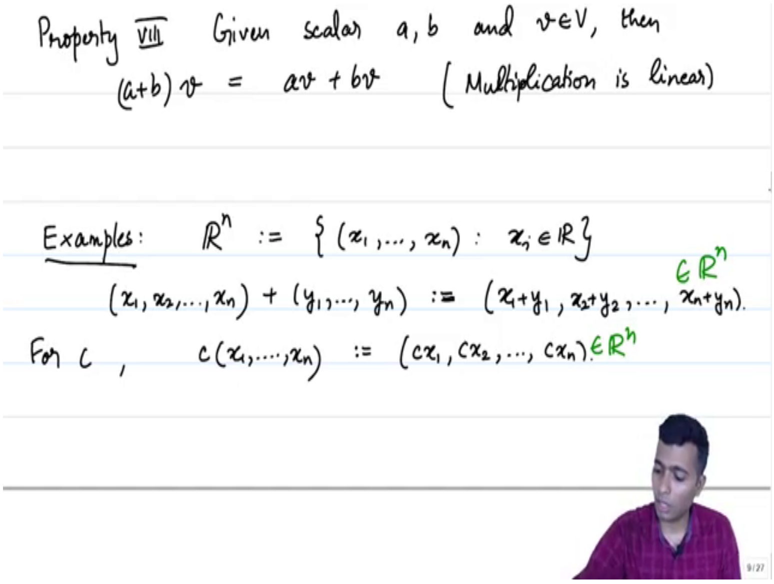
Linear Algebra

Professor Pranav Haridas
Kerala School of Mathematics, Kozhikode

Lecture 1.2
Examples of Vector Spaces

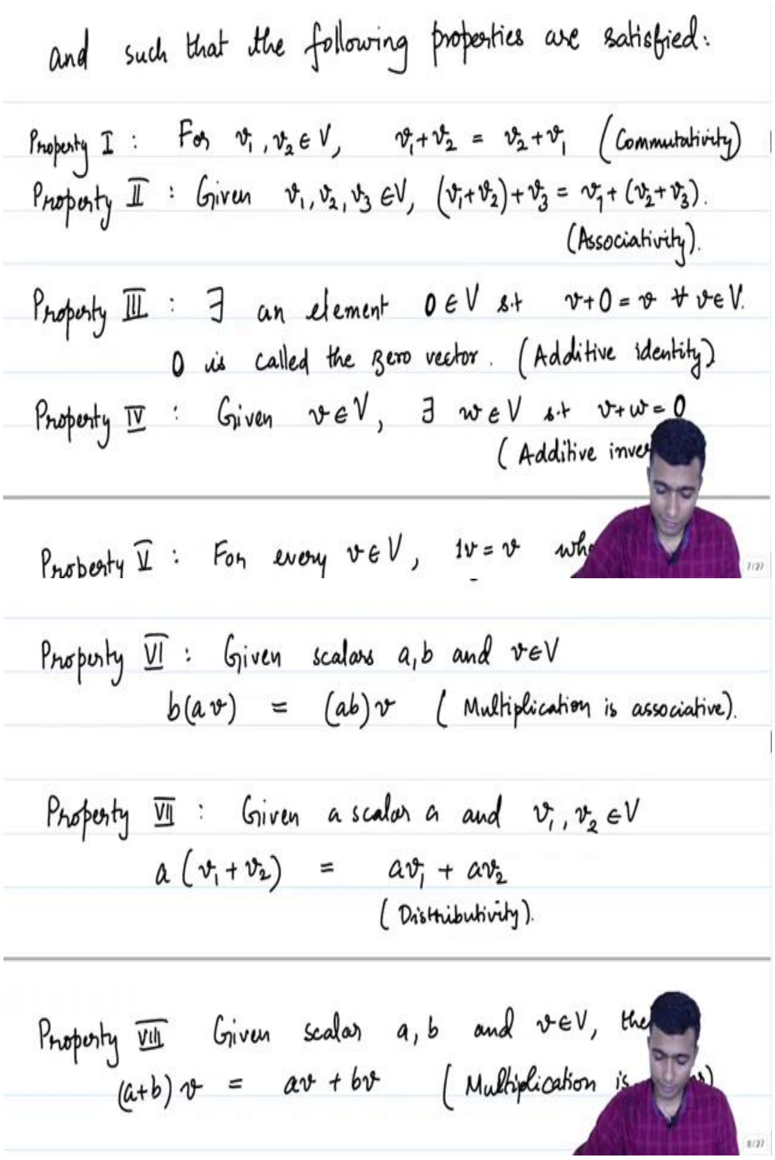
Let us now give many examples to get a grasp of this concept.

(Refer Slide Time 0:29)



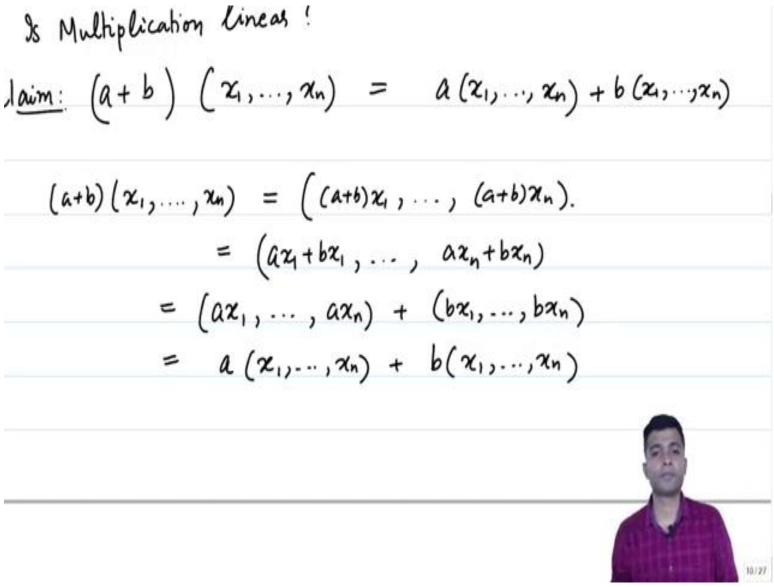
Examples: The first example is…. we have already seen a couple of examples. In fact, most of these properties are obtained by noting what are the properties in say R^2 or R^3 which are special and which we would like to generalize. Right? So, let me now just give you the more general Rn. Rn is just the Cartesian product of R with itself n times. It is x_1, x_2 up to x_n where each of the xi are real numbers. So, this is an ordered tuple. Again (2, 3, 4) is not the same as (4, 3, 2). The order matters. And what is the addition and the scalar multiplication? As is to be expected, suppose x_1, x_2 up to x_n is an element in Rn and y_1, y_2 up to say yn is an element in Rn and we define. So, when I put a colon followed by an equal, it means that we are defining something. Yeah, this case we are defining the vector addition. What is this? This is x_1+y_1, x_2+y_2 so on up to x_n+yn. Right? So for a scalar or Real numbers c, c times (x_1,….,x_n), this is being defined -- again I will put a colon followed by an equal to -- this is (c x_1, c x_2,…, c x_n). So that the vector addition of two elements in Rn is giving back an element in Rn is quite straightforward. It is clear that this is an element in Rn. Right? This is also an element in Rn, so Rn is certainly closed under these two operations.

(Refer Slide Time 2:44)



We are however left with the properties I to VIII that are to be checked to say that Rn is indeed a vector space. Let me not go over all the properties. Let me just focus on maybe one or two properties… Maybe let us check property 8.

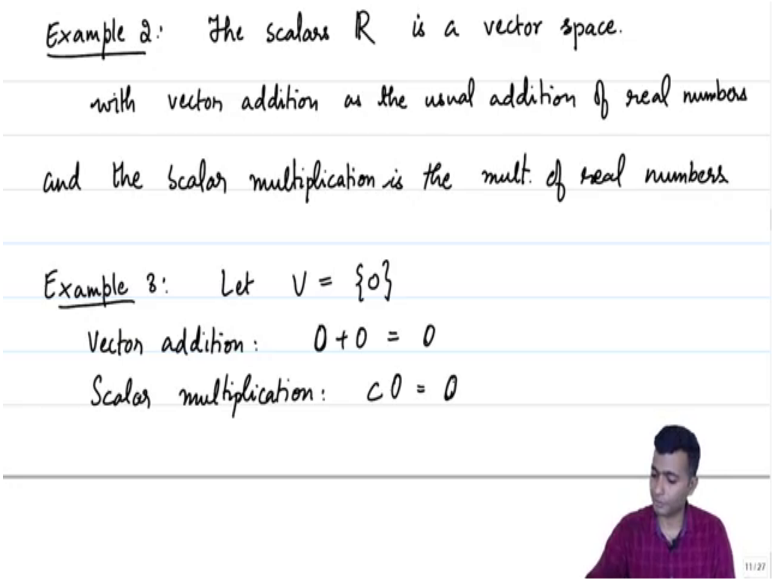
(Refer Slide Time 3:14)



Is the multiplication linear? What do we have to check? We have to check that a+b times x_1 up to x_n is this equal to….. This is what we have to check, (that) this is equal to a(x_1,...,x_n)+b(x_1,...,x_n). Right? This is exactly what we would like to check. But what is (a+b) times…. Let us start with the left hand side. What is (a+b) (x_1,...,x_n)? This is component wise multiplication. Right? Scalar multiplication by definition is ((a+b) x_1,...,(a+b) x_n). But each of the components, if you observe a, b and x_1, all real numbers and we are just looking at the distributivity property of the Real numbers. Right? So, this is equal to (a x_1+b x_1,..., a x_n+b x_n). But if you observe carefully by the definition of vector addition, this is exactly (a x_1,...,a x_n)+(b x_1,...,b x_n). And again, now we will use the definition of what the scalar multiplication is. If you look at the scalar multiplication of a with the vector (x_1, x_2,...,x_n), we get back (a x_1, a x_2,...,a x_n). And similarly, if you look at the scalar multiplication of b with (x_1, x_2,...,x_n), you get back (b x_1, b x_2,...,b x_n). And that is precisely what we were trying to prove. Right? This is exactly equal to what we have written here. Okay, so we have established the question mark. So claim..... We have made a claim and we have given a proof.

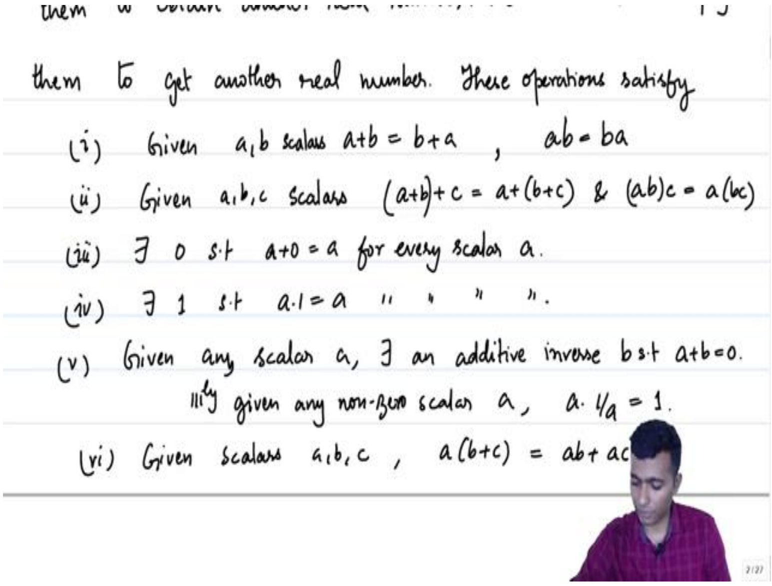
Now, this is just the property VIII that we have checked. There are seven other properties and the proof of each of the seven properties or the seven properties getting satisfied is very similar to how we would have done it in say in R^2 or R^3. So let me leave that as the first exercise for you and I would strongly suggest that you really sit down and write each of these steps. Check that property I, II, III up to VIII are indeed getting satisfied. Okay?

(Refer Slide Time 6:29)



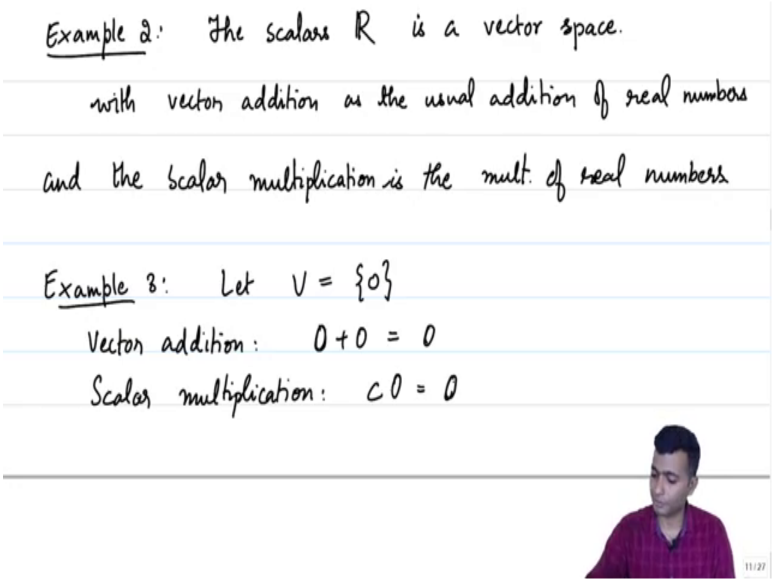
A second example, example 2. Example 2 is the scalars, the field of scalars. The scalars, F or R, let me call it R, is a vector space. Well, I think we just have to go up a bit and recall some of the properties which we were capturing.

(Refer Slide Time 6:51)



If you observe carefully, (i), (ii), (iii) to (vi) here satisfy most of the properties….. I did not even mention what are the vector addition and scalar multiplication here. Let me go in an organized manner.

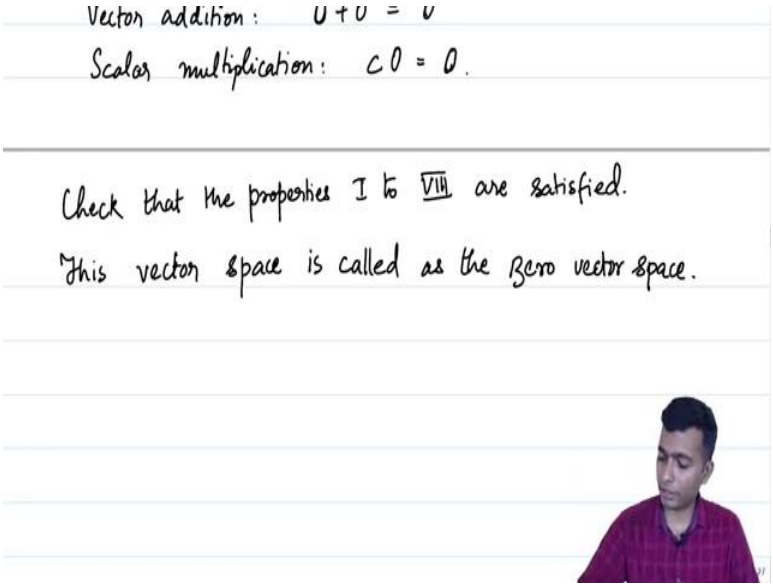
(Refer Slide Time 7:12)



So, R is a vector space with the vector addition as the usual addition -- addition of real numbers. And the scalar multiplication….. again what is expected of a scalar multiplication here? A scalar and an element of the set V should give you back an element of this set V. Here our set V is also R. So you get a scalar and a real number. We can now just talk about normal multiplication. Right? Scalar multiplication is the multiplication of real numbers. I was scrolling up, let me not do that again. The scrolling up part was to show that the various properties….. the fact that R is closed under addition and multiplication is known to us. And the various properties, many of the properties have already been checked above when we described what scalars are and what properties of scalars we are interested in. This is a boring exercise. This is a boring example but nevertheless, it is an extremely important example.

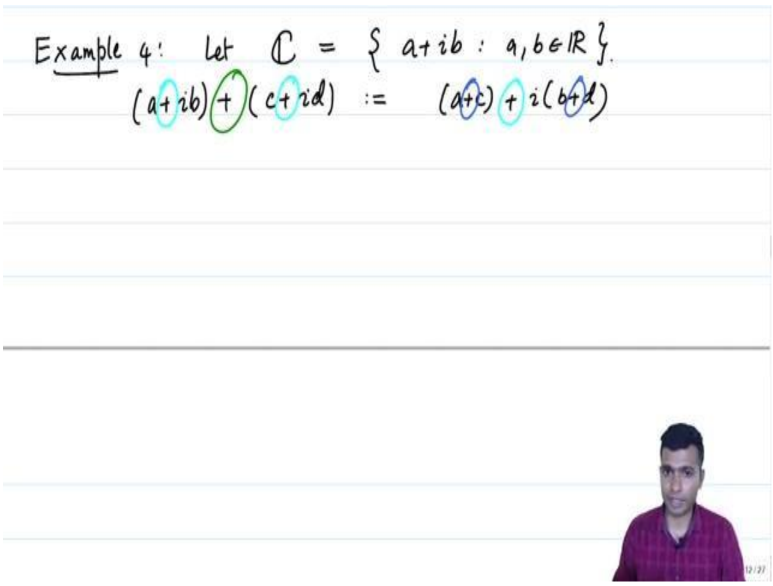
A similar, not very interesting but very important example, is what is called as the zero vector space. Let V be the set which has just the 0 element. What are the operations? Vector addition? You do not have much choice in talking about vector addition. There is only one element in the set V. So, we have to talk about what is the sum of 0 with itself and we have not much of a choice again. There is only one element. It has to be 0 itself. And scalar multiplication, you take any scalar c and multiply it by the only element in V, it has to certainly give you back the only element. Check that all the properties are satisfied. This is clearly…. the vector addition and scalar multiplication is being written in such a manner to ensure that V is closed under these operations.

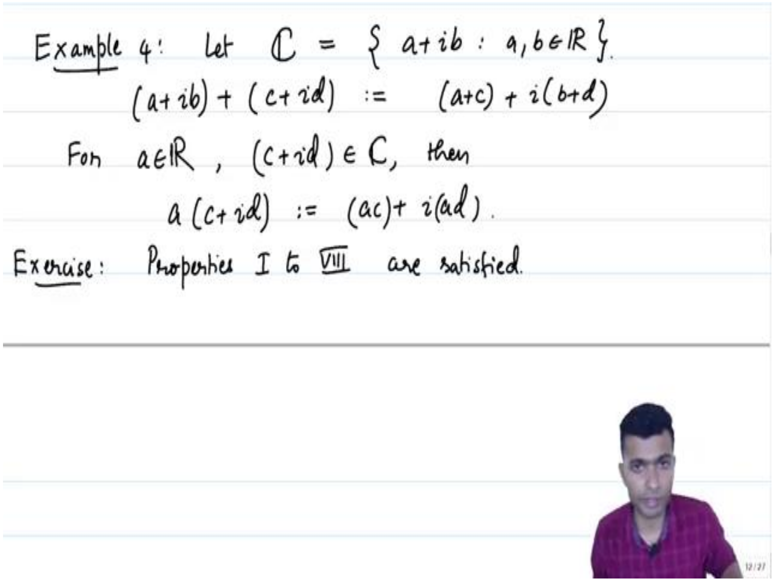
(Refer Slide Time 9:59)



The properties I to VIII are trivially satisfied. This vector space is called as the zero vector space. Okay, so till now we have only seen examples which either are familiar or are not very interesting.

(Refer Slide Time 10:48)





So let us know consider an example…. this is example four. Let us consider an example which is a bit more exciting. You might be familiar with the set of Complex Numbers. Let C, let me denote that set by C. So our vector space V is what is being denoted here by C. This is the set all a+ib, where a and b are real numbers. If you have seen some operations in complex numbers, we have defined -- we have not defined in this course but you might have seen -- that if you take say 2+3i, and if you take say 4+6i, and if you add it, you will get….. 2+3i and 4+6i, you will get 6+9i, right? We will do it component wise in some sense. (a+id)+(c+id), this is being defined…. Again, I am putting a colon…. which is equal to (a+c)+i (b+d).

Again, this might look like a bit of manipulation of symbols. But it is not like exactly manipulation of symbols. let me just show you what is happening here. The ones which I am circling in blue, they are just notations. Right? The set of all a+ib, we could have as well written it as (a, b). Right? This is just a notation. The one I am circling in green is what is being defined. That is the addition that is being defined right now, a+ib, which is a notation and c+id which is a notation for complex numbers that is being added here by what is described in the right. In the right let me use another colour now. The yellow is being used, maybe yellow is a bad choice…. let me use blue and let me circle the two additions that are written to the right hand side of the equation. Those are addition of scalars, those are addition of Real Numbers.

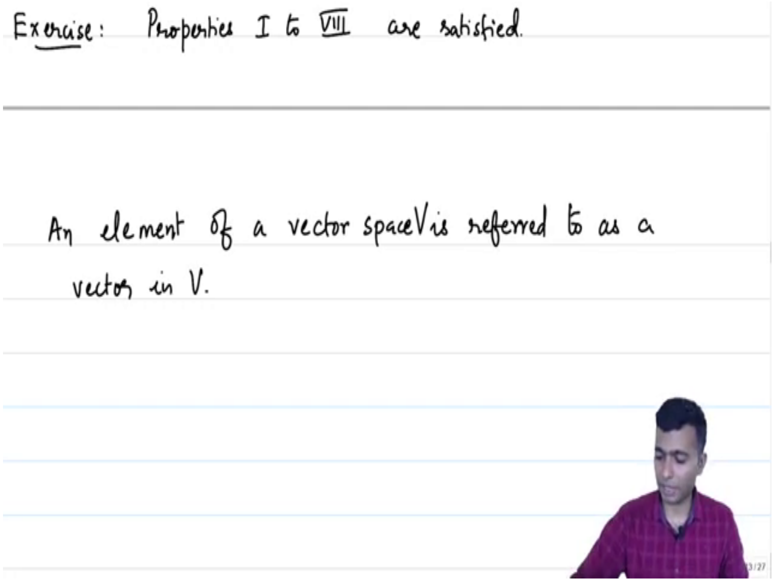
Remember that a and b, and c and d all are Real Numbers. So, a+c is a Real Number, b+d is a Real Number. So we get back some real numbers. Oh, there are many pluses which are featuring in this equation and different pluses have different meanings. It is very important to keep in mind that when we are doing some abuse of the notation, the context should make it clear and that is something which we should familiarize (ourselves with). Okay. Enough is said about the use of notations here. We have defined addition. So this is our vector addition, okay, that we would like to consider. Observe that if you are taking two Complex Numbers and adding it, we are getting back a Complex Number.

Now for a scalar a or a real number a in….. Let me note as R…. so F is also used generally for the field of scalars. But in our case, in this course, most of our examples are cases when our field of scalars is Real Number. So, I will use interchangeably between R and F. In fact, I will mostly use R. And c+id be an element of C. Then define this scalar multiplication as ac+i ad. So, scalar multiplication is also the most straightforward definition that we can think of. Again notice that C is closed under this operation of scalar multiplication. And again I leave it as an exercise for you to check that all the properties I to VIII are satisfied. Let me leave it as an exercise. Properties I to VIII are satisfied by the vector addition and scalar multiplication.

If you have seen some operations….. or if you have worked with Complex Numbers, you would have seen that we can also talk about multiplication of two Complex Numbers, and we will get back another Complex Number. For example, (a+ib)(c+id) is (ac-bd)+i(ad+bc). But to talk about a vector space, again, let me reiterate, given two vectors v1 and v2, we do not talk about the product of v1 and v2. That operation is not being defined in a vector space. The only operation that we are defining is a scalar multiplied to a vector.

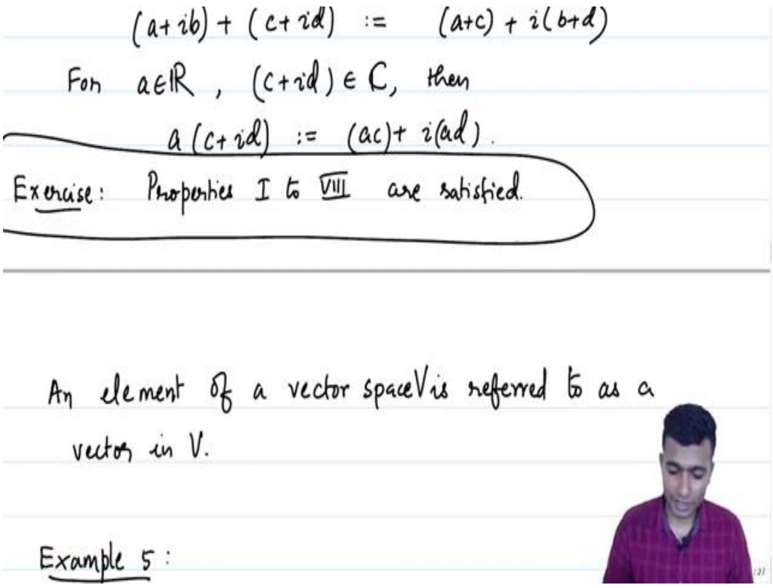
So even though it is possible to describe the multiplication of two Complex Numbers, for the purpose of studying the Complex Numbers -- the set of Complex Numbers as a vector space -- we will restrict our attention only to what is the scalar multiple of a Complex Number by scalar. It is a good observation to keep in mind that this is just the special case of the complex multiplication, multiplication of complex numbers that we might be already familiar with. Nevertheless, I spent a couple of minutes exclusively to push forward the notion that, given two vectors, given two elements v1 and v2 in V, we are not talking about the product of the elements v1 and v2.

(Refer Slide Time 17:06)



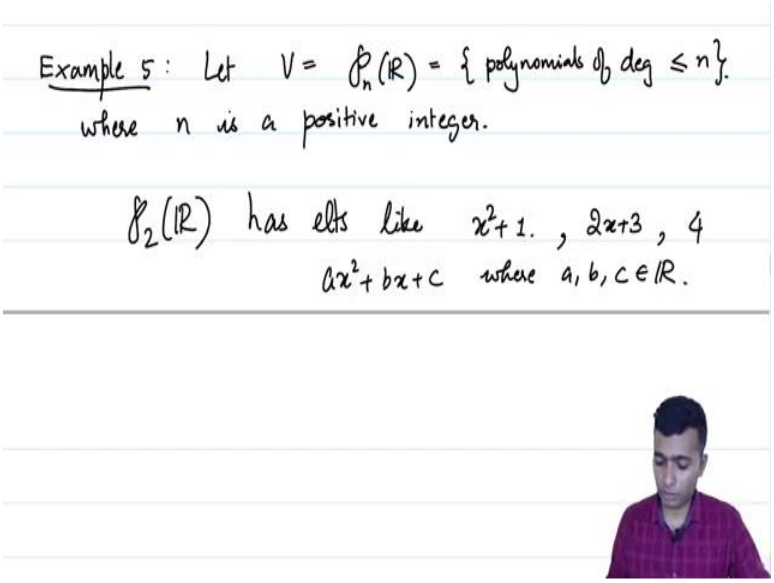
I am using the word vectors. Let me…. Before going into the next example, let me just give a definition. An element of a vector space is sometimes referred to as a vector. An element of a vector space is referred to as a vector. of a vector space V is referred to as the vector in V. So when we say something is a vector in V, we just mean that it is an element in V. Okay, let us look at the next example, example 5.

(Refer Slide Time 17:44)



I leave it as an exercise for you to check that all the properties I to VIII are satisfied and I strongly recommend that you really sit and write down and check each of these properties.

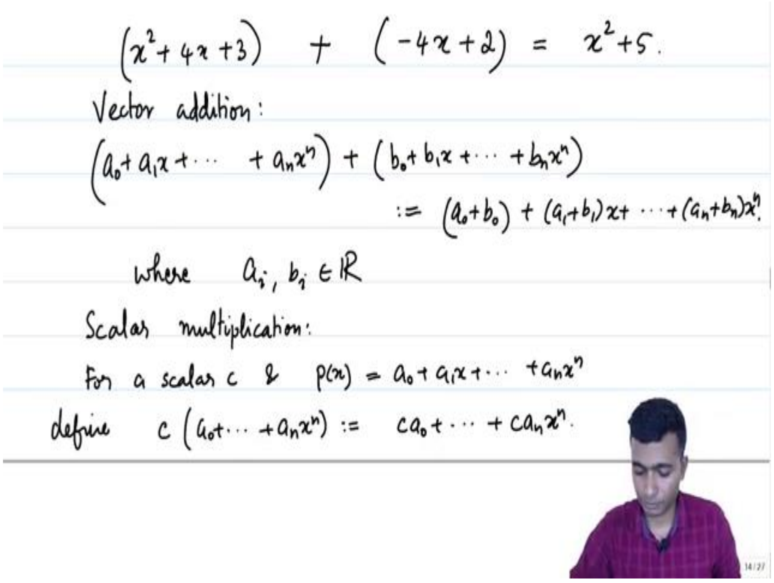
(Refer Slide Time 17:56)



Okay, what is the next example? consider P_n(R). Let V be equal to script P_n(R). What is this? This is the set of all polynomials of degree less than or equal to n. So, when I am asking you to check for all the properties, you are expected to come up with the candidate for what is the additive identity and what will be the candidate for the additive inverse of a given vector. Not just taking the remaining properties…. you will also have to talk about what these are the additive identity and the inverses. So, let us now look at the example 5, which is polynomials of degree less than or equal to say n, where n is a positive integer.

For example, if you look at P_2(R). (It) has elements of the type... like…. say x^2+1 or 2x+3. What else? Well, the constant 4 is a polynomial that is an element in P_2(R). All polynomials ax^2+bx+c, where a, b, c are in the field of scalars…. are Real Numbers. Right? So, P_2(R) will be the collection of all polynomials (of degree) less than or equal to 2.

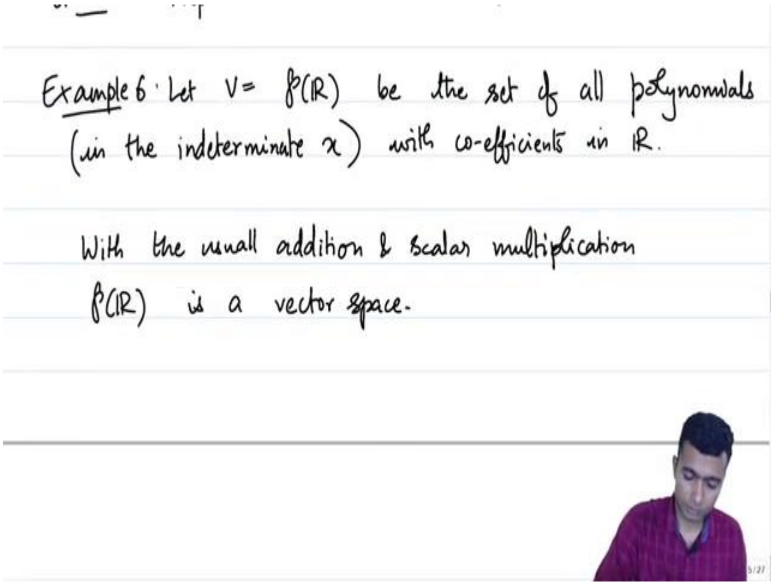
(Refer Slide Time 20:33)



Now we are familiar with the notion of addition of polynomials. So given two polynomials, we know how to add them. For example, if I say x^2+4x+3, and I am to add it to minus of $4x+2$. We know that by basic algebra... we know that this is going to be x^2+5. Right? So the addition is exactly generalizing this or putting it in a more formal framework. This exact vector addition which we are familiar with. So, $a_0+a_1 x+\ldots$.. Let me just write the definition of vector addition. Maybe I should leave that as an exercise as well. The next one we will. $a_0+a_1 x+\ldots+a_n x^n$, this is the general expression for element here. Right?

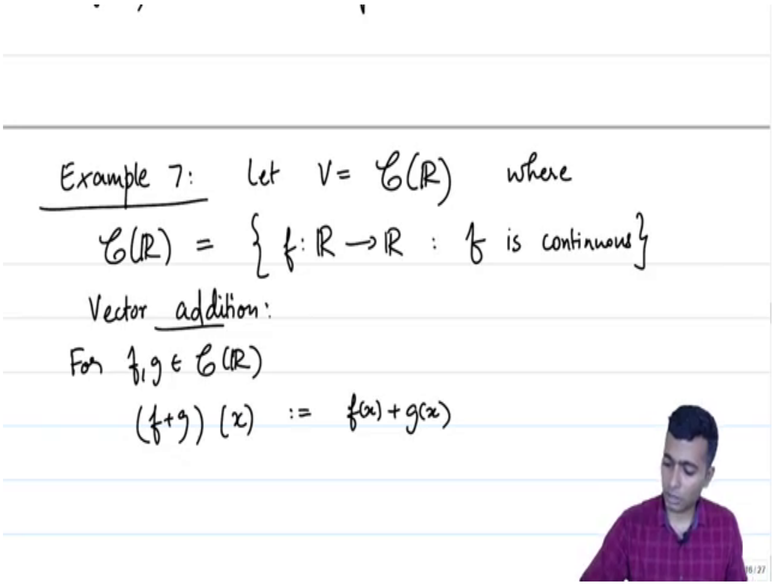
Some polynomial of degree less than or equal to n can be represented as $a_0+a_1 x+\ldots+a_n x^n$, where $a_0, a_1, a_2, \ldots a_n$ are all Real Numbers. This is being added to another element say $b_0+b_1 x+\ldots+b_n x^n$. And what do we get? we get this as $(a_0+b_0)+(a_1+b_1)x+\ldots+(a_n+b_n) x^n$. When we are writing it like this, the doubt that can crop up is whether we are considering the cases like in the example here. The polynomial to the right minus $4x+2$. That does not have degree two. Right? That is a degree one polynomial. But that is okay because we have allowed b_1, b_0 (to be 0). So where ai and bj all are in the field of scalars. Right? So in particular, ai could be 0. All ai's could be 0, a few a i could be 0, a n could be 0 and so on, and still we can talk about the sum here. Okay, how about scalar multiplication? Scalar multiplication also... what if we take x^2+4x+3 and multiply it by say 2. We will get $2x^2+8x+6$, right? So scalar multiplication is also doing exactly the same thing that we expect. For a scalar c, and p(x), a polynomial which is equal to say $a_0+a_1 x+a_n x^n$, we find c times (this polynomial). So remember that, and let me put a colon here because we are defining it, we are defining the scalar multiplication here. This is being defined to be $c a_0+\ldots+c a_n x^n$.

(Refer Slide Time 23:40)



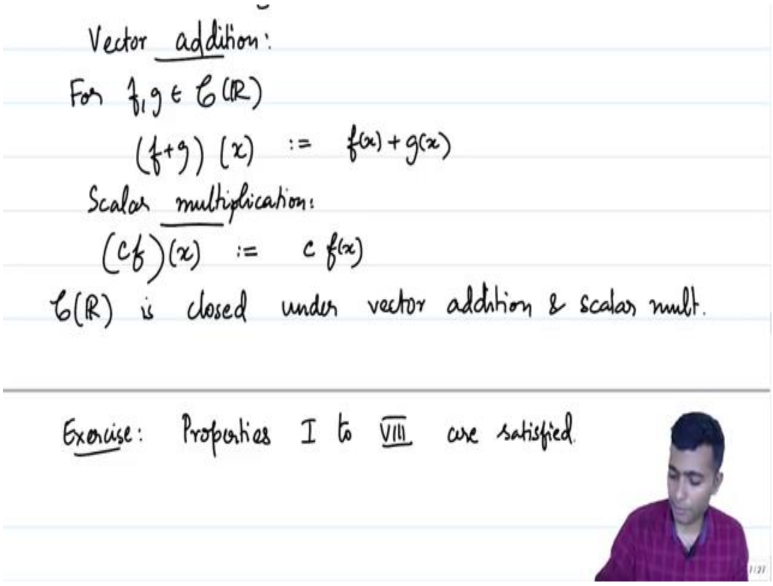
Okay, so again, I leave it as an exercise for you to check that all the properties listed in the definition of a vector space are satisfied. Okay, let us make it a bit more interesting. So we were looking at polynomials of degree less than or equal to n. Let us not put any restrictions. Let us consider all polynomials. So let V = P(R) be the set of all polynomials in the indeterminate x. Yeah, the previous one also was something like this. Well, let me just put it in bracket…. with coefficients in Real Numbers. The previous one also was the set of all polynomials of degree less than or equal to n with coefficients in R. So how do we define addition? We have already given you a glimpse of how we define the addition. I will not write it down explicitly. We can add any two polynomials. We can also talk about the multiplication of a scalar to a polynomial. We get back polynomials. Right? And with the usual….. let me just write it like this….. with the usual addition and scalar multiplication P(R) is a vector space. Let me give numbers here….. this was example six.

(Refer Slide Time 25:58)



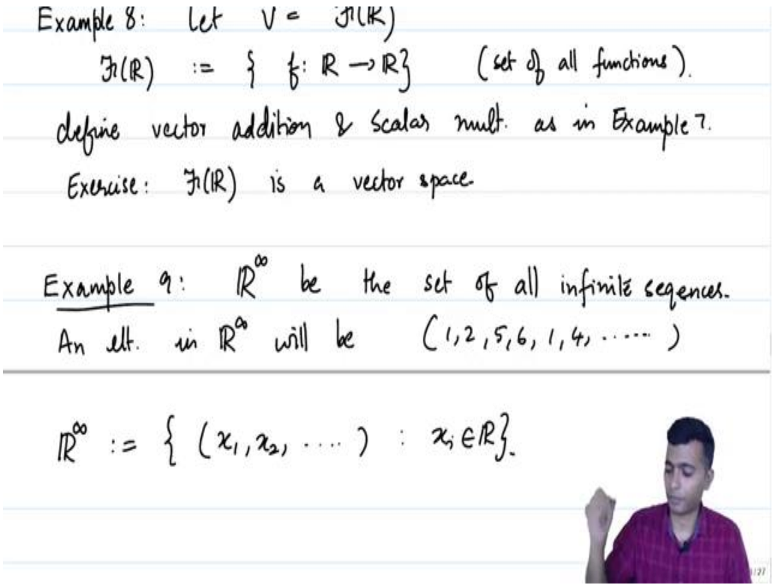
Okay, example seven. This is an example which is not similar to what we have seen till now. So let V = C(R) where C(R) is the collection of all functions f from R to itself such that f is continuous. So if you have seen the notion of continuous functions…... Let us look at the set of all continuous functions from R to itself. Okay. We define the vector addition here. Let us define, for f,g in C(R), f+g. We need to describe…. f+g should be some function from R to R. Right? So we will define (f+g)(x) to be f(x)+g(x). Now notice that this makes f+g into a function from R to R. And some more background in Real Analysis will tell us that whatever we have defined turns out to be continuous as well. So, the closedness is something which I will not verify now and leave it for you to verify later after doing a course in Real Analysis.

(Refer Slide Time 27:44)



And the scalar multiplication…. so this is the vector addition we have just defined. Scalar multiplication…. Again cf. This is as of now just a notation and let me say how this is a function of R. This is defined as c times…. the scalars c times…. Now f(x) is a function... f is a function from R to R. f(x) is hence a real number. So (cf)(x) is another Real Number. Define cf to be this particular function. Again, I will leave it as…. it is not an exercise. Let me leave it right now to check that multiplication of a scalar by a continuous function gives you back a continuous function. That is something which I will rest right now and let you verify later when you do a course on Real Analysis. So, the fact that C(R) is closed under vector addition and scalar multiplication will be taken for granted right now. Let me just write that down, that C(R) is closed under vector addition and scalar multiplication.

(Refer Slide Time 29:15)

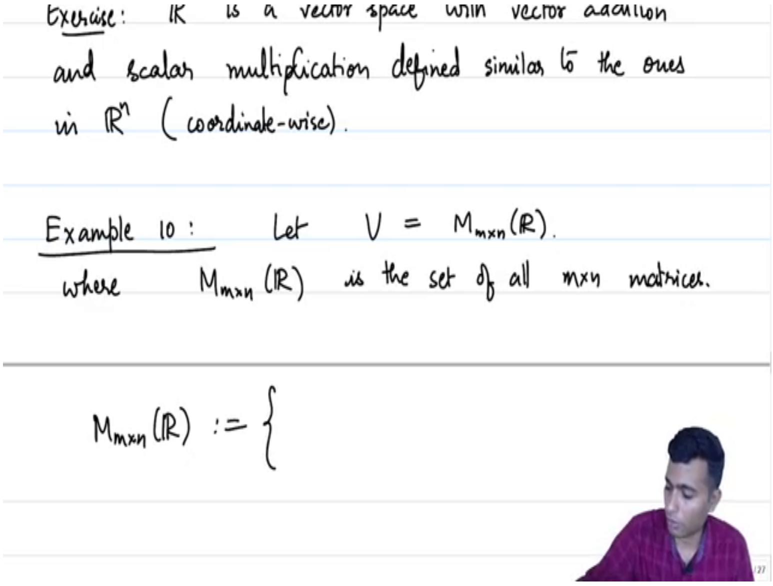


An exercise is to check that properties I to VIII are satisfied. An interesting question to ask might be, what will be the zero-vector of C(R)? I will let you think about it. Let me give you more examples. So example nine, or not….. Example eight. Let V be equal to what is usually denoted as F(R). You might be wondering why I am giving strange notations. These are all classical notations and F(R) is used to denote the set of all functions from R to R. The set of all functions from R to R….. And define vector addition and scalar multiplication as in example seven; point wise. It is good to again sit down and check…. already most of the things have been checked if you have done the exercise, previous exercise…. if you do the previous exercise, this will also follow, that F(R) is a vector space. It is an exercise to be checked. The fact that vector addition and scalar multiplication will give you back an element in F(R) is quite straightforward. It is the properties which you have to check but I think the previous exercise would have already done…. you would have done most of the work. So, with that, F(R) is also a vector space.

We started off with just very basic examples like R^2, R^3, R^n and now we are looking at spaces which has functions in it. And we are saying that these are vector spaces. They are similar in some sense. At least some of the properties of the addition and scalar multiplication are the same as the properties of the addition and scalar multiplication in R^n.

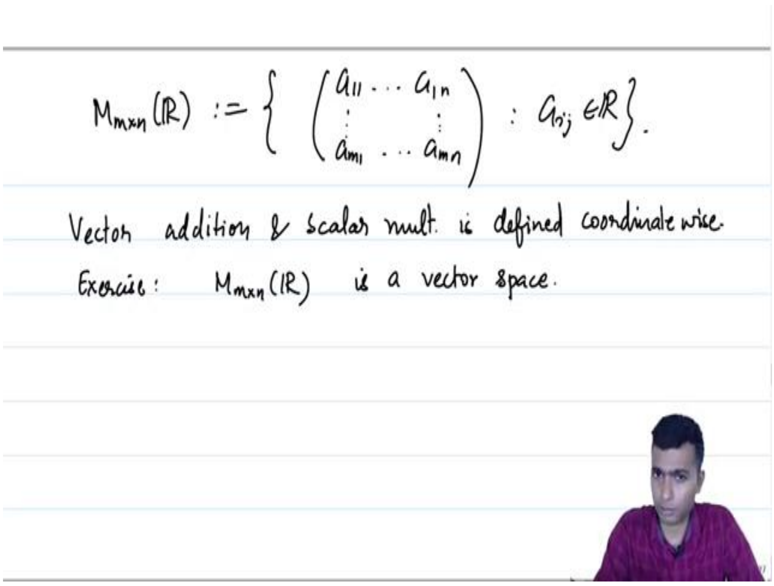
I will maybe give a couple of examples more. They are important examples and hence….. Let R^infinity be the set of all infinite sequences. An element in R^infinity will be something like say (1, 2, 5, 6, 1, 4, ……) an infinite sequence. So formally R infinity is just the set (x_1, x_2 ….) such that x_i belongs to R.

(Refer Slide Time 32:35)



And how do we define vector addition? Again, point wise; just like how we define it in R^n, define addition and scalar multiplication in R^infinity as well and check that it is a vector space. So I will just write it as R^infinity is a vector space with vector addition and scalar multiplication defined as in…. similar to…. define similar to the ones in R^n, coordinatewise. Okay, Let me give you just one more example finally before going ahead. So example ten is again the classical notation is M_{mxn}(R). This is an example which you will be quite familiar with. Where M_{mxn}(R) is the set of all m cross n matrices.

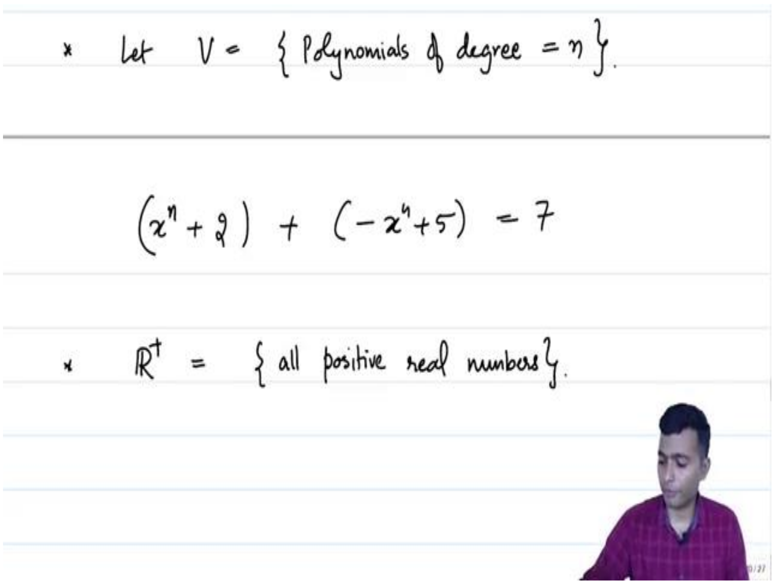
(Refer Slide Time 34:36)



What is the set of all a_11, a_22 up to a_n1 ... a_mn ….. So M_{mxn}(R) formally is the set of all elements of this type, a_11, a_m1 up to a_mn where a_ij are real numbers. So, what would be the vector addition and scalar multiplication? Again vector addition and scalar multiplication is coordinate wise. It is defined coordinate wise. I will not write it down. It is just going to make it cumbersome and something which you can very easily do. So let me leave it as an exercise for you to write down formally what the vector addition and the scalar multiplication is. And again I leave it as an exercise for you to check at all m cross n matrices over R is a vector space with these operations.

So, yes, we have now defined what a vector space is and we have seen many many many examples of vector spaces. There are actually plenty of examples, infinitely many examples of vector spaces. But let us not go into it anymore. We will be keeping on visiting more examples during the course. Many new vector spaces will be defined from existing vector spaces and so on. Okay, so now that we have spent so much time to talk about examples, let us also discuss a few cases which will be non-examples. Let me just write it down Non-examples.

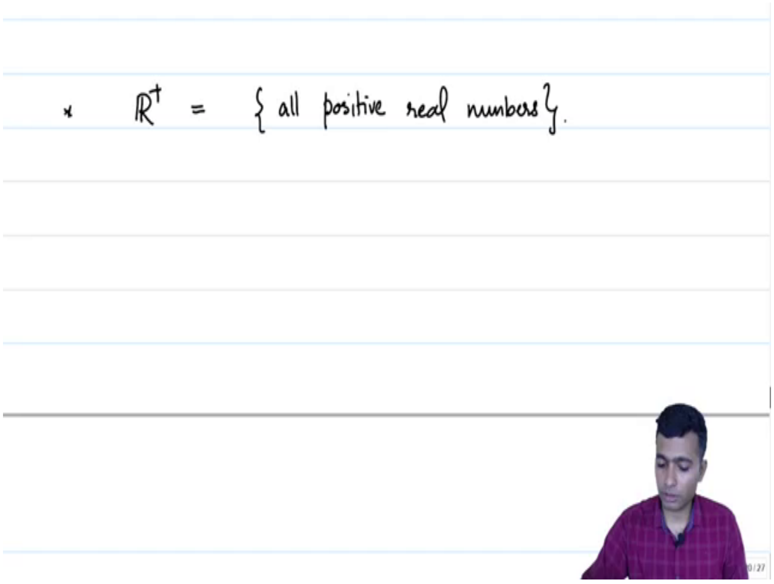
(Refer Slide Time 36:26)



One example would be….. consider all polynomials of degree equal to n. So, let V be equal to the set of all polynomials of degree equal to n. Well, I will give you your favourite pick of which property is getting violated. In fact, even before we enter into properties, notice that V is not closed under addition, for example. If you take a polynomial, say x^n+2 and if you add it to -x^n+5, what we end up with is 7. Right? If n is say 2, if you are looking at P_2(R), polynomials of degree equal to two, 7 is certainly not a polynomial of degree two. So this set is not even closed under vector addition. So this is certainly not a vector space.

The first requirement before we start looking at the properties is that it has to be closed. If you take two elements in the vector space, in the set, and if you add it, you should get back an element in the set itself. It is not getting satisfied here. What could be another example? Yes, another example is…. consider R^+. R^+ is the set of all positive real numbers. So if you take R.

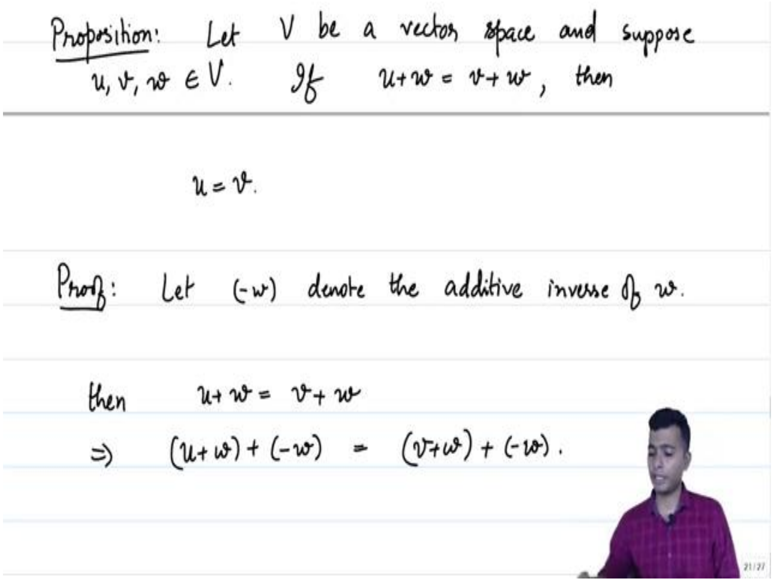
(Refer Slide Time 38:10)



Here, the addition and the scalar multiplication is defined, like we define it in the vector space R. So if you take two positive real numbers, if you add it, you get back a positive real number. So, R plus is closed under addition. And if you do the scalar multiplication, now the problem comes, is it closed under scalar multiplication? Answer is no. It is not closed under scalar multiplication. If you take say 2, and if you look at the scalar multiple of -1 to 2, then scalar multiplication should be -2, which is not in R^+. This is not closed under scalar multiplication. So, with vector addition and scalar multiplication as in R, this is not closed under scalar multiplication. It will not be a vector space. Okay.

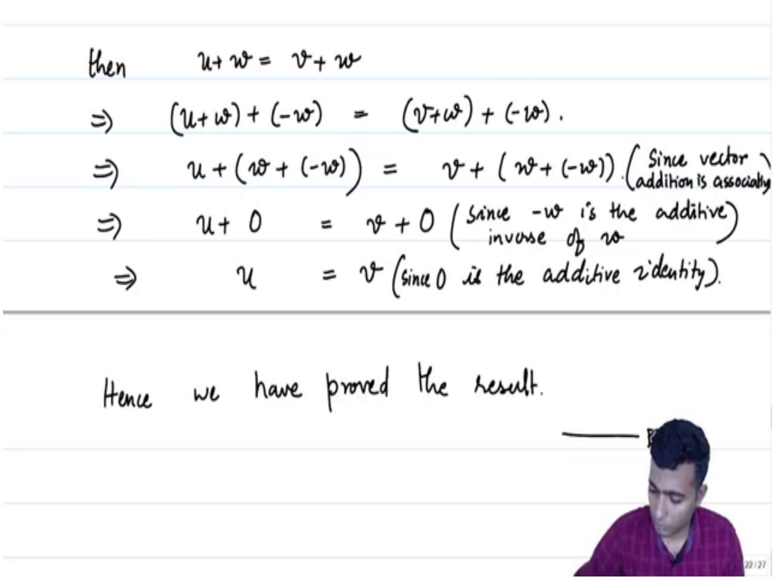
We have discussed quite a few examples now. We have seen a couple of non-examples. Let us see what the impact of the properties are on the vector addition operation and the scalar multiplication operation. So, maybe let me give a proposition. This is the first proposition of this course.

(Refer Slide Time 39:50)



Let V be a vector space and suppose u, v, w are vectors. Suppose they are elements in the vector space V. Then u+w = v+w, then if…. then if does not sound good. If u+w = v+w, then u = v. What does this proposition tell us? The proposition tells us that if you have a vector w which is……. the vector basically cancels, that is what it says. Right? You can cancel out vector w which is featuring in the left hand side and the right hand side and we get u = v. Let us give a quick proof of this. This is the first proposition of this course. So let us spend some time to talk about how a proof can be given rigorously. Okay so, what is it that we know? We know that V is a vector space and given any vector space, we have that every vector has its additive inverse right. So, let -w denote the additive inverse of w, the element w has an additive inverse let us denote it by -w. Then u+w is equal to v+w implies, by adding the vector -w both sides nothing changes. u+w+(-w) = v+w+(-w). Right? But our vector addition is associative.

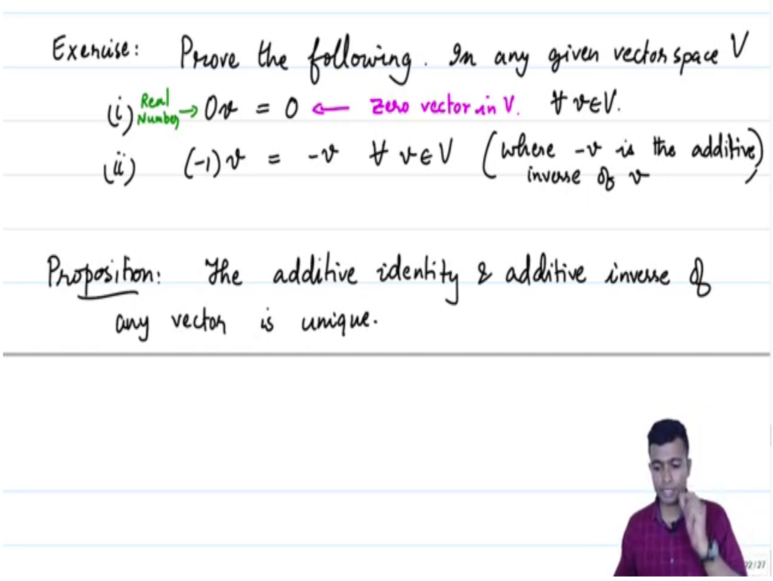
(Refer Slide Time 42:16)



This implies: this is equal to u + ….. the above u+w+(-w) is the same as u+(w+(-w)). Which of the two vectors are added first does not matter. Similarly here, v+(w+(-w)). But minus w is the additive inverse of w. So if you add w and -w, you get u+0, which is the same as v+0. Notice that when you write the 0 here it should not be confused with the scalar zero, the real number 0. This is an element in the vector space V and it is clear from the context what it is. u is a vector in V, and therefore you cannot make sense of u + the real number 0. So here when we write u+0, it is clear from the context that 0 is the zero vector of V. But what is u+0? It is just u and this is just v, and hence we have proved the result.

Let us just…. for the sake of completion, even though I said it very orally, what is this implication coming from? Since addition…. vector addition is associative…. important to note all the reasons…. Why is this happening? Since -w is the additive inverse of w. You should go back and check which property of the definition of the vector space we have used and this since 0 is the additive identity. Okay, this symbol will generally denote that we have completed the proof.

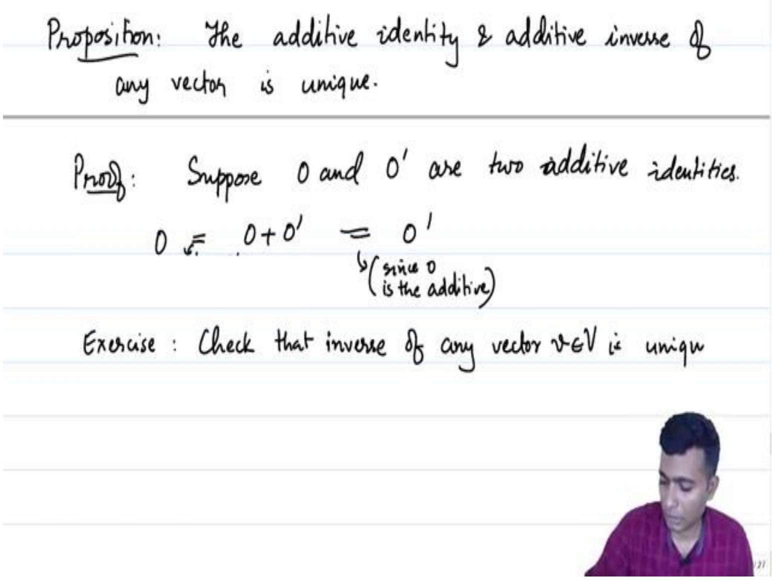
(Refer Slide Time 44:58)



I will give a few exercises here before concluding this session. Prove the following: 1) 0 times any vector v is equal to 0. This is good exercise for you. with the abuse of notation that we are familiar we are getting familiar with. So if you observe carefully, the left hand side is the scalar 0 multiplied to be the scalar multiplication of the real number 0 to v. So this is the set v, just note that this is the real number 0 and this is the 0 vector in v. To proof the following in any given vector space v, so this is true for all v in capital V. Okay what next, proof that, if you look at minus 1 times v this is give you minus v. So, for all v in capital V, what is minus v? where minus v is the additive inverse of v, okay.

So we have just written the additive inverse. So, that is not something acceptable yet because what first is the inverse to be unique could have been an additive investment would have been more appropriate here. So to do that, let us do one thing, before we go into other examples or other exercises let us proof the following, the additive identity and additive inverse of any vector of any vector is unique. So, first we will show that the additive identity is unique and then we will show the next part okay.

(Refer Slide Time 49:00)



So suppose there are two different so suppose 0 and 0 prime are 2 identities additive identities. What is the property of an additive identity? If you add any vector to that it should give back the vector itself, so 0 in particular is equal to 0 plus 0 prime, right? But what is 00? This is because going by the assumption that both 0 and 0 prime are additive identity. So in particular, 0 is also an additive identity, so if you treat our 0 prime as our vector v, and if you add it to 0, because 0 is an additive identity, you should get back v right, so this is equal to 0 prime and why is this equality coming up? So this equality is because of this, this equality is because since 0 is the additive identity.

So it may not give a proof of the inverse being unique, I leave it as an exercise for you to check that inverse is unique it is more or less similar. Check that the inverse that the inverse is not yet right. That is precisely what we are trying to do inverse of any element, v any vector v in capital V is unique alright let us stop here.