Linear Algebra
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Lecture 5.4
Quotient Spaces

So in this video we discuss the quotient space of vector space V with respect to a subspace U. We will then define a vector addition and scalar multiplication on this quotient space, make it into a vector space and then notice that the vector space V mod U very naturally interacts with the vector space V. So, let us begin by considering a vector space V.

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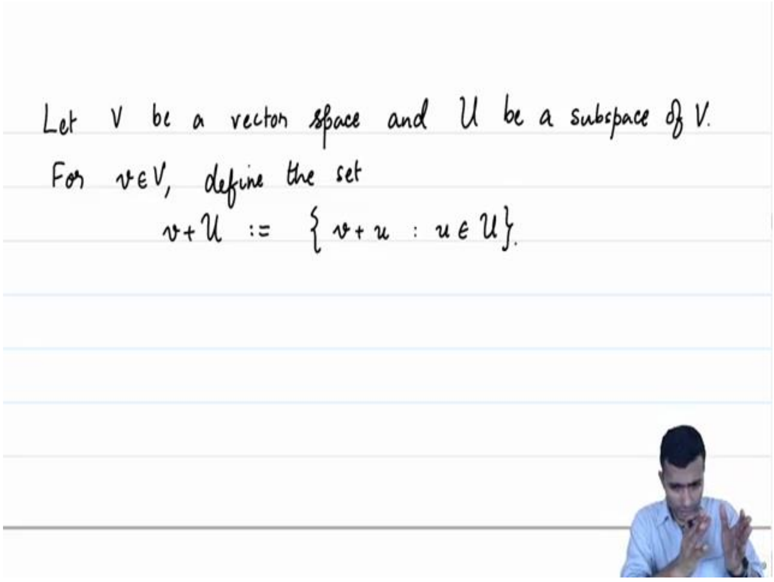

So let V be a vector space and U be a subspace of V. Consider the following set. Define the set, for an element small v in capital V, for small v in capital V, define the set small v plus capital U. Remember that, as of now small v plus capital U does not make any sense because capital U is a subspace, small v is just a vector in capital V. We are defining it to be a set, to be a set of in some sense translates of U.

This is the collection of all small v plus small U where small u belongs to capital U. In some sense, we have translated our capital U by V. So, let us look at an example to understand what we have just done. Let us look at R2.

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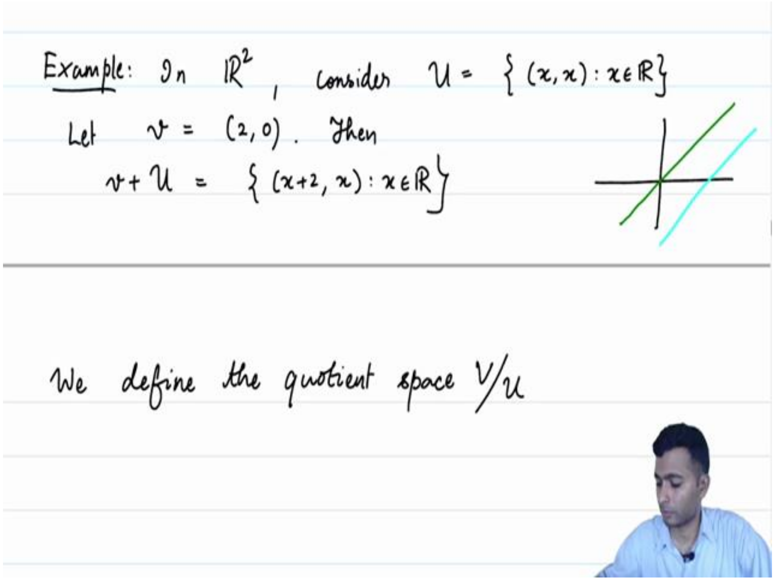

In R2, consider U to be this subspace which is x, x, where x belongs to R. So let me just draw it for you in the Cartesian coordinates which you will be familiar from high school.

So this, there is a Cartesian coordinate like this and the green denotes the subspace U. The line x is equal to y. So, let us pick some v. Let v be equal to say 2, 0 then, v plus U is nothing but x plus 2, x such that x belongs to R. Let me use some other colour, let me use blue to tell you that that is going to be this line. It did not come out well, but you can imagine that there are parallel lines.

The green and blue are parallel lines, the blue is actually v plus u and the green is basically u. So that is why it makes sense to think of v plus u as in, as a translate of u in some sense.

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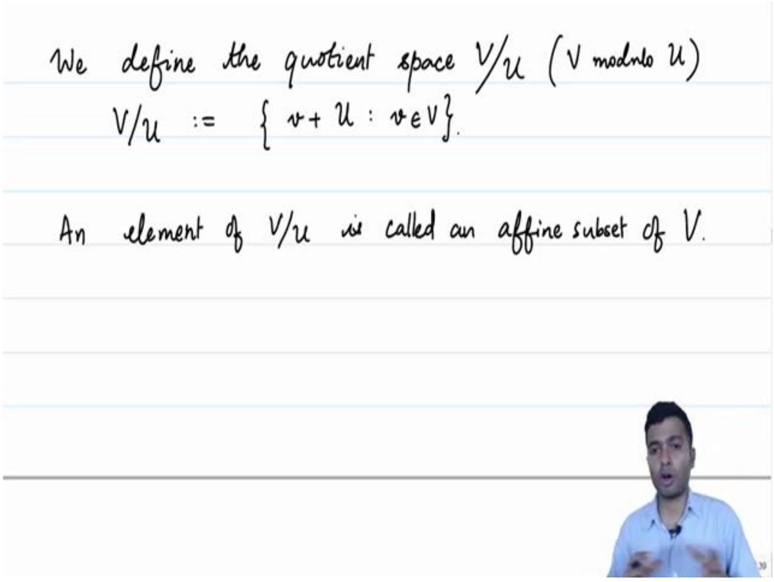

So, we define the quotient of v by u to be the collection of all such sets of the type (())(4:16). So, we define the quotient space v modulo u, it is many times called, V mod u in short. V modulo u, that is what the terminology is pronounced as or called. We define the quotient space V mod u to be it is defined as the set of all v plus capital U, such that v is varying in capital V. So, let me give it a name, an element of V mod u is called an affine subset, affine subset of V corresponding to u and V mod u is the collection of all such affine subsets.

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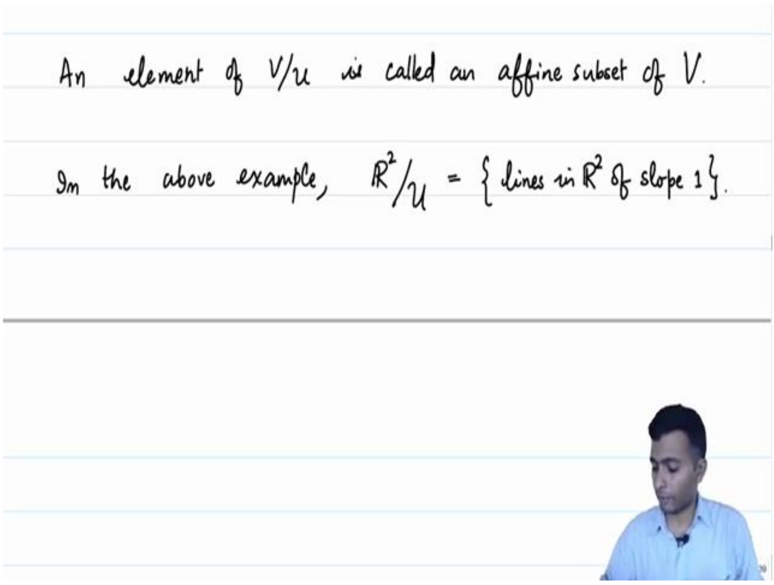

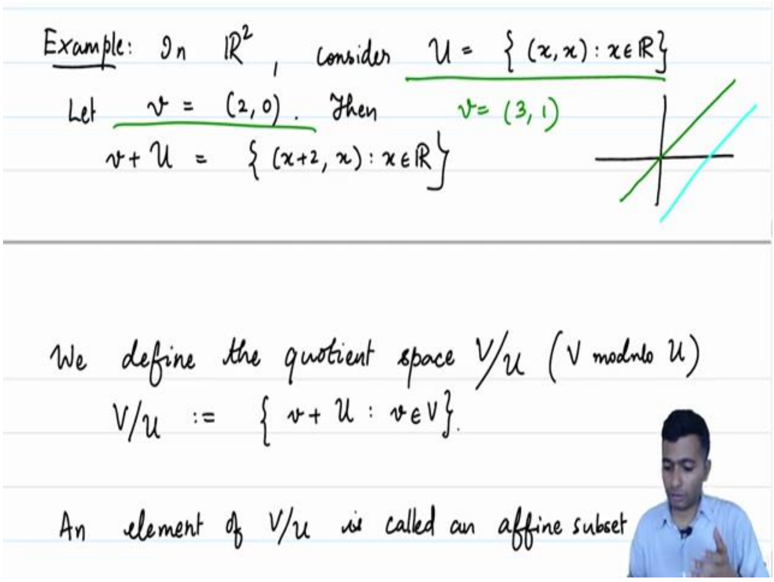

So, in the example we just described, what would be V mod u? So in the above example, R2 and u is x, x, the above example R2 mod u will just be all the translates. So it is going to be the collection of all lines in R2 of slope 1. You carefully think about it and the Cartesian coordinates, it will capture all those lines which has slope 1. So we have described what the quotient space is.

As of now remember that this is just a set. There is no vector space structure which we have defined on R2 mod u or rather V mod u in general. That will be our next goal, but before we get into the definition of vector space and vector addition and scalar multiplication, let us make a few observations. So observe that when we talked about the affine set in this case where v was our 2, 0 and u was this, what we obtained is an entire set.

If instead of 2, 0 if we had started off with, say V is equal to 3, 1 my claim is that we would have ended up with the same affine subset. It is for you to go back and check that there is nothing unique about the vector with which we have translated it. It can happen that two different vectors give us the same affine subset. Our next question is to answer when can that happen? So let us answer that question through a proposition.

The next proposition answers the question of when does two different affine subsets give us the same element? Or when are two different affine subsets the same?

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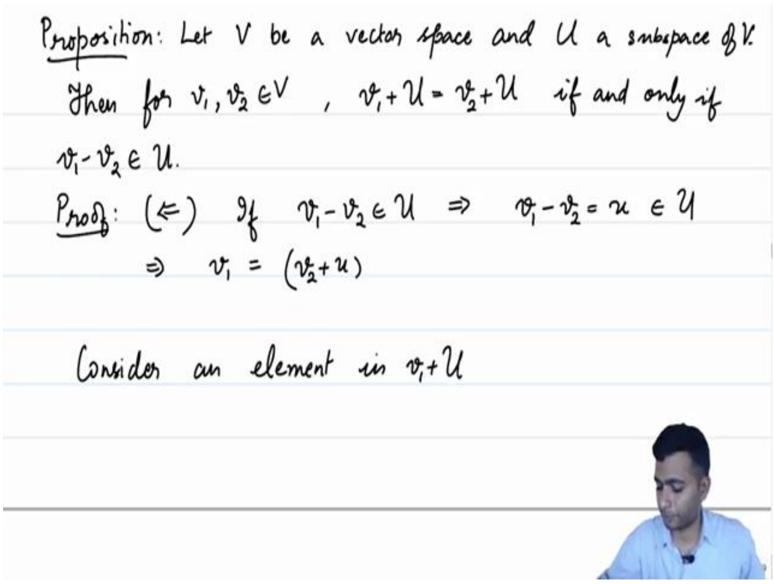

Proposition. Again the setup is as above. So let me just write it down. Let v be a vector space and u be a subspace of v. Then, for V1, V2 in capital V, let us consider v affine subsets of v corresponding to u obtained with v1 and v2. And let us ask when are they the same? V1 plus u is equal to V2 plus u if and only if it is a necessary and sufficient condition as you can see, V1 minus V2 belongs to capital V, if V1 minus V2 belongs to capital U, sorry.

If V1 minus V2 belongs to capital U, then the corresponding affine subset are the same. Let us have a look at the proof, one side is easy, so let us both sides are easy let us solve the easier side maybe this is not easier said. Anyway, if V1 minus V2 is in capital U, let us assume that V1 minus V2 is in capital U. What does this imply? This implies that V1 minus V2 is equal to some small u which is in capital U. After all it is in u means that it is some vector u in capital U. This implies that V1 is equal to V2 plus u.

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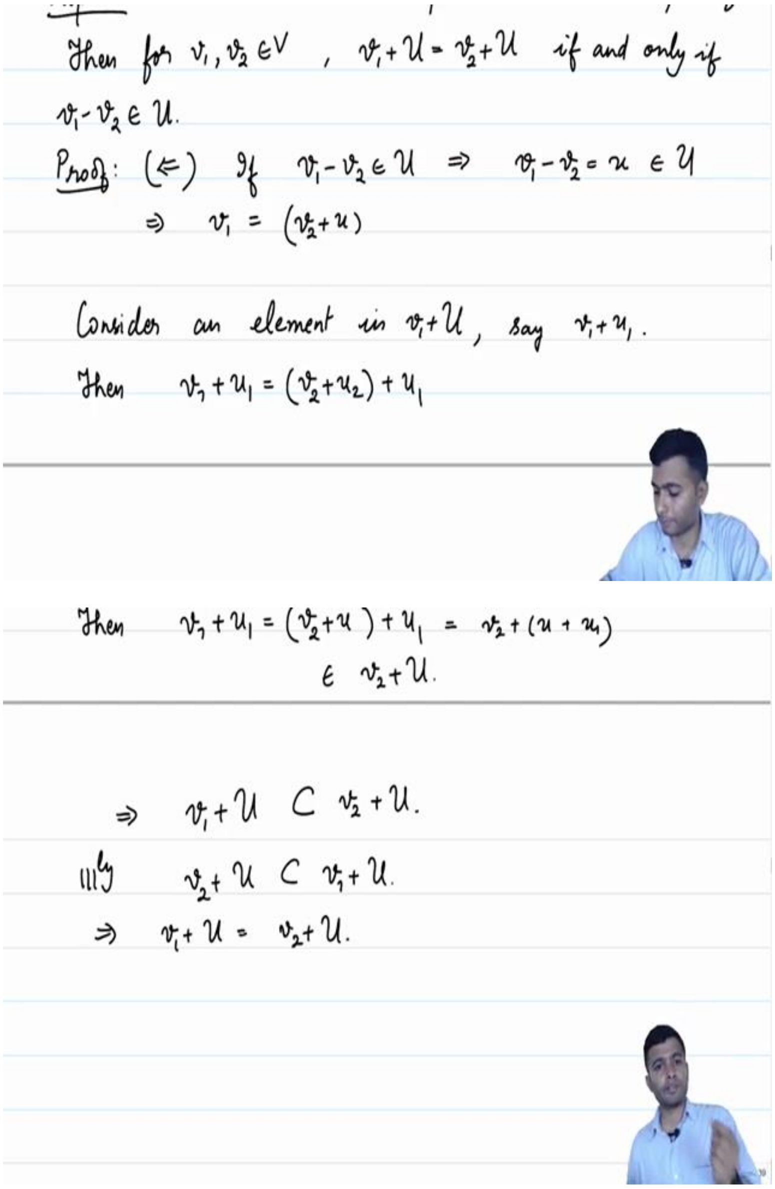

Now, consider an element in V1 plus capital U it will be something like V1 plus U1 only, say V1 plus U1. Then, V1 plus U1 as you can note, V1 is v2 plus u. So V1 plus U1 is equal to V2 plus u plus u1 but what is this? This is equal to V2 plus u2 plus, there is no u2. It was v2 plus u.

So that is here (())(11:27) again. It is u, u plus u1 but u plus u1 is an element of capital U. This hence belongs to V2 plus capital U. So essentially what we have shown is that, V1 plus capital U is contained in V2 plus capital U but this is a symmetric argument instead of starting with V2 and V1, we could have done the other way, and similarly, it can be shown that V2 plus u is contained in V1 plus U.

That implies, V1 plus U is equal to v2 plus U. So if v1 minus v2 is an element of the subspace U, then the corresponding affine subsets, it is also called as cosets, the corresponding affine subsets are the same.

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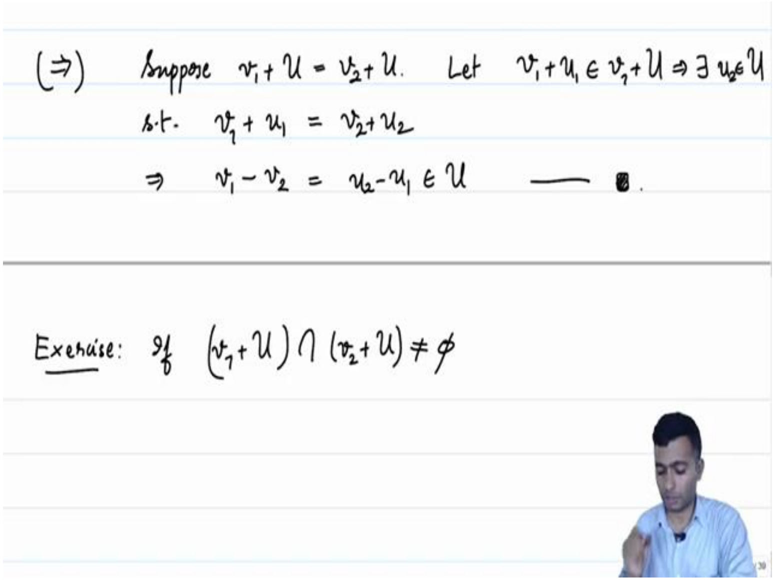

Let us now consider the forward direction. Suppose, the assumption is that V1 plus u is the same as V2 plus u. Suppose V1 plus u is equal to V2 plus u. This implies that V1 plus any element, say U1 is equal.

So, let V1 plus u1 plus be in V1 plus capital U, take some arbitrary element. That is equal to V2 plus u. The affine set V1 plus U is equal to V2 plus u implies that there is some U2, this implies there exists u2 in capital U such that, V1 plus U1 is equal to V2 plus u2. We are almost done because this implies V1 minus V2 is equal to u2 minus u1 but both u2 and u1 belong to capital U and hence, u2 minus u1 belongs to capital U and that is precisely what we had said.

But so we have essentially proved that two affine subsets v1 plus capital U and v2 plus capital U are the same if and only if V1 minus V2 is an element of capital U. At this juncture let me give u an exercise which tells you something even stronger. It says that, this exercise tells us that if two affine subsets intersect even in one point, suppose the intersection is non empty, then they should necessarily be the same.

So, if V1 plus u intersected with V2 plus u is not empty, suppose, there is even one element in the intersection, then V1 plus u is equal to V2 plus u. So the ideas are quite similar to the ideas used in the proposition we just proved. This can be established in a similar manner. So we have done this paid work to now define a vector addition and a scalar multiplication on v mod u.

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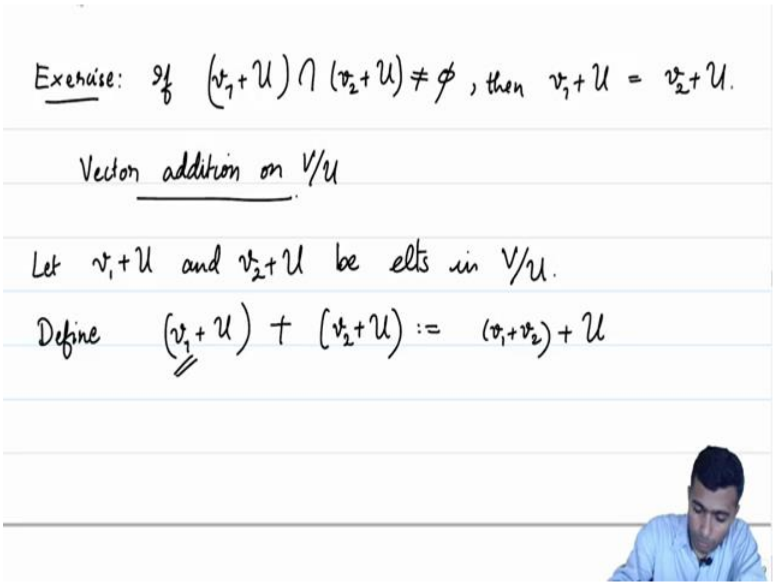

So, let us now define a vector addition on v mod u. So let V1 plus capital U and V2 plus capital U be elements in v mod u. Let us define V1 plus u plus, this is the plus we are defining. And V2 plus u, remember whatever is there inside the brackets, they are elements in v mod u. So V1 plus capital U as a symbol is something which we have already defined, an affine subset.

Similarly, V2 plus capital U in the bracket is a symbol which actually defines an affine subset corresponding to u and V2. We are now defining the sum, this sum of these two vectors. We will define the sum of V1 plus u and V2 plus u to be V1 plus V2 which is a vector capital V and the affine subset corresponding to u and V1 plus U2. So this is our definition of the vector addition.

Now, if you think about it carefully, this definition has a serious problem because we already noted that, remember in R2 when we considered the subspace U, which was x is equal to y or set of all x, x we already noted that for v equal to 2, 0 and v equal to 3, 1 both the affine subsets were the same. So when we are talking about the addition of two such elements, what is the choice of the vector involved? So let me repeat what I just said. The vector V1 that we have here is not necessarily picked in a unique manner.

We already saw that if V1 minus V, if two vectors difference is in u, then the corresponding affine subsets are the same. So there could be many, many, many V1s which will give you the same affine subsets. Similarly, there could be many, many, many V2s which could give the same affine subset here. So there is a very serious problem with respect to this addition in terms of the choice of v1 and v2 involved.

If we V1 and V2 changes, what if the right hand side here depends on that choice. So we have just now proved a proposition which essentially tells us that this choice does not matter.

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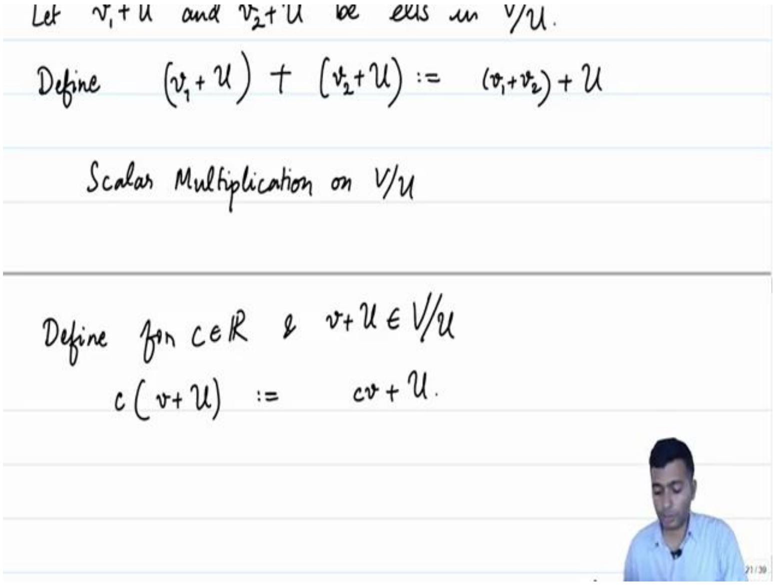

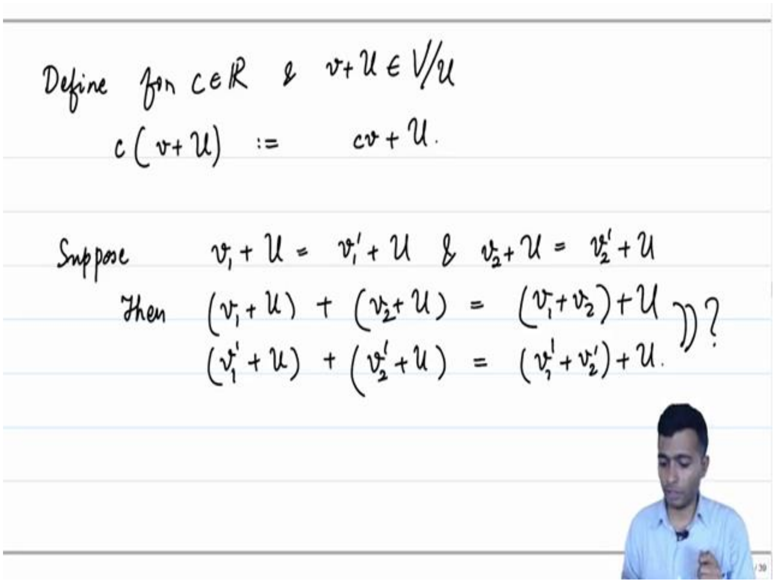

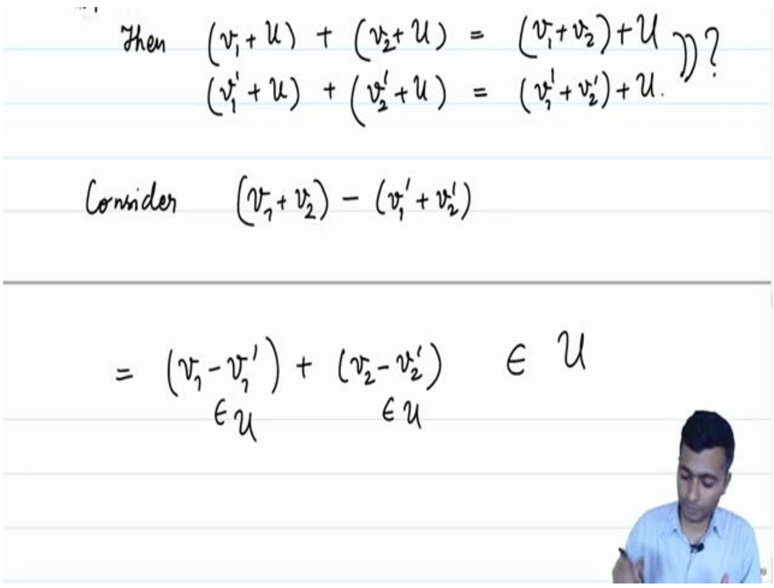

Before we get into that, let me also define what the scalar multiplication on v mod u is. Scalar multiplication on v mod u. Observe that V1 plus u and V2 plus u by the above definition if at all the definition makes sense, if you add it, it is giving us back an implement in v mod u and therefore, addition so v mod u is closed under that addition, if at all it makes sense. But we will come to that in a minute.

Before that, let us define or c in R and v plus capital U in capital V mod u, c times small v plus capital U, this is defined to be CV plus U. Again the same problem arises, the choice of V is a problem here. So let us first get that out of our concern. Suppose, V1 plus u is equal to V1 prime plus u. And V2 plus u is equal to V2 prime plus u. So in other words, V1 to V1 prime give us the same affine subset, and V2 and V2 prime give the same affine subset.

Then we could define addition in two different ways. Then V1 plus u plus V2 plus u, this is equal to v1 plus v2 plus u, but this is also the same as V1 prime plus u plus V2 prime plus u, because both are after all the same affine subsets and this is equal to V1 prime plus V2 prime plus u. The question is, are these two objects equal? If they are equal, then we have shown that the vector addition we have defined does not depend on the choice of V1 and V2. But what does it mean to say that these two are equal?

We have shown a proposition earlier, that two vectors give us the same affine subset if its difference belongs to u. So consider V1 plus V2 minus v1 prime plus V2 prime, this is equal to V1 minus V1 prime plus V2 minus V2 prime, but what was our assumption to begin with? Our assumption was that V1 plus u and V1 prime plus u are the same affine subsets, hence this belongs to u. Similarly, V2 plus V2 prime also belongs to u, which implies that this is sum of two vectors in u which is having the neck in U.

Again by using the preposition if the difference of two vectors belongs to u, then the affine subset corresponding to the two vectors will be the same.

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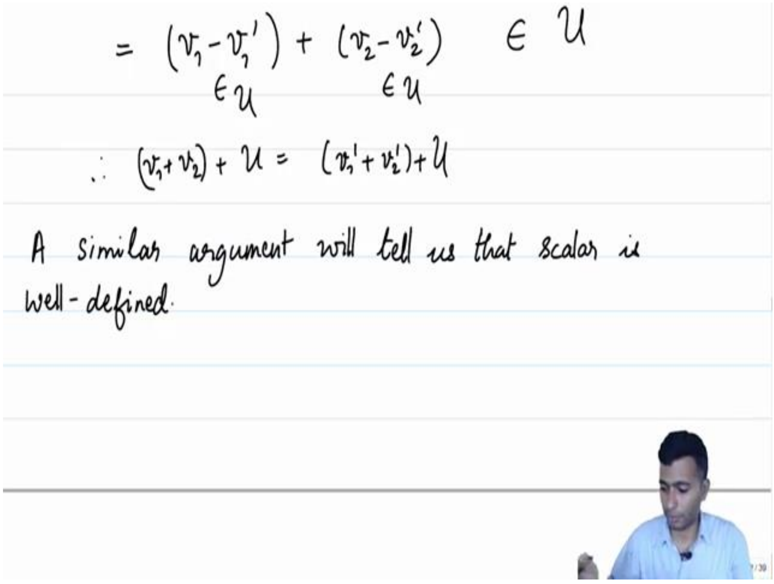

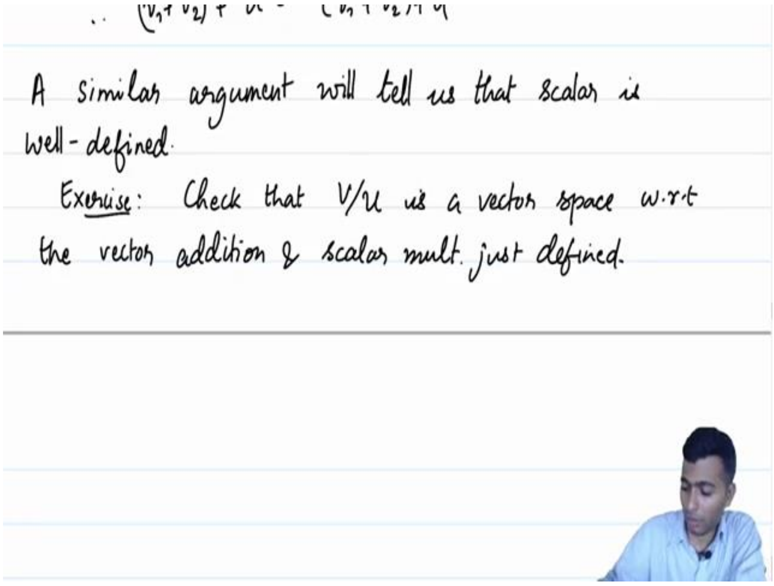

Therefore V1 plus V2 plus u is equal to V1 prime plus V2 prime plus U. Therefore, our vector addition is well defined, it makes sense to define it like this. Well, it is a vector addition and a similar argument, let me just write it and leave it as an exercise for you. A similar argument will tell us that scalar multiplication is well defined that it does not depend on the choice of the vector small v with respect to which they affine subset is being described.

I would like to leave again as an exercise for you to check that v mod u is a vector space corresponding to the vector addition and the scalar multiplication which we have just defined. Again, I should stress here that it is a check which you should certainly do. It might be a routine check, but it is important to do all these routine checks. So, exercise is to prove that check that v mod u is a vector space with respect to the vector addition and the scalar multiplication just defined.

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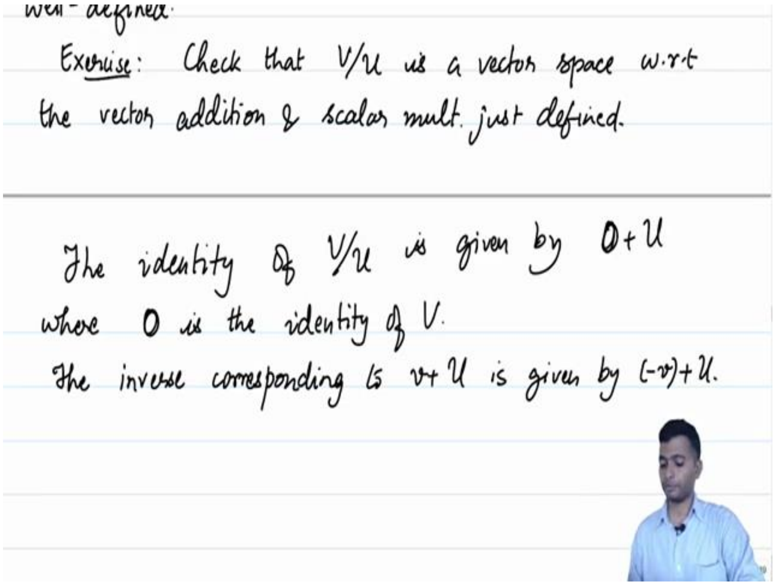

Again, like in the case of product spaces, I would like to draw your attention to what would be the identity here and what would be the inverse. So the identity as is to be expected will be corresponding to the 0 vector, the identity of v mod u is given by 0 plus u, where 0 is the identity of V. Well, that is quite straightforward.

If you look at small v plus capital U and add it to 0 plus u, by the very definition it is going to be small v plus 0 plus u, which is v plus u. Therefore, this is the additive identity. How about the inverse? The inverse, again you should check this out, inverse corresponding to v plus u is given by minus of v plus u. v plus minus v will give you 0 and hence 0 plus u is the additive identity of v mod u. So yes, this makes sense.

So why are we considering v mod u? Why are we giving it vector space structure? Of course, the vector space structure which we defined, directly is obtained by using the vector space structure of v, the vector addition and the scalar multiplication as you can see, which was defined was using the vector space addition and scalar multiplication operation, but why are we doing all this?

The reason is that in the study of linear transformations, quotient spaces come naturally. Before that, let me define a map for you.

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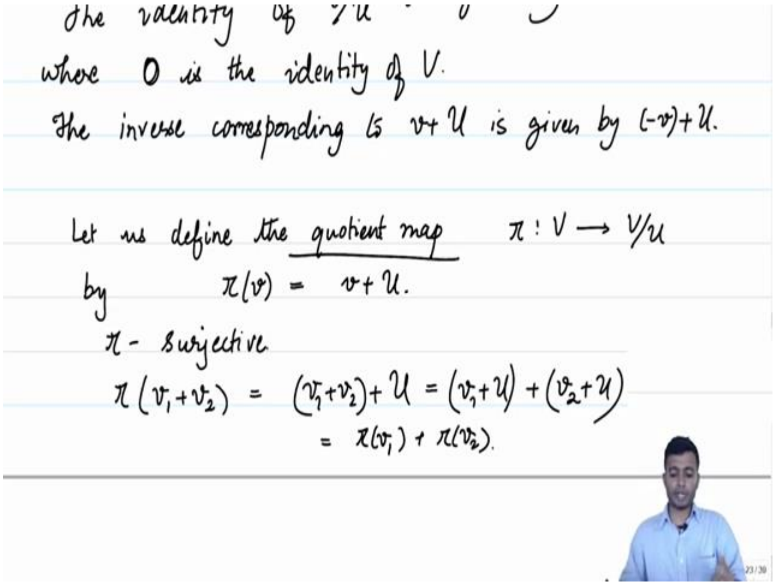

Let us define the quotient map. In some sense it is going from v to the quotient of v by u. Well, it is not that way but maybe the name was generated that way. So let us define the quotient map by from v to v mod u by pi of v is equal to v plus capital U. Again, there is a question of whether, no it is not again, there is a question of whether pi is a linear map, pi is certainly a map onto the v mod u if you carefully observe, pi is a subjective map and it is an easy check.

Maybe I will just check it for you. Pi of V1 plus V2 is equal to V1 plus V2 plus capital U which by our definition is V1 plus u plus V2 plus u which is equal to pi of V1 plus pi of V2. Similarly, you can check that by pi of scalar times vector small v is c times, is scalar times the vector pi of v. So pi is actually a linear map.

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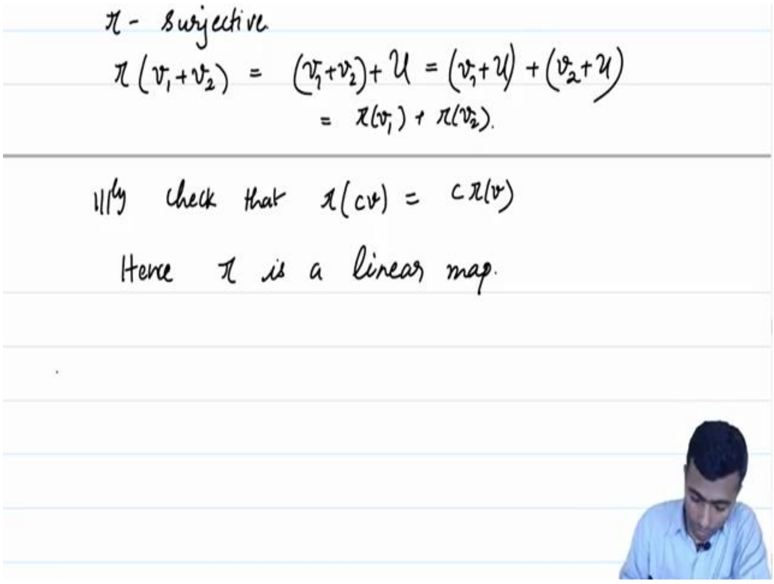

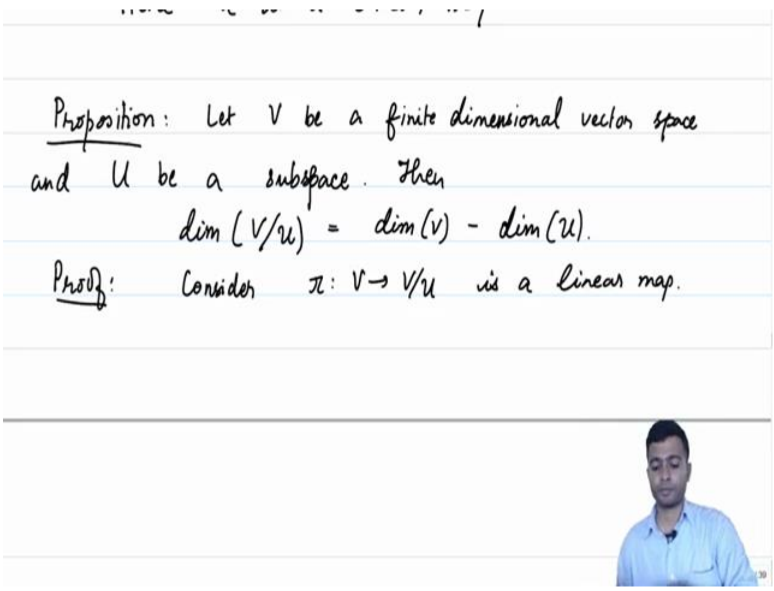

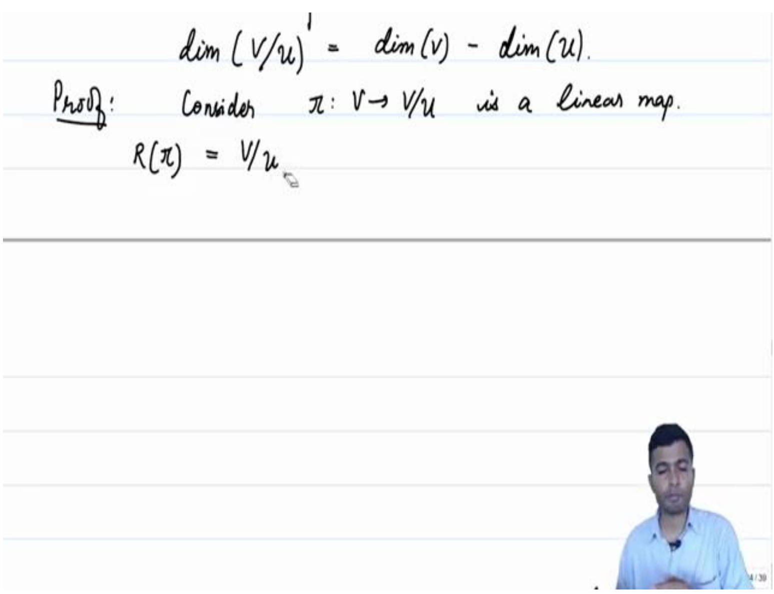

Similarly, check that pi of CV is equal to c times by pi. Hence, pi is a linear map. So the first question is, what are the implications of the definitions that we have given on v mod u in terms of v? So just like in the case of say product vector spaces we had that dimension of V1 plus V2 up to Vn was dimension of v1 plus dimension of V2 up to dimension of Vn. Can we say something similar here? The answer is, yes.

So that is captured in this proposition. Let V be a finite dimensional vector space. So as you can observe, till now, whatever we did, did not demand that V be a finite dimensional vector space. We need to demand a finite dimensionality here in order to talk about dimension which is interesting (())(28:46) finite. So let V be a finite dimensional vector space and U be a subspace, then dimension of v mod u is equal to dimension of v minus dimension of u.

Let us give a quick proof of this proposition and as one would expect, there is the dimension theorem involved in the proof of this. So let us see, dimension theorem applied to what. We just noted that the map pi, quotient map pi from v to v mod u is a linear map. We just proved it. We proved that it is a linear map from v on to v mod u. We did not check that it is subjective but checked that R of pi is equal to v mod u where R of pi is the range of pi.

So subjective map, so the range will be the entire space. So R of pi is v mod u. What does our dimension theorem tell us?

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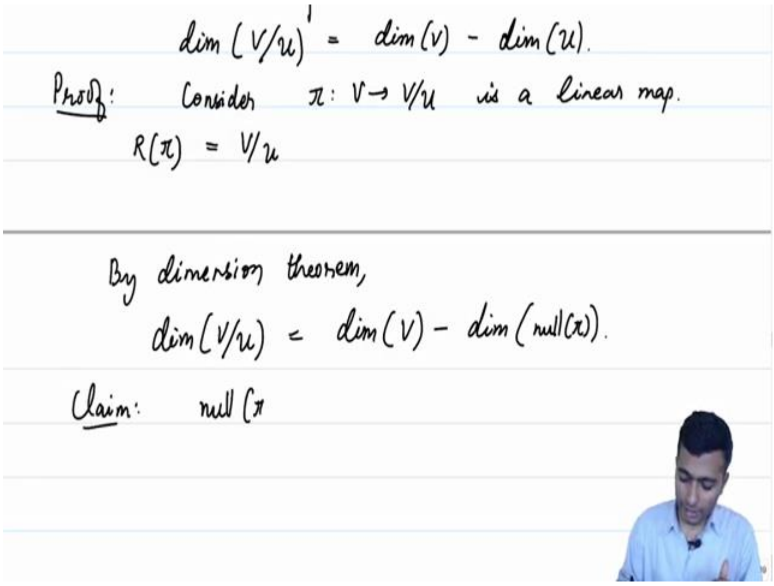

By dimension theorem, dimension of v mod u is equal to dimension of V minus dimension of v null space of pi. So we are almost done. We have that dimension of v mod u is equal to dimension of v minus the null space of, dimension of the null space of pi. And our goal is to show that dimension of v mod u is dimension of v minus dimension of u. So one should expect that null space of pi and u have something to do with each other.

In fact, it does claim null space of pi is exactly equal to u. So let us just have a look at the proof of this.

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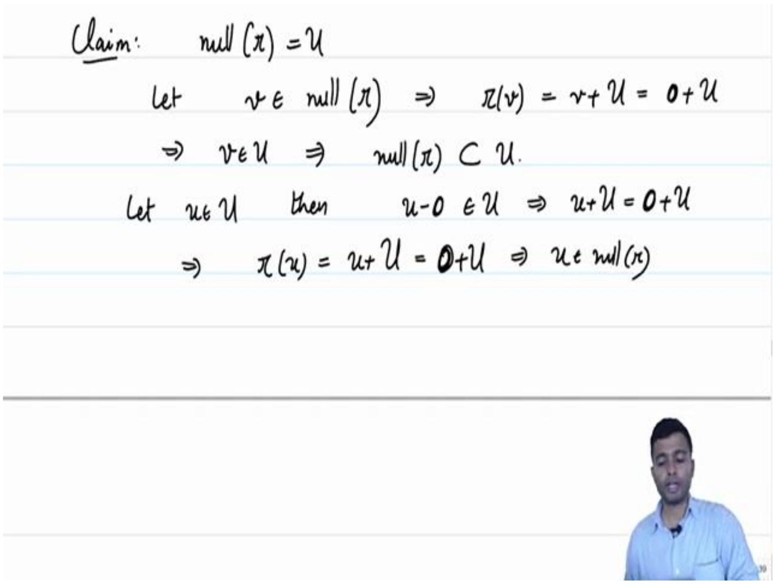

Consider, v be an element of null space of pi. What does that mean? This implies that pi of V which is equal to v plus u belongs to or is equal to the 0 vector. This is equal to the 0 vector plus u, but what does that mean? This implies that v belong to u which implies null space because v minus 0 belongs to u to be more precise and v minus 0 is just v. This implies that null space of pi is contained in u.

Now if let u be an element in capital U, then u minus 0 which is equal to u belongs to u, which implies u plus capital U is equal to 0 plus capital U, which implies u belongs to, which is equal to u plus capital U is equal to 0 plus capital U which implies that u belongs to null space. And hence, we are done. So we have essentially proved that the dimension of v mod u is equal to dimension of V minus V dimension of the null space vector which is u. Now, let us see how this setup can be used to study linear transformations more effectively.

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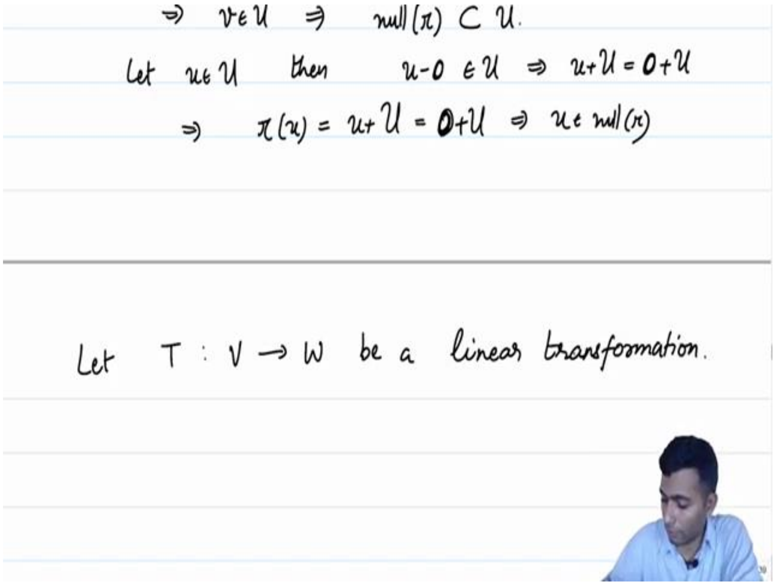

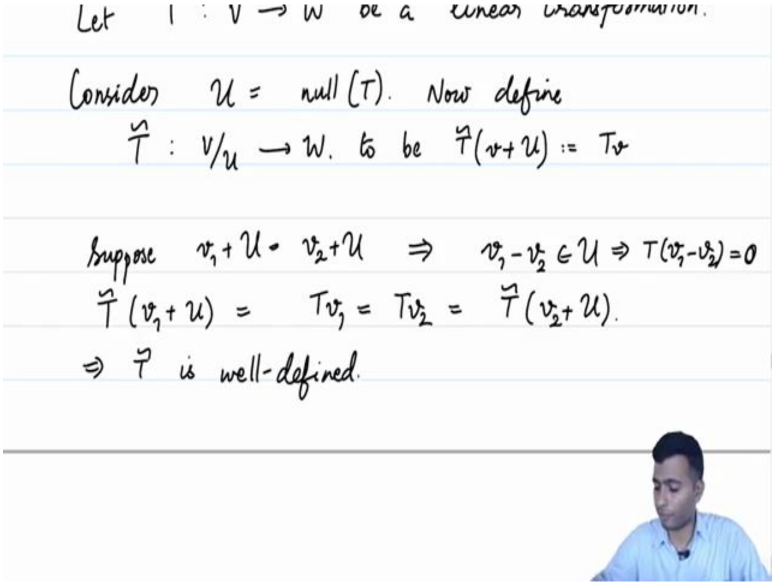

So let us start with a linear transformation. So let T from say V to W be a linear transformation and there is vector subspace of V which is very naturally associated to T which is basically the null space of T. So consider u to be equal to the null space of T, then define, now define T tilde to be a map from V mod u to w. First thing to check is that, leave it as an exercise for you to check that, oh what is, I did not define T tilde. T tilde of say small v plus capital U, the guess should be correct which is T.

Just send the corresponding affine subset to TV. Yes, that is a question of before we check anything more, let us check that this definition makes sense again. Like in the case when we were defining a traditional scalar multiplication, we were worried about whether the choice made any difference, whether it would make sense to define it like this at all. Here also the same question is relevant and the answer is, it does not matter.

So suppose, let us check that out. Suppose V1 plus u is equal to V2 plus u, then T of V1 plus U plus u should be the same as T of V2 plus u because they are the same vectors in v mod u. So T tilde rather, so T tilde of V1 plus u should be equal to T tilde of v2 plus u. What is T tilde of V1 plus u? That is equal to T of V1, but what is the meaning of V1 plus u being equal to V2 plus u? By proposition proved earlier this is if and only if v1 minus v2 belongs to u.

Recall what u is, U is the null space of T, which implies T, put a plus, this would be minus. T or V1 minus V2 is the 0 vector. It is in the space of u, null space of T. And that means TV1 is equal to TV2. This TV1 is equal to TV2. That precisely is T tilde of V2 plus u. Therefore, to define T tilde in this manner based on a choice does not affect the definition. It is still well defined.

This implies that T tilde is well defined. It makes sense to define it like this.

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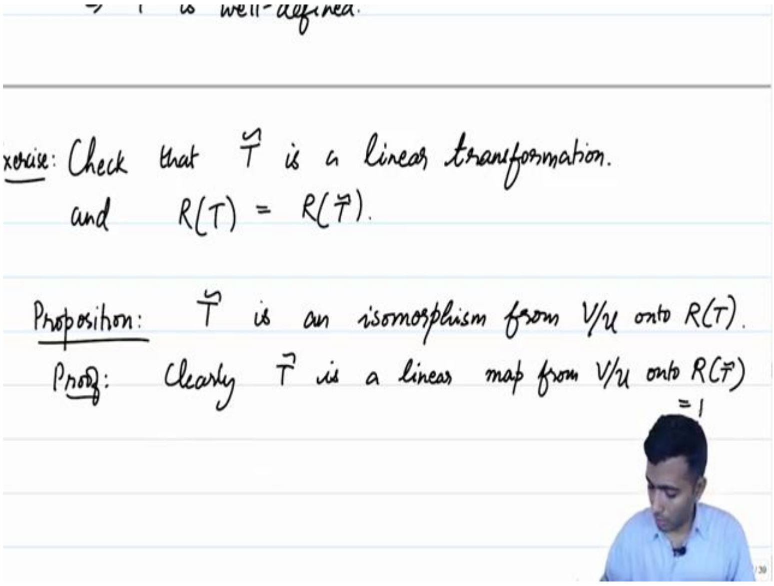

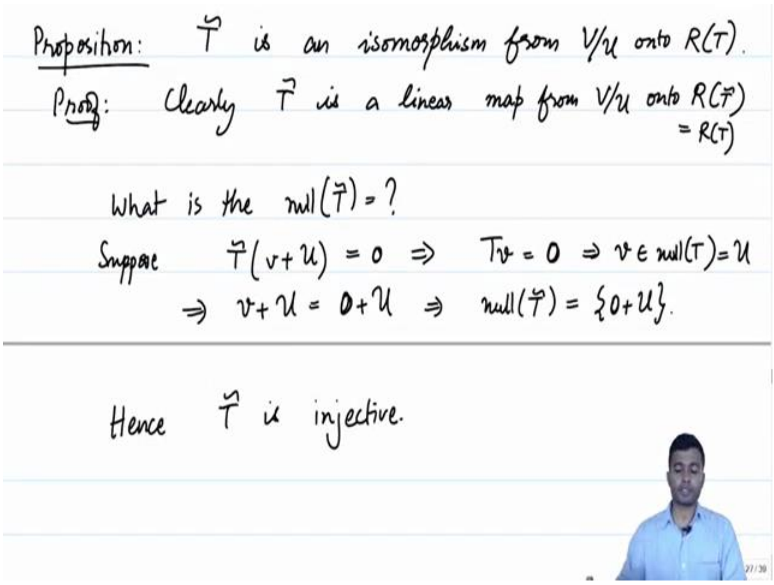

Now, it is a check for you to see that T tilde is a linear transformation. Not just that T tilde is a linear transformation, and so let me put it as an exercise for you. Also check that the range of T is equal to the range of T tilde. So why are we going to such lengths to discuss T tilde? The reason is the following proposition. Proposition. T tilde is an isomorphism from v mod u on to R of T. So T tilde is a linear map which is clearly let me give a proof, clearly T tilde is a linear map by our exercise.

So clearly is all based on whether you have solved the exercise or not. Assuming that you have done the exercise by now, T tilde is a linear map from v mod u on to R of T tilde. And again the above exercise tells us that R of tilde is the same as R of T. So this gives v mod which is equal to R of T.

To check that this is an isomorphism on to R of T, it is enough to show that this linear map is both injective and subjective onto R of T. You have already seen that it is subjective onto R of T. It is enough to now check that T tilde is injective. So claim is that T tilde is injective. To show that T tilde is injective, let me invoke a result we have proved earlier to show that the null space of T tilde is, it has the 0 space. So what is the null space of T tilde?

So suppose, T tilde of v plus u is equal to 0. This implies that TV by the very definition, T tilde of v plus u is TV. TV is equal to 0. Remember this 0 is in W. But that implies that v belongs to the null space of T which is equal to u, which implies v plus u is equal to 0 plus u which is the 0 vector of v mod u, which implies space of T tilde is the singleton set 0 plus u, which is 0 element of v mod u and hence T tilde is injective.

So we are in good shape now. If we start off with a linear transformation from a finite dimensional, yes, maybe we do not need to even talk about, let me be a bit more careful. Finite dimensionality is needed only in the proof of this theorem where we are using dimension theorem. In the proof of this proposition we do not need any finite dimensionality.

If you carefully observe the entire proof goes through for any vector space v and any linear transformation T from v to w. So we have proved that if T is a linear map from v to w, there is the subspace, null space of T which is called u and then v mod u is isomorphic to the range of T. So if T is a subjective map from v to w, then v mod null space of T is isomorphic to w. That is what we have proved. So next, we will be discussing dual spaces in the next video.