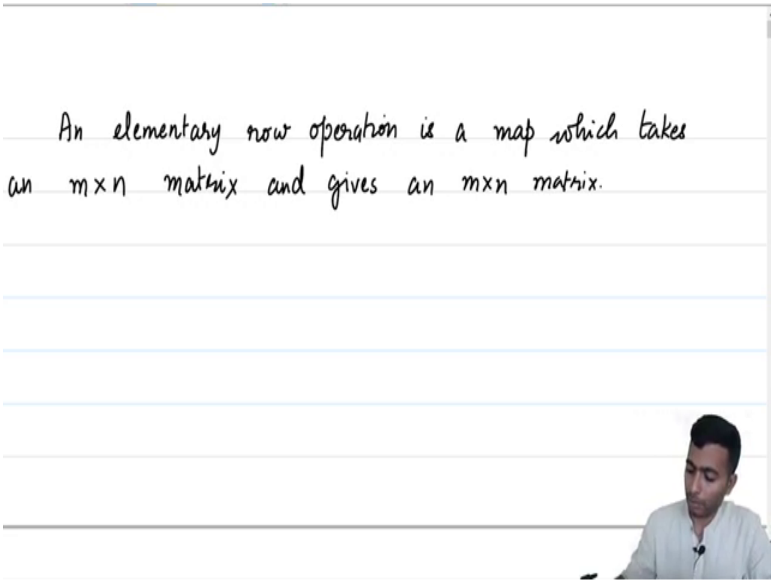


So an elementary row operation is a map which takes an m cross n matrix and gives another m cross n matrix. But what is the output like, they are very special maps. So row operations are of three types. So let us discuss them one by one.

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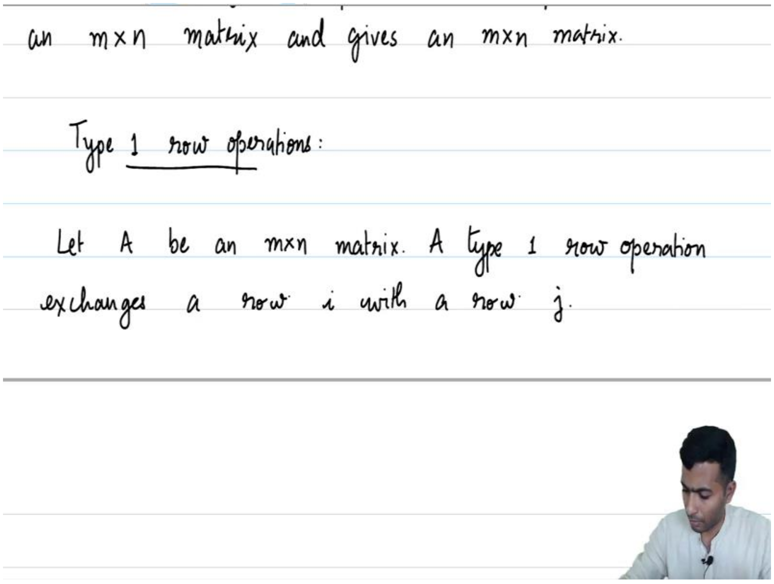


So first one is the elementary row operation of type 1. Let me just write it as type 1 row operations. Type 1 row operations are obtained by exchanging two rows of a given matrix. So let A be an m cross n matrix. A type 1 row operation exchanges row i with row j.

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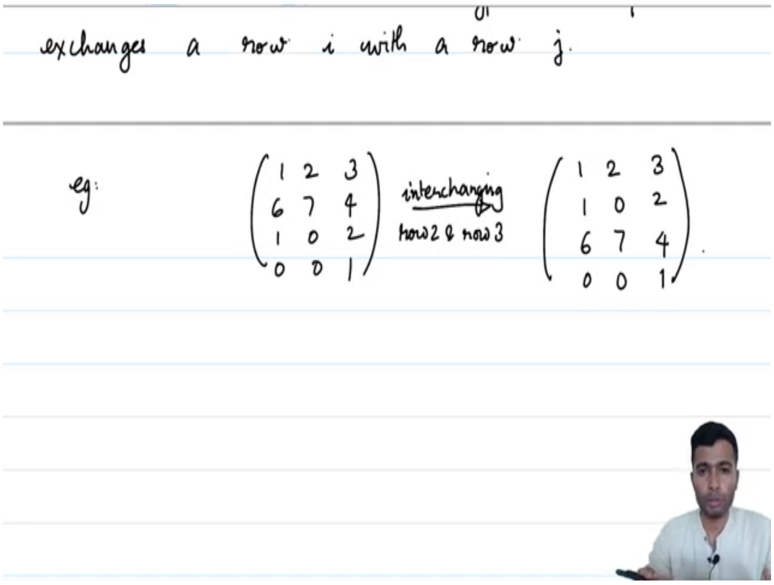


So let us look at one example of what I just said. So if you are to take some matrix, say 1, 2, 3, 6, 7, 4, 1, 0, 2 and 0, 0, 1. Let us take a 4 cross 3 matrix. Then, let us look at a simple row operation, let us swap row 2 and row 3. Interchanging row 2 and row 3. This is an elementary row operation of type 1 we get the matrix 1, 2, 3, 1, 0, 2, 6, 7, 4 and 0, 0, 1. Now the row operation that was just described can also be realized in an alternate manner.

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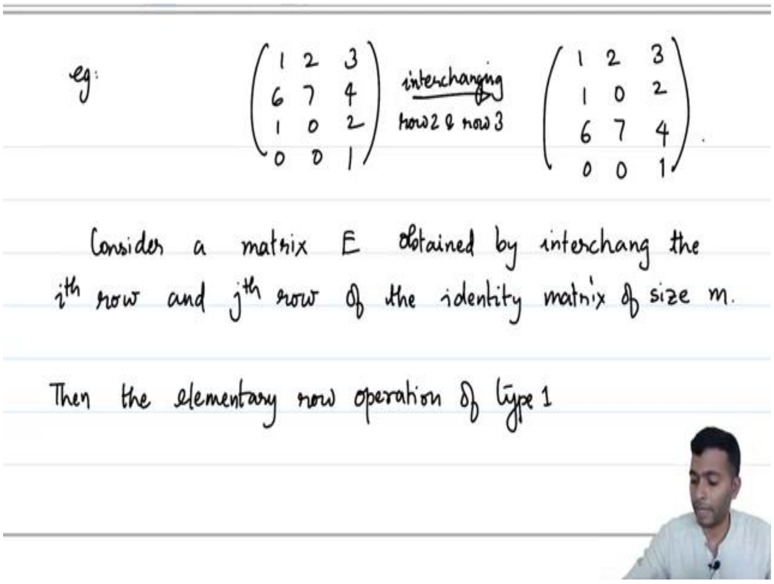


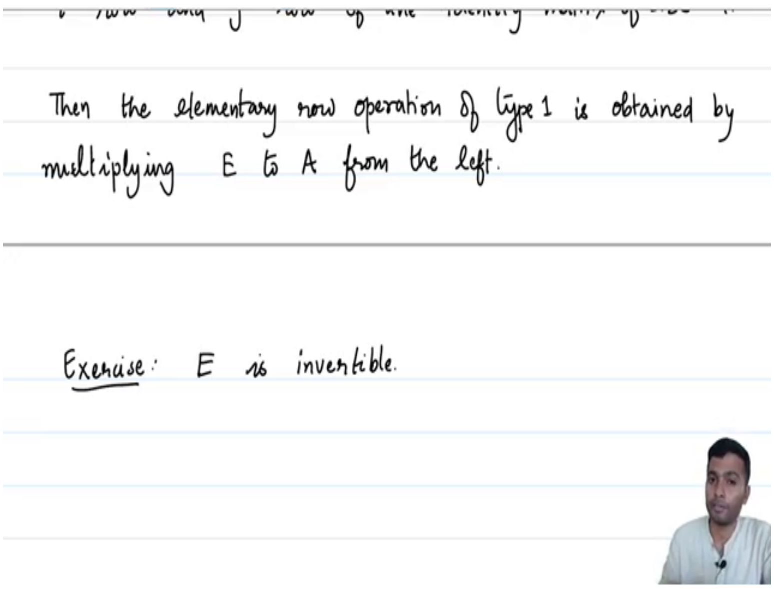


Now consider a matrix. Consider a matrix E obtained by interchanging the ith row and the jth row of the identity matrix of size n, size m, sorry. Our matrix A is m cross n, so you look at the identity matrix of size m and interchange the row i with the row j of the identity matrix call that matrix E. Then notice that or check that, so this is an exercise for you. Then the elementary row operation of type 1 is obtained by left multiplication by this matrix E is obtained by multiplying E to A from the left.

Multiplication makes sense because E is an m cross m matrix and A is an m cross n matrix, what we get back is an m cross n matrix. The thing to note here is that the matrix E that we have just obtained is an invertible matrix. So exercise for you. In fact, I would say that in this case, E is its own inverse.

E is invertible, I will leave it as an exercise for you to check that if you multiply E with itself, it will as to be expected because E swaps Ith row with the Jth column. So if you multiply E with itself, it will swap the ith  row with the jth column, but if you swap ith row with the jth column and swap again the ith row with the jth column, you get back what was initially the case and this is the identity mapping. So E is in invertible map.

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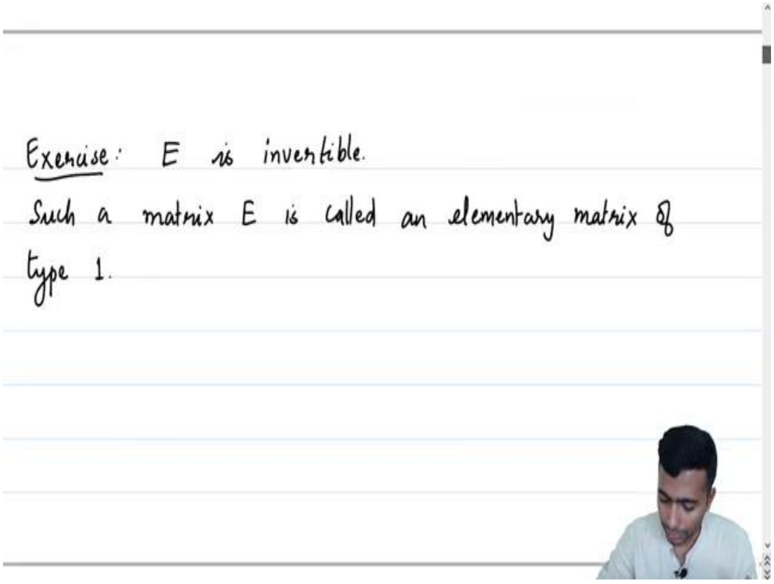


This matrix E or this type of this family of matrix, such a matrix E is called an elementary matrix of type 1. It might be a good check to see what happens in this example.

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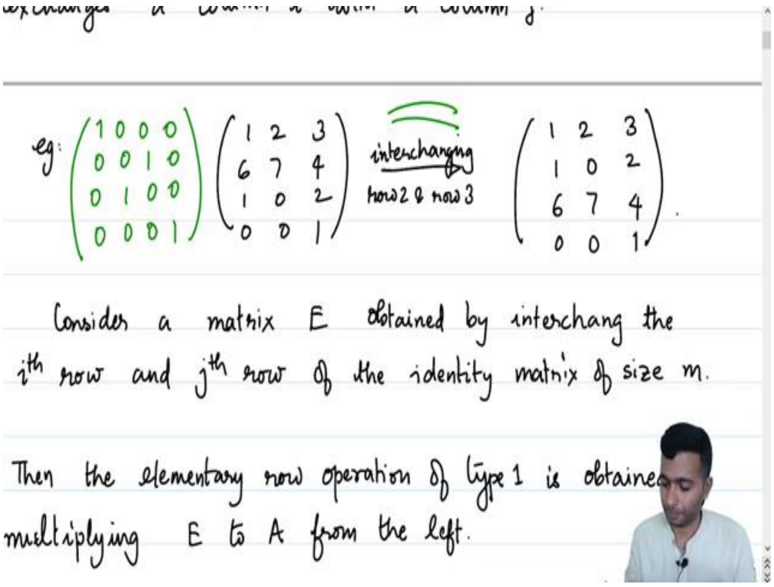

I would say that if you multiply, let me put a green here to show you that it should be a 4 cross 4 matrix that we are multiplying. And if you look at 1, 0, 0, 0, we have exchanged row 2 and row 3, so this is going to be 0, 0, 1, 0. And the 3rd row will now be the 2nd row which is 0, 1, 0, 0, 0, 0, 0, 1. This is the example of E that was being mentioned below. And if you multiply, you should check that this is going to be equal to the matrix on the right. It will just exchange the 2nd row with the 3rd row. Alright, so that was one of the row operations we are familiar with. Next, we will discuss row operations of type 2.

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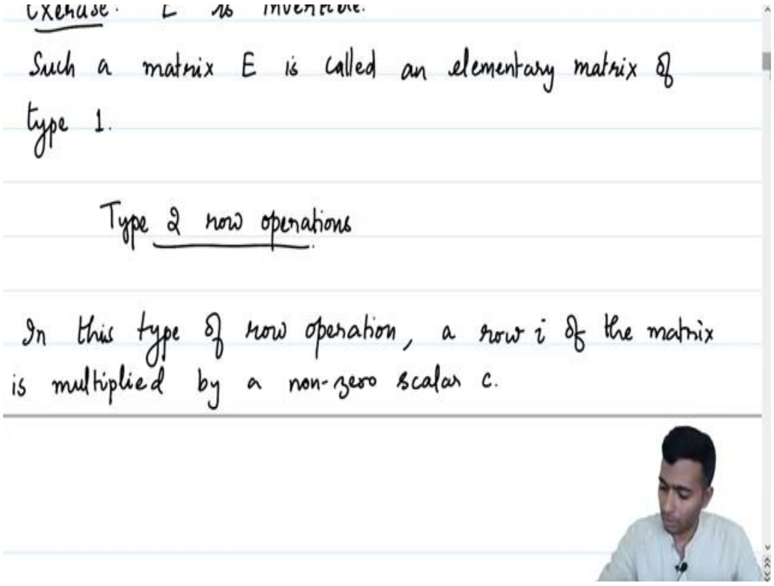


So type 2 row operations. So in the first type of row operation we just interchanged rows. Now, we will multiply a row by a non-zero scalar, so in this type of row operation, a matrix A or a matrix, okay so let me write in this manner. A column of the matrix, so this is all happening to a particular matrix, right. So column of, oh, why am I writing column sorry, a row of the matrix. We are doing row operations. Row of the matrix is multiplied by a non-zero scalar. So all such operations are put in the type 2, category of type 2 row operations scalar c. So, suppose let us say a row I of the matrix is multiplied by a non-zero scalar.

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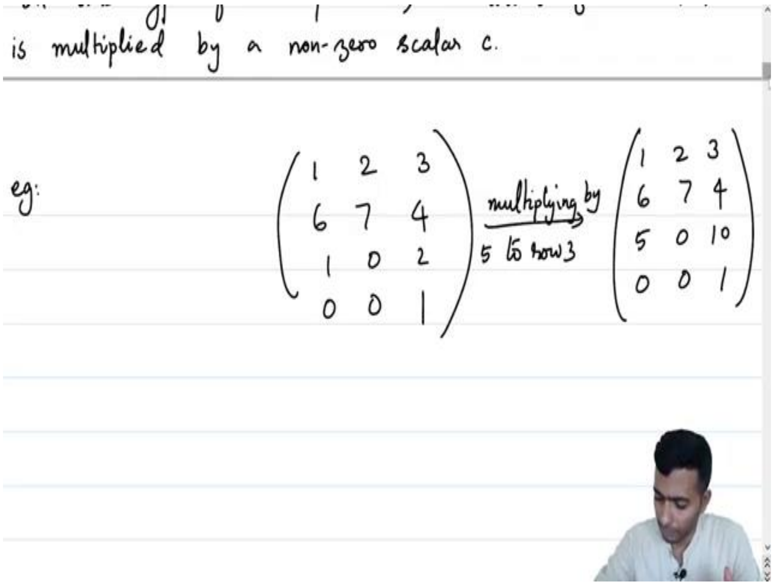


So let me just give you an example from the previous example itself. Let me just see what the example was 1, 2, 3, 6, 7, 4, 1, 0, 2. There was more, and there was one more row and 0, 0, 1. Maybe I should just write it here to give it more space 1, 0, 2 and 0, 0, 1. So this matrix by multiplying by say, 5 to row 3 what do we get by multiplying 5 to row 3? This will give you 1, 2, 3 this is unchanged, 6, 7, 4 is unchanged but now it will be 5, 0, 10 and the final row will be unchanged. So this is an example of type 2 row operations.

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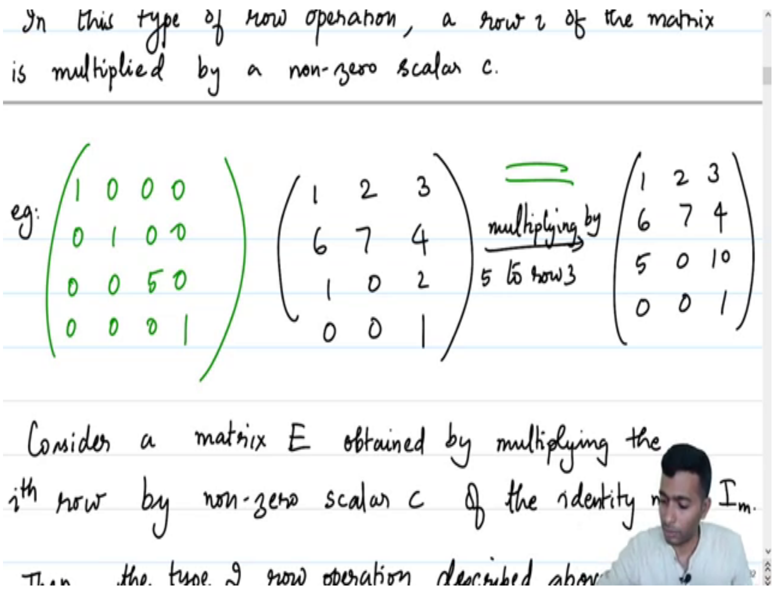

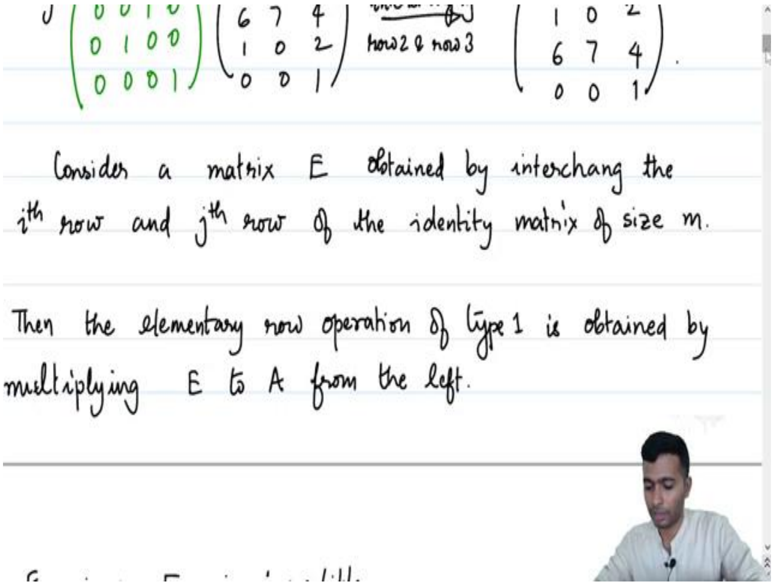


And as is to be expected, this type of row operation can also be obtained by multiplication by a matrix E. So as you can see, here, this matrix multiplication is giving us this row operation. So what type of a matrix will give us the row operation that we are concerned with? So we will consider now a matrix E. So, recall that here maybe I did not mention it. Of size m, I did not mention, so look at the identity matrix of size m then multiply the ith row of the identity matrix by the non-zero scalar. So consider the matrix E obtained by multiplying the ith row by the non-zero scalar c. ith row of what? Of the identity matrix. I will just write Im to indicate that it is of size m.

Then the type 2 row operation described above is obtained by multiplication of E to A from the left, multiplying E, A. So, let us again get back to our example, we multiplied here five times to row 3 right, so again let me use green to tell you that, first write down the matrix, this is going to be multiplication by this is, a 4 cross 3 matrix. So m here is 4, this is going to be 0, 1, 0, 0. And this is what will change now to 3rd row is being multiplied by 5 and the 4th row is unchanged. If you carefully observe multiplication by this will give you the matrix on the right.

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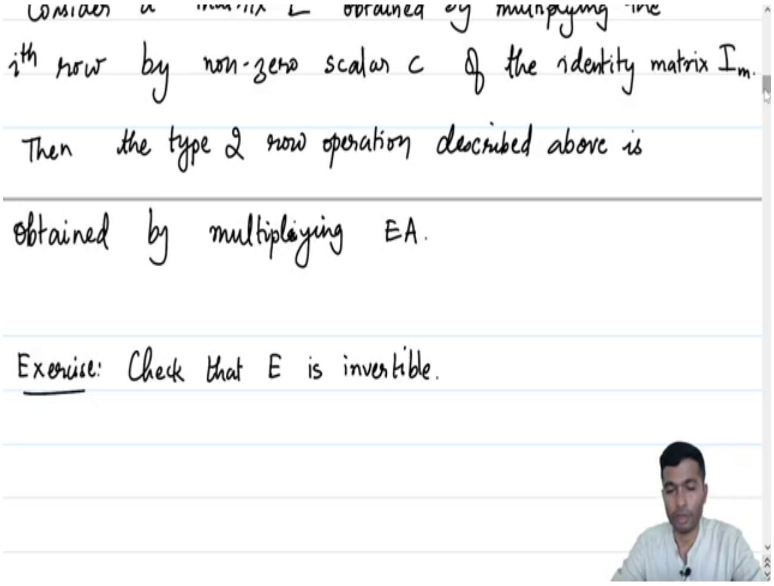


So again, note that again I will leave it as an exercise. Check that E is invertible. In fact, the inverse of E, we will use the idea that when multiplied by E, the ith row is multiplied by c. So you look at the matrix E prime, which is obtained by multiplying the ith row of E by 1 by c, c is a nonzero scalar, so we can multiply by 1 by C and look at E prime. Now notice that if you multiply E by E prime to the left, then the ith row of E gets multiplied by 1 by c, but the ith row has only c in the ith column, and therefore 1 by C times C will be 1 which will give you back the identity. So it is quite easy to check that E is invertible, and we know exactly what the inverse of the E is.

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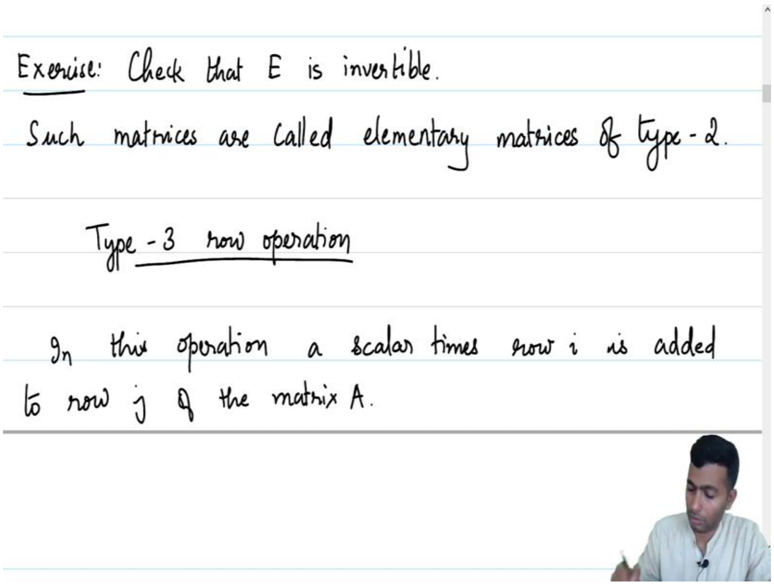

So note that E is invertible and such matrices, all such matrices are put in the category of type 2 elementary matrices. Such matrices are called elementary matrices of type 2. Alright, so we have now noted what type 2 elementary row operations are. And there is one 3rd type of row operations. The 3rd type of row operation basically deals with multiplying a row j with a scalar times row i, you take 2 rows i and j, look at c times, say row i and add it to row J that will give you the mu jth row of the matrix that is the elementary row operation of type 3. So, type 3 row operation. So let, in this operation scalar times row i is added to row j of the matrix.

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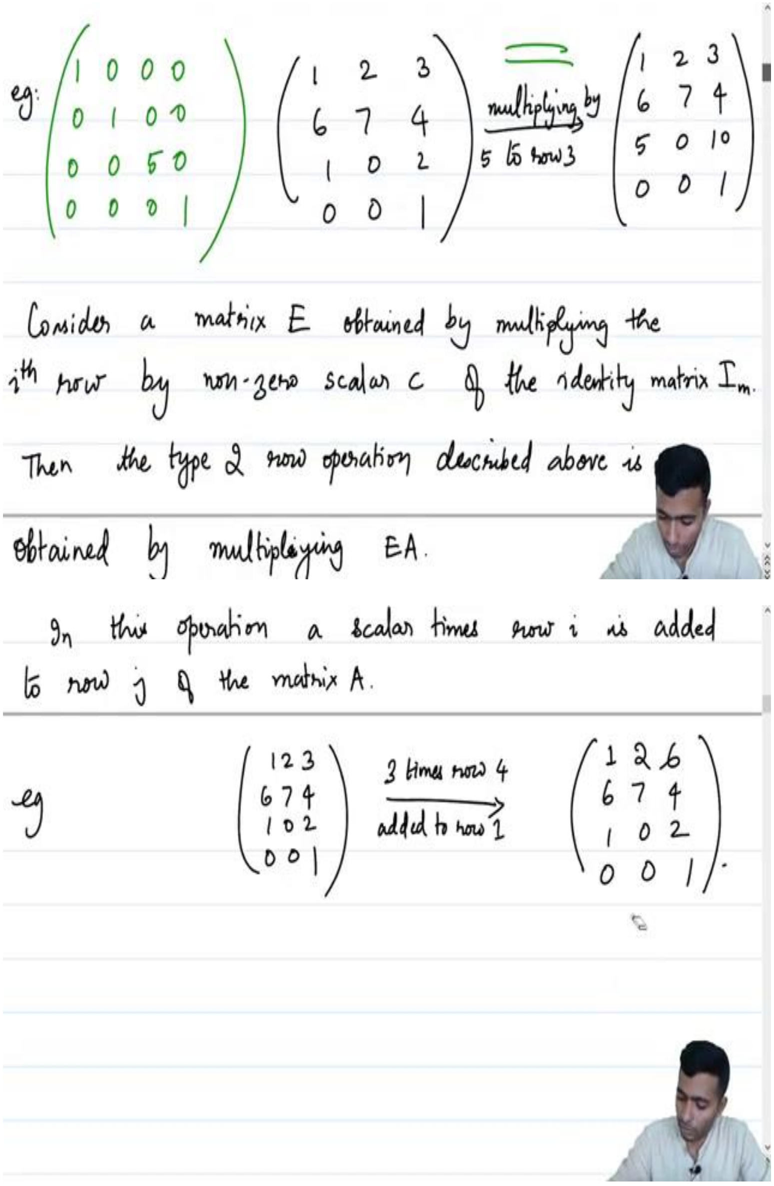


So, let us give an example with again the same example as before 1, 2, 3, 6, 7, 4, 1, 0, 2 and 0, 0, 1 this was our matrix if you recall. And what is the row operation, maybe we should do maybe 3 times row 4 is added to row 1 let us say. Let us see what happens in this case, so this will give you row 1, 1 plus 3 times 0 is 1, 2 plus 3 times 0 is 2, 3 plus 3 times 1 is 6. This is what the first row will be, the other rows here it will be unchanged. And as in the previous cases, one should expect that we can get hold of some matrix E, which when multiplied to this matrix gives us the matrix on the right, this matrix okay.

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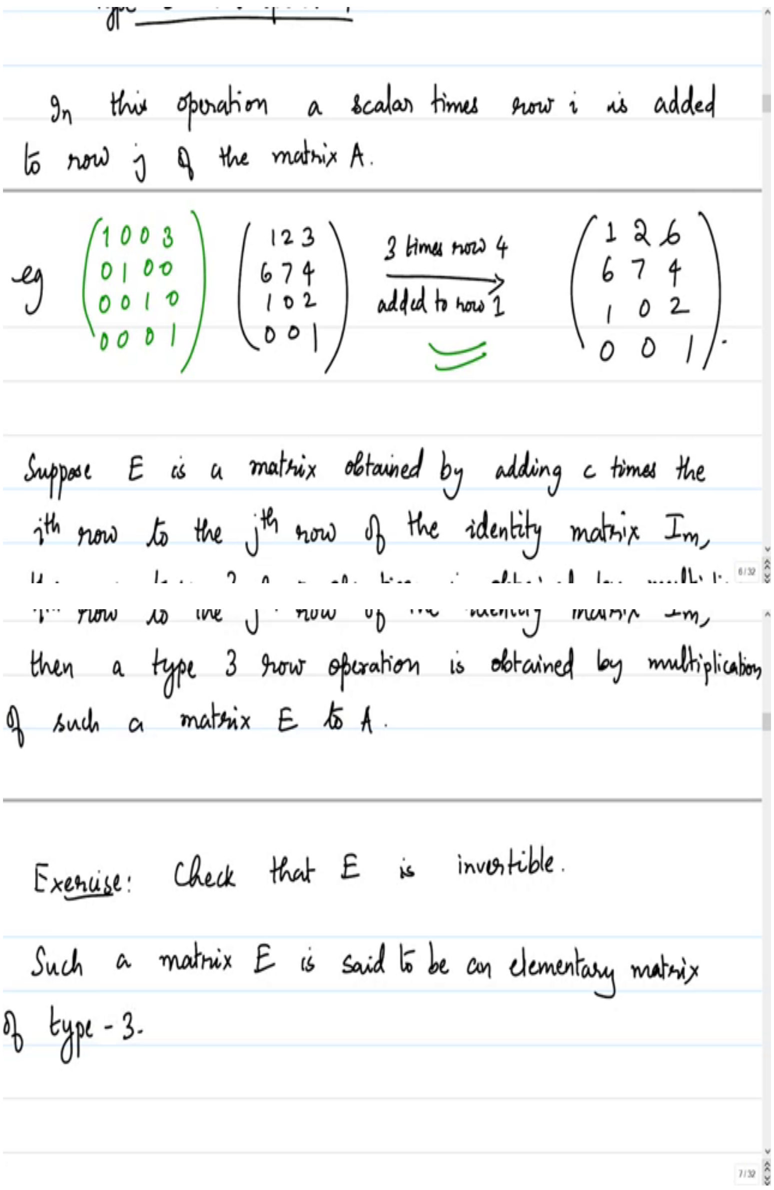


Let us try to see what that is. The right guess will work here. Suppose, E is a matrix obtained by adding c times the ith row to the jth row of the identity matrix Im. Then a type 3 row operation is obtained by multiplication of such a matrix E from the left to A. So what would be the case here? Let us make a guess, we are not making a guess we will know exactly what we are going we are expecting here.

So 3 times row 4 is added to row 1, so this will be 1, 0, 0, 3, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1. This matrix when multiplied, you should check that this is going to give us exactly our matrix to the right here. Again, this is an invertible matrix. So note that I will leave it as an exercise for you. This time let me not even give you a hint of what the inverse would be, check that E is invertible and such a matrix E is set to be an elementary matrix of type 3, all right.

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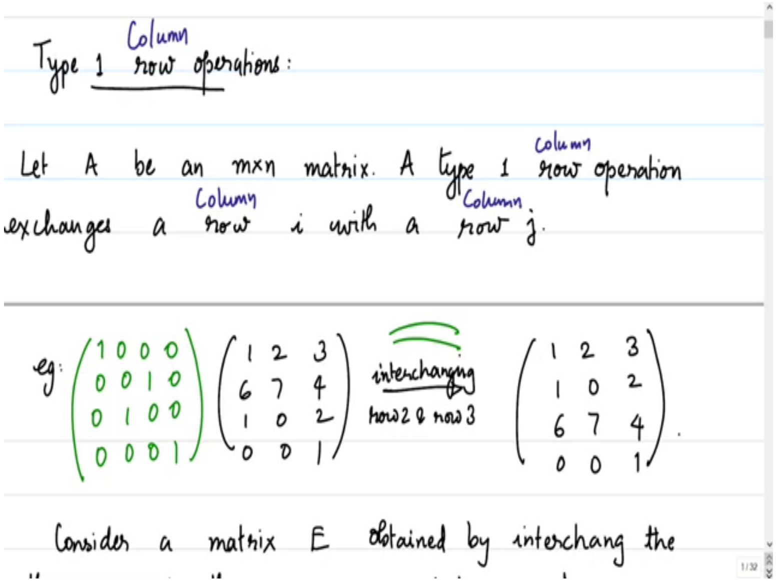


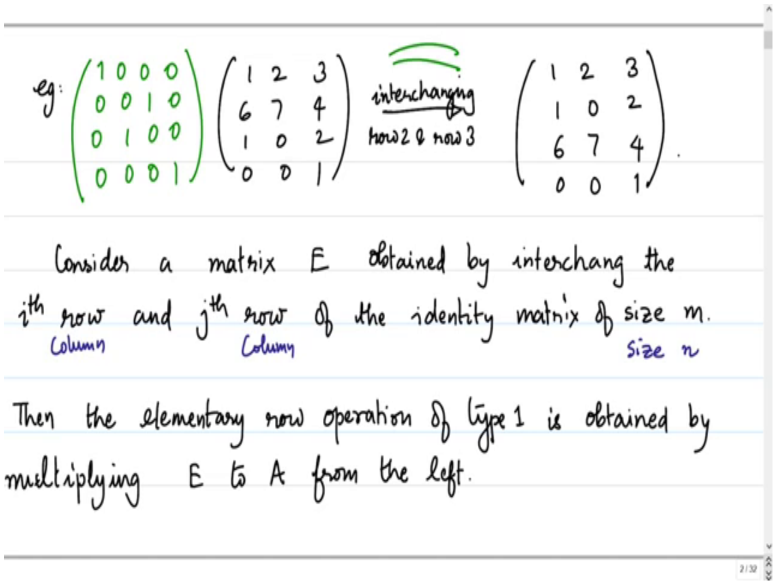


So we have now discussed 3 types of elementary row operations. Of course, there was no need to start off with a row operation, we could have described elementary column operations instead of row operations, but then what would be, let us just do a quick review of what we did, and parallelly let us look at what would be the corresponding row operations. So let me just use some other color here, let me use red here.

A column, a type 1 column operation would be just in the same definition, just replace the word row and see what happens. So, here it will be a column if we exchange columns, then it will only be column operation. So I would like, I would request you to go back to when we define type 1 row operations and see that the definition should be like this, a type 1 row operation is obtained by exchanging row i with the row i. A type 1 column operation is what you will obtain when you replace column i with column j.

And in the case when it is a column operation the multiplication will not be from the left, multiplication will be from the right. So it will then turn out to be of size n because it is an m cross n matrix, the elementary matrix that we consider will be of size n. Again, it is obtained by interchanging the relevant column of the identity matrix of size m. A row operation is obtained by interchanging rows, column operation is obtained by interchanging column. Row operation can be realized as multiplication from the left of an elementary matrix, a column operation is obtained by multiplying from the right by a elementary matrix, all right.

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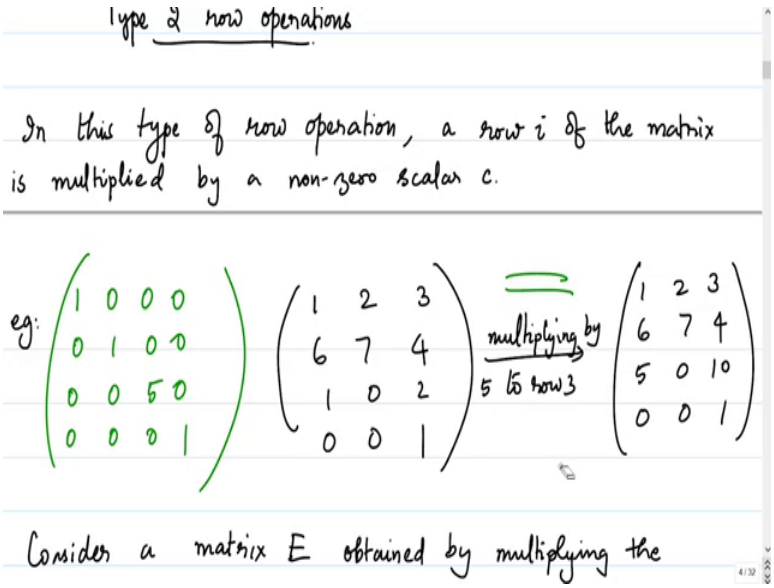

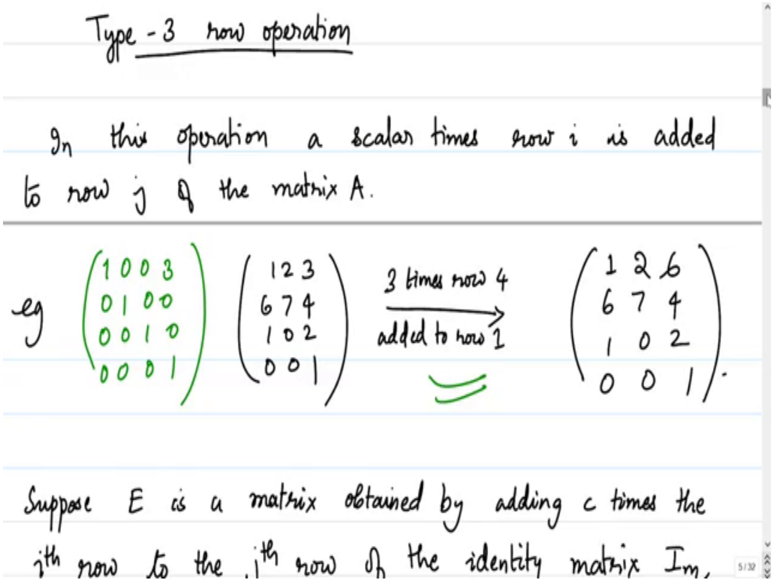


Similarly, a type 2 column operation is obtained by multiplying say, a column i by a non-zero scalar and that will be obtained again by multiplying from the right by an n cross n matrix, an elementary matrix which is obtained by multiplying the ith column of the identity matrix of size n. Those are also called elementary matrices of type 2 and what about the column operation of type 3, it is obtained by adding to say column j, c times column i. So if you add column j, c times a column i to column j that will be an example of a column operation. And this will be again obtained by multiplying to the right an n cross n matrix which is obtained by adding c times the ith column of the identity matrix to the jth column okay.

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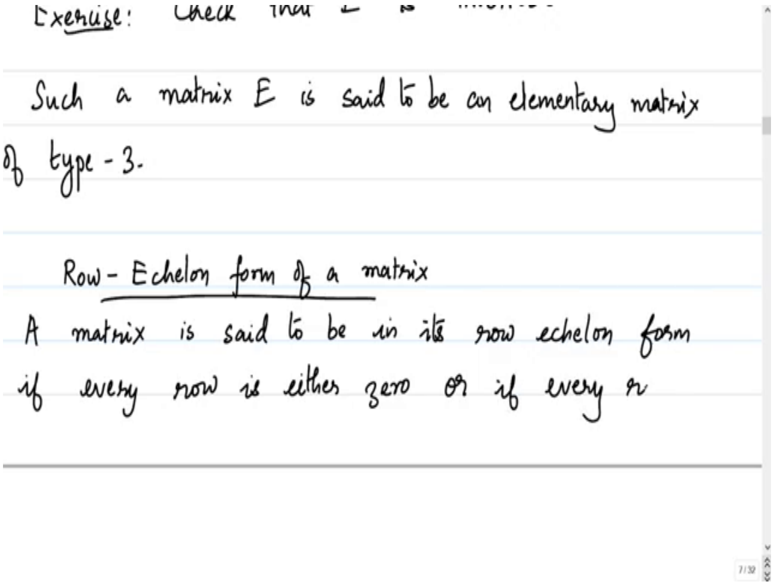

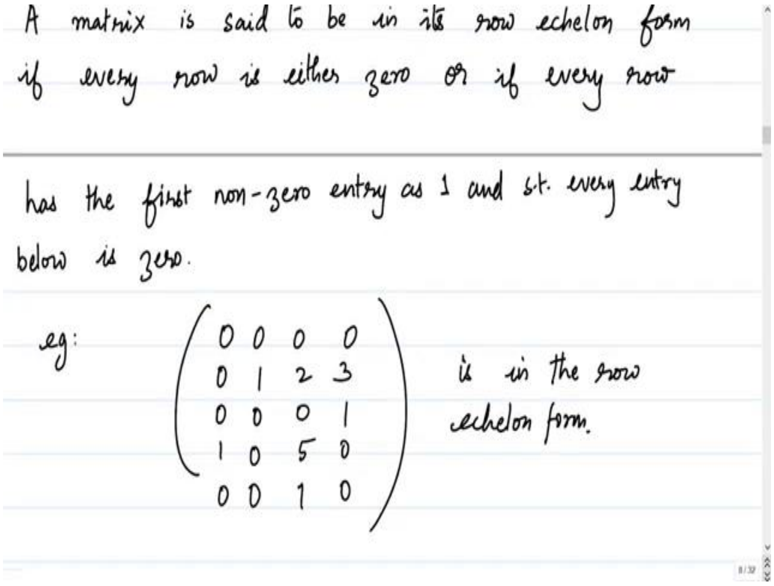


So we have discussed what elementary row operations are. So, why are we discussing all this and in fact, let me give you one more definition before we talk about why we are discussing this and that is of a row echelon form. So a matrix is said to be in its row echelon form, so let me write it down and then we will discuss after the definition is ascertained. A matrix is said to be in it is row echelon form if every row is either zero, all entries are either zero or if every row it starts with a 1 has the first non-zero entry as 1 and not just that, and such that every entry below is zero.

So if you have a column it is first non-zero entry is a 1 and every entry below that 1 will be a zero. Let me give you an example. So suppose this is a matrix we would like to consider 0, 0, 0, 0. Suppose the first row itself is all zero, second row is something like say 0, 1, 2, 3 but then this 1 should have all entries below as zero. So let us say it is, let us say this is a 5 cross 4 matrix, 1 will have all entries below it as zero. And suppose our next matrix is 0, 0, 1 there is nothing left but this 1 will have all zeros below. And our next entry is 1, 0, 5, 0 and this 1 will have zeros below 0 ,0 say 1, 0. This matrix is in the row echelon form.

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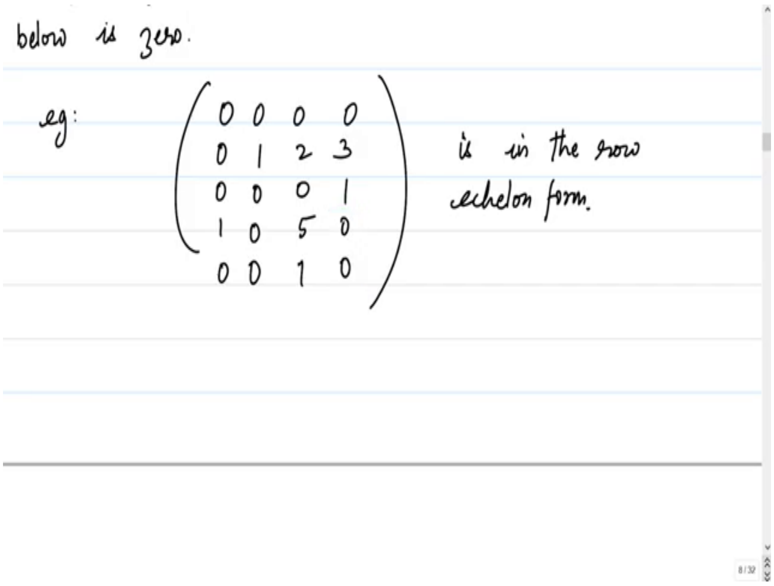


So again, let me tell you why this is in the row echelon form, either every row is zero, that is one of the conditions or the first non-zero entry is a 1 and all entries below, c is the first non-zero entry is 1 and all entries below are zero. Similarly, in the next column, this is the first non-zero entry, the last column has the first non-zero entry and all entries below are zero. That is true with every row, either it is all zero or the required condition is getting satisfied. So this is a 5 cross 3 matrix which is in the row echelon form.

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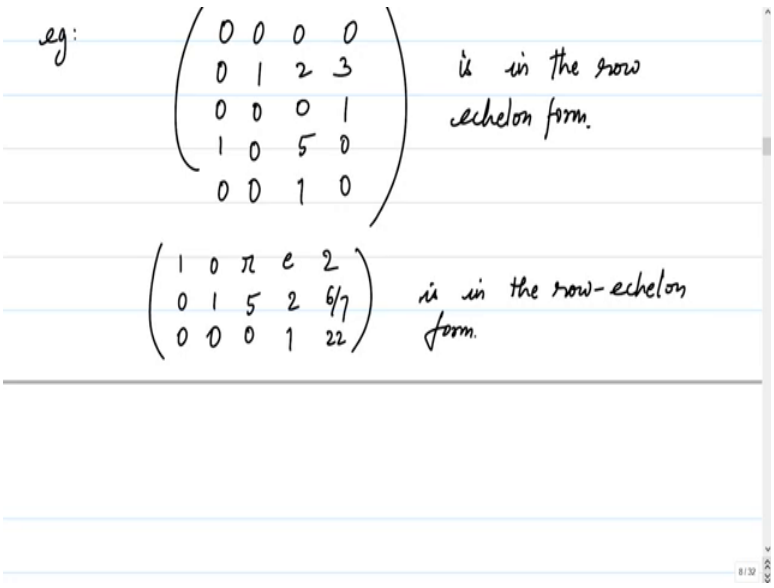


Maybe one more example might help. Let us look at say, 1, 0, Pi, e. Why should we just write integers 1, 0, Pi, e, and say 2. And then 1, below 1 it should be zero, so let us look at a 3 cross 5 matrix. And in the 2nd row we have 0, 1, 5, 2 and say 6 by 7. Let us make it complicated and this 1 should have zero below right, that is the row echelon form and suppose the last row is entirely zero. There is no necessity that everything should be a zero, so let us just do one thing. Let us put a 1 here and 22. This is a, this is in the row echelon form.

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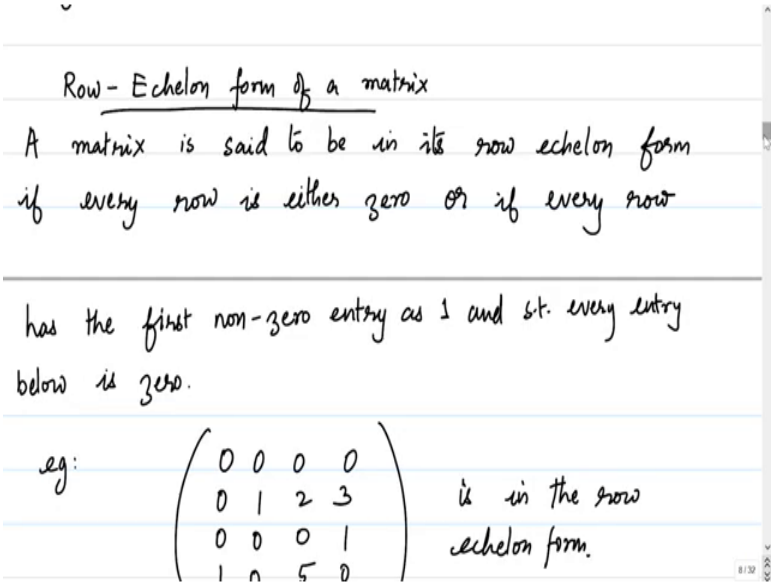


Okay, so I will not give more examples but I would like to point out that there is a similar notion of a column echelon form as well. I will not write it down, but let me just tell it in words what the column echelon form of a matrix is. A column echelon form matrix is that either every column is zero or the first non-zero entry in the column from the top is either 1 or it is entirely zero. So the first non-zero entry should be necessarily 1 and all entries to the right in the row should be zeros. And that is exactly the dual idea of what we have just written down right.

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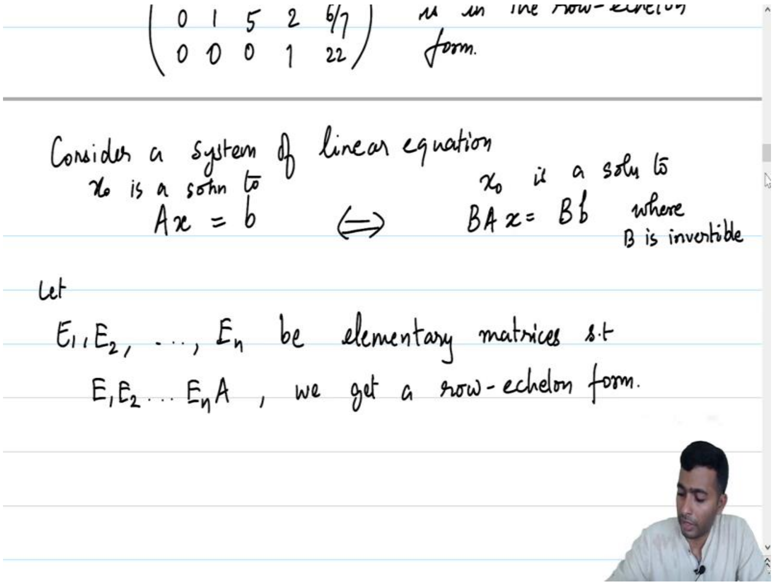


Okay so row echelon form is also, and I would like to point out that a matrix can be reduced to its row echelon form by repeatedly applying the row operations of say type 1, 2 and 3. Okay, so why are we really making so much of noise about row reductions or row operations and echelon forms? One good justification can be by considering a system of linear equations. So consider a system of linear equations, say m equations in N=n variables. So when written in the matrix format A will just turn out to be an m cross n matrix, m equations in n variables.

Note that any solution to Ax is equal to b is a solution or let me put it this way. A solution x naught to Ax is equal to b or rather x naught is a solution to Ax is equal to b if and only if, suppose, we multiply both sides by an invertible matrix BCx is equal B times, okay BAx is equal to b times, so small b, where b is invertible. Think about it if we just multiply by the inverse you get back. So, x naught is a solution to Ax is equal to b, if and only if x naught is a solution to BAx is equal to b times, where b is an invertible matrix.

And we already noted that each of our elementary row operations are multiplication from the left by invertible matrices, so suppose E1. So let E1, E2 up to say En be the elementary matrices, elementary matrices of various types. Let me not mention what types, if we do not know what it could it could be any elementary matrices such that E1, E2, E1 times E2 times up to En after all these row operations acting on A, we get a row echelon form.

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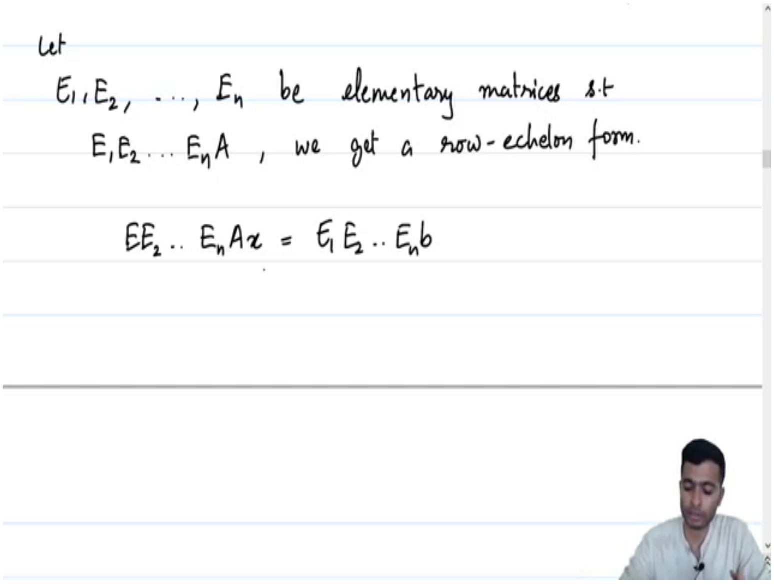


Then if you notice, E1, E2, En times Ax is equal to E1, E2, En times b is a system of linear equations. If we solve this and if we get hold of an x naught which solves this that will be a solution to Ax is equal to b. And if this is in row echelon form, there is a concrete way to either get solutions or decide that there could be many solutions or there are no solutions whatever the case is. So this is a effective way to tackle this problem.

We will not be, however addressing this problem much. Our motivation stems from the fact that row echelon form of a matrix which is obtained by various elementary operations is useful in studying the rank of the matrix. That will be the content of the next video where we will be discussing the rank of a linear transformation, which we already know and the rank of a matrix which will also link to each other, all right.