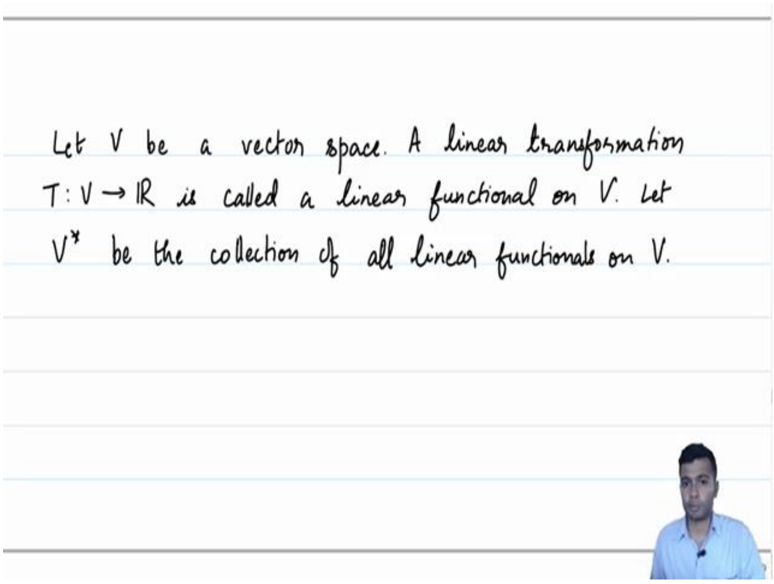
Linear Algebra
Professor Pranav Haridas
Kerala School of Mathematics, Kozhikode
Lecture - 5.3
Dual Spaces

So we have seen the product of vector spaces obtained from vector spaces v1, v2 up to vn. We have also seen what is the quotient of a vector space V by a subspace u. Let us now look at what is meant by the dual of a vector space. Again we are in the process of obtaining newer and newer vector spaces from the given ones. So in order to study the dual, let us consider all linear transformations from a given vector space V into R.

So remember that R is a vector space over itself. It is a vector space of dimension 1. So it makes complete sense to talk about linear transformations from V into R. Such linear transformations are called linear functions, functionals.

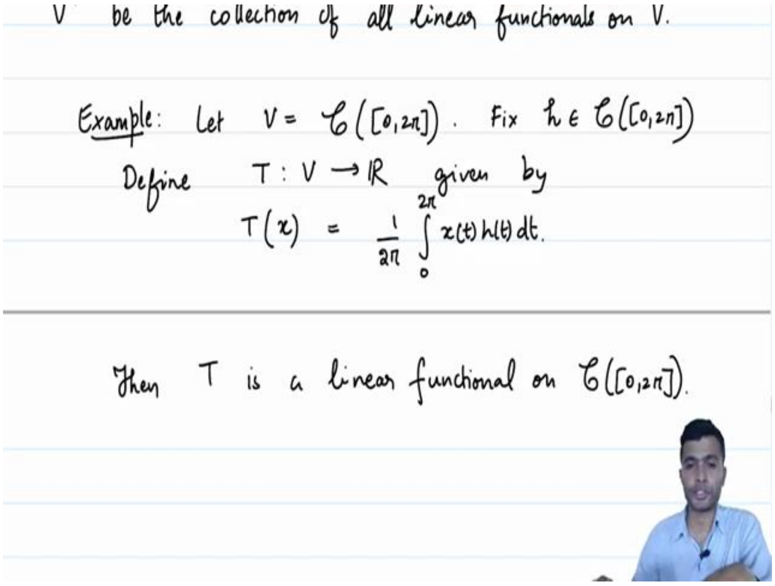
(Refer Slide Time 1:00)



So, let us begin with the definition. Let v be a vector space a linear transformation T from V to R very specifically into R, the codomain or the target or the range is R, is called linear functional on V. Let V star be the collection of all linear functionals on v. So in the previous week discussed, we how we can add two linear transformations. So if v and w are two vector spaces over R and suppose s and t are linear transformations from v to w, we discussed what is the meaning of s plus t.

We also discussed the scalar multiplication of a scalar to a linear transformation. So if T is a linear transformation from v to w, we talked about c times T which also turned out to be a linear transformation from v to w. So, if our w is our R here, as given here, so all those things can be used, all those definitions can be used here to talk about addition of linear functions and scalar multiplication of linear functions. So let us do that and give a vector addition on v star and a scalar multiplication and on v star.

(Refer Slide Time 3:23)



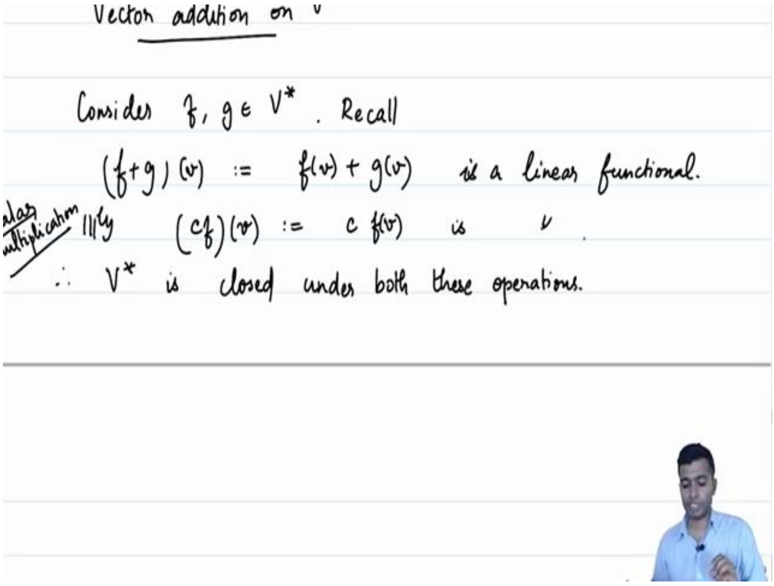
So before we go into that, let us look at an example of a linear functional, which will tell us certain indicate us, why we should be considering these objects. After all, they do not look very natural or maybe they do, but there is a very concrete reason why these objects are studied. The dual is a very well studied object. So one example might help us in justifying that, so consider the vector space v of continuous functions on the closed interval 0 to pi.

So let v be c of 0 to pi and fix some continuous function, let us call it h on C of 0 to 2 pi. Let us define a linear functional on v. Define T from V to R given by T of function x. This is equal to 1 by 2 pi times the definite integral from 0 to 2 pi of x of t h of t dt. You have already seen such examples of linear maps. And it is easy to check that this is indeed a linear map linear transformation.

So then T is a linear functional on C of 0 to 2 pi. It is quite well studied as mathematical objects, especially when we consider h to be the functions say sign nf, sin of nt, where n is some natural number or where h is cos of nt where n is some natural number. Such linear functionals will give us what is called as the nth Fourier coefficient of x. So in later analysis courses, later courses in mathematics you will see that these objects are very very well studied and it gives us a lot of information about the function itself.

So, this might be thought of as a partial justification on why we should be studying linear should be studying okay I will say that again. So, this is one example of how the study of linear functional on a vector space tells us a lot about the vector space itself. So let us as I was discussing earlier, let us discuss how we can talk about the addition and the scalar multiplication.

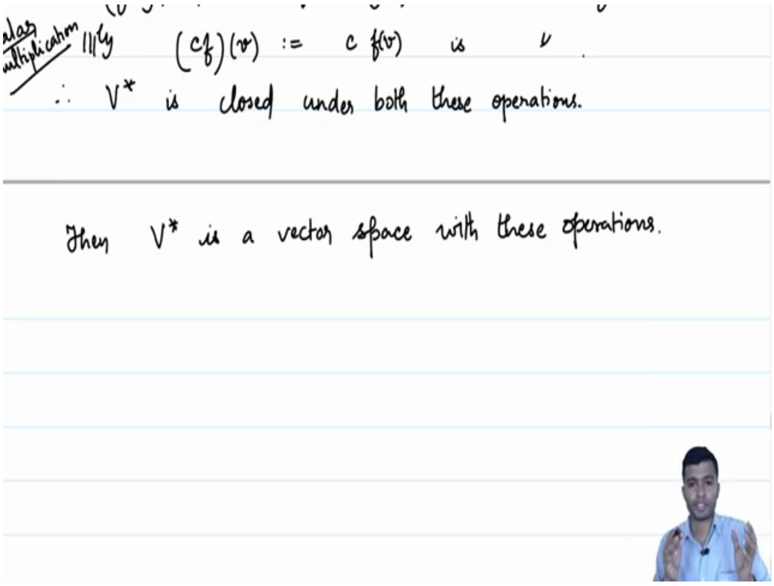
(Refer Slide Time 6:19)



So let us talk about vector addition first on v star. So consider f and g belonging to v star. Then, f and g are linear transformations from v into R. And we have already defined what it means to say f plus g. So recall, f plus g of x by definition, is equal to f of x plus g of X and we had checked that f plus g defined in this manner is a linear transformation from V to R. Therefore, it is a linear functional.

Recall that this is linear functional. Similarly, CF acting on a vector v, so let me not use x here, is defined as C times f of P is also a linear functional. This we had discussed and therefore, v star is closed. So this is scalar multiplication. This is the vector addition and this is the scalar multiplication. So v star is closed under both these operations. And as usual, I am going to leave it as an exercise for you to check that v star is a vector space with these operations.

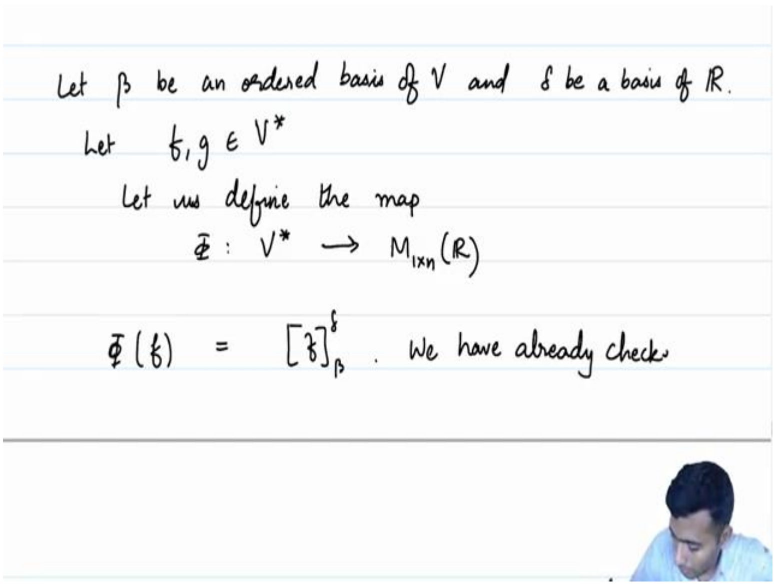
(Refer Slide Time 8:30)

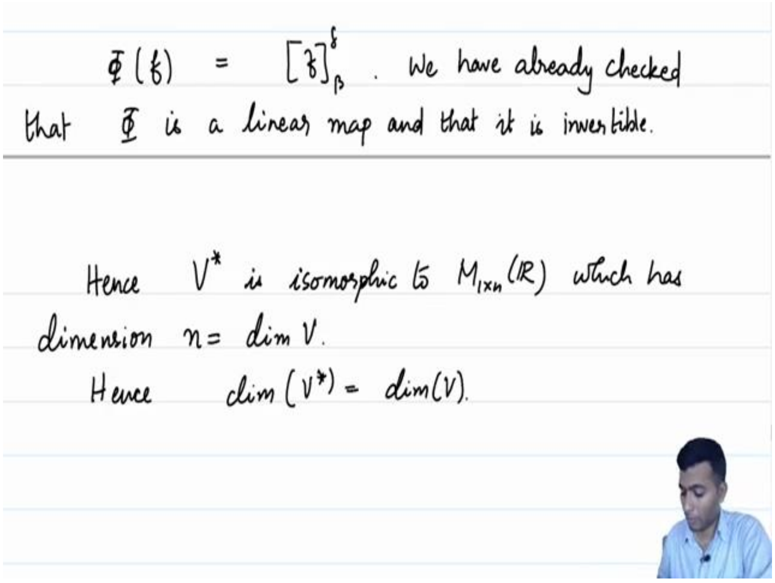


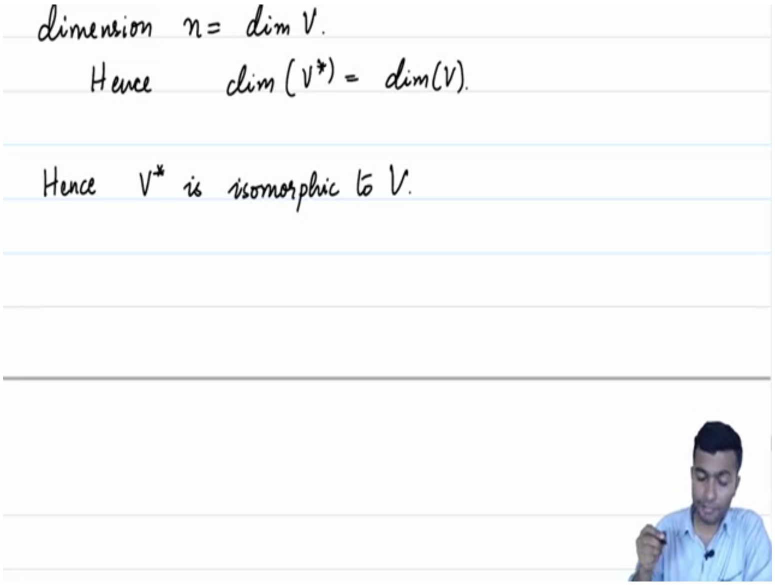
The zero element of v star will just turn out to be the zero linear functional, every element being mapped to the zero element of R. And for f, minus f will turn out to be the additive inverse. So you should check that v star is indeed a vector space with the two operations we have just defined. And we did more, suppose we fixed an ordered basis. In the last week, we also discussed how the sum of two linear transformations manifests in the corresponding matrices.

We showed that matrix of s plus t is the matrix of s plus the matrix of t. So let us just observe that here.

(Refer Slide Time 9:31)







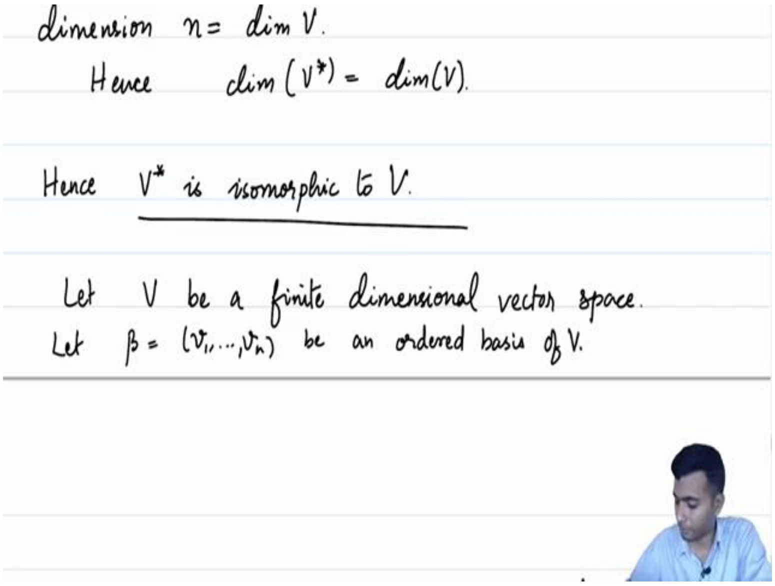
So let beta be an ordered basis of v and delta be a basis of R. Remember that R is a one dimensional vector space over itself. Therefore, delta will consist of one element. It does not matter though. Let f and g be two elements in v star. Remember, we did a lot more than just associating matrixes and checking that the sum of our linear transformations will manifest as the sum of the corresponding matrices.

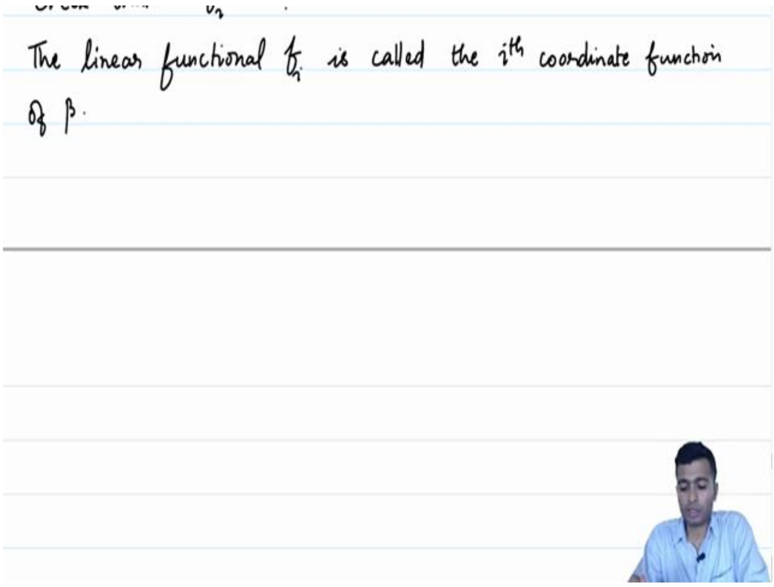
We showed that this operation is invertible in the sense that the matrix associated to the linear transformation, when we consider the linear transformation corresponding to this matrix, we get back the linear transformation. So let us define the map, let us call it phi. From v star into the matrices of size 1 cross n of R which can be identified with Rn, but we will come to that later, and what was this map? We defined it to be, phi of some f is the matrix of f with respect to beta delta which is a 1 cross n matrix.

And we checked that if we consider L subscript f corresponding to f, we get back f. So we checked, we have already checked that phi is a linear map and that it is invertible because this tells us that v star can be identified with matrices of size 1 cross n which is of dimension n. So hence v star, matrices of size 1 cross n is nothing but v is isomorphic to M1 cross n of R which has dimension of n.

But what is n? n is the dimension of v, it is equal to the dimension of v. Hence, dimension of v star is equal to the dimension of v, but what do we know about two vector spaces V and w of equal dimension? We know that if two vector spaces V and w have the same dimension then they are isomorphic and since v and v star have the same dimension, v star is isomorphic to v. Now let us consider some very specific linear functions which will be of interest to us.

(Refer Slide Time 13:27)



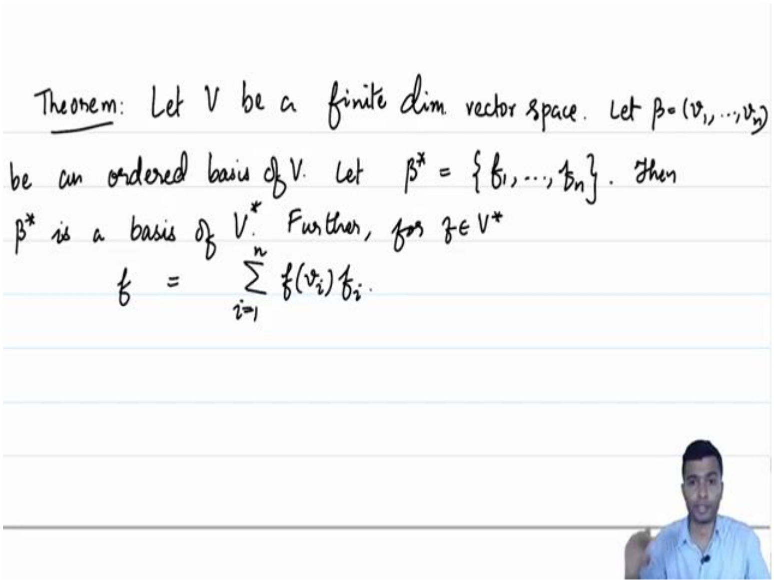


So in order to do that, let us let v be a finite dimensional vector space, let us fix an ordered basis of V. So let beta which is equal to say v1 to vn be an ordered basis of v, then for any v, there exists a corresponding column vector corresponding to beta. Then for v in capital V consider v beta to be equal to the column vector a1 to an. Now let us define fi from v into R given by fi of v is equal to ai.

It is an easy check to show that fi is a linear map. Check. It is an exercise that fi belongs to v star. So given any ordered basis, for each basis vector, we have now constructed a corresponding linear functional. This linear functional has a name, it is called the ith coordinate function. Then fi, the functional fi, the linear functional fi is called the ith coordinate function as should be the case, is called the ith coordinate function.

But it very heavily depends on the ordered basis beta. So this is of beta or with respect to beta. So these linear functionals fi that we just defined the very explicitly used our knowledge of beta. It so happens that these fi's which are n vectors in an n dimensional vector space happens to be a basis of our v star as well.

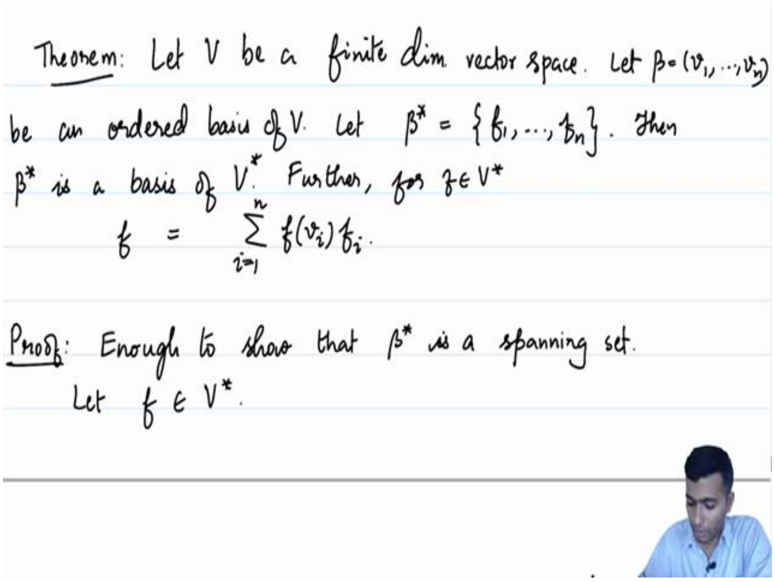
(Refer Slide Time 16:15)



So let us give a theorem here. So let me just rework the entire setup so that it is completely clear. Let v be a finite dimensional vector space and suppose beta is an ordered basis, v1 to vn be an ordered basis of v. Then for each vi we have, for beta we have a ith coordinate vector. So let the dust R be given by f1 up to fn which we just defined or beta fi captures the ith coordinate of any vector v when written in the column representation.

Let beta star be f1, f2 up to fn, then statement says that beta star is a basis of v, of v star. Further, f is equal to for f in v star, f is equal to summation i is equal to 1 to n, f of vi times fi. Let us prove this theorem.

(Refer Slide Time 18:11)

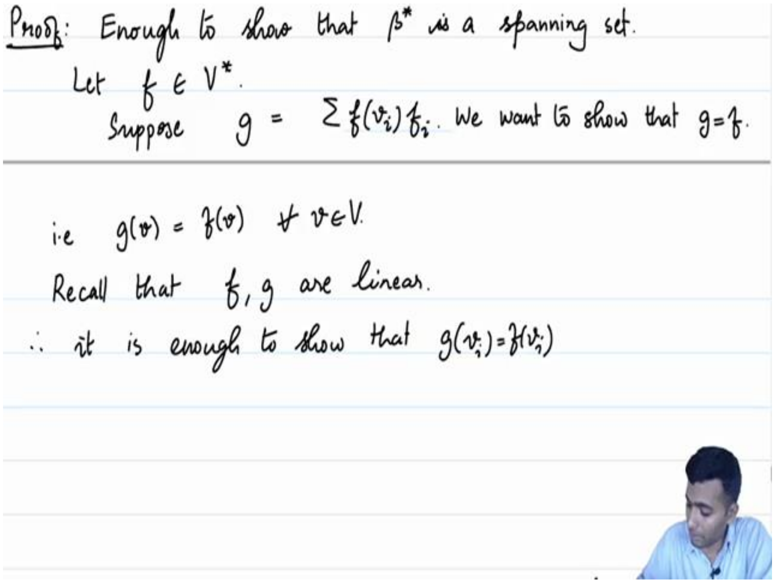


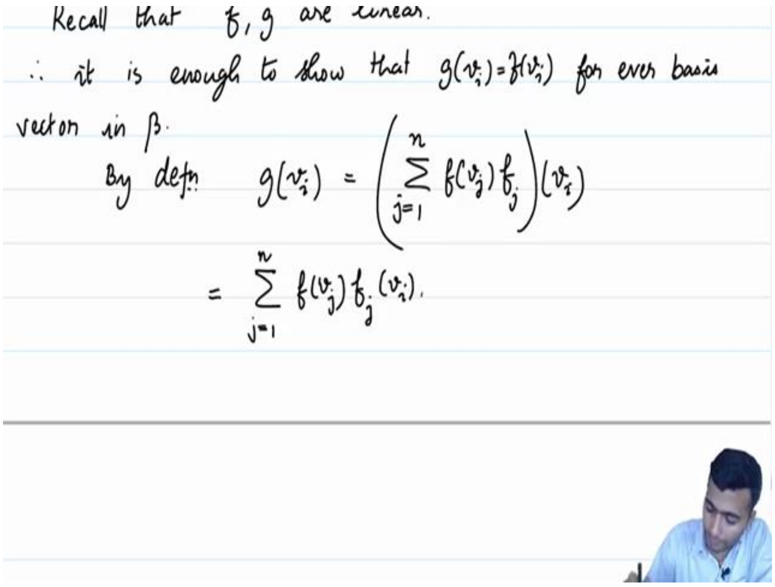
Let us give a proof. So we have n vectors f1, f2 up to fn in an n dimensional vector space v star. So, if we show that f1, f2 up to fn is a spanning set, then by a corollary, it will be replacement theorem. This is necessarily linearly independent and hence a basis. So first observation is that enough to show that beta star is a spanning set. The theorem already describes how to show that it is a spanning set.

For any F, we know that we have to prove that f is equal to summation f of vi times fi. So, we are being we are being prescribed exactly the method to say how f is in the span of f1, f2 up to fn. So, let us take an element in v star. Let us show that it is exactly this, but we do not know apriori whether the right hand side is equal to f, is not it? So let us do one thing.

Let us say that this right hand side is some element, g in v star, of course, it is an element in v star. Let us say that it is some g in v star and we will show that g and f are equal on every element of t. So that is the strategy.

(Refer Slide Time 19:53)

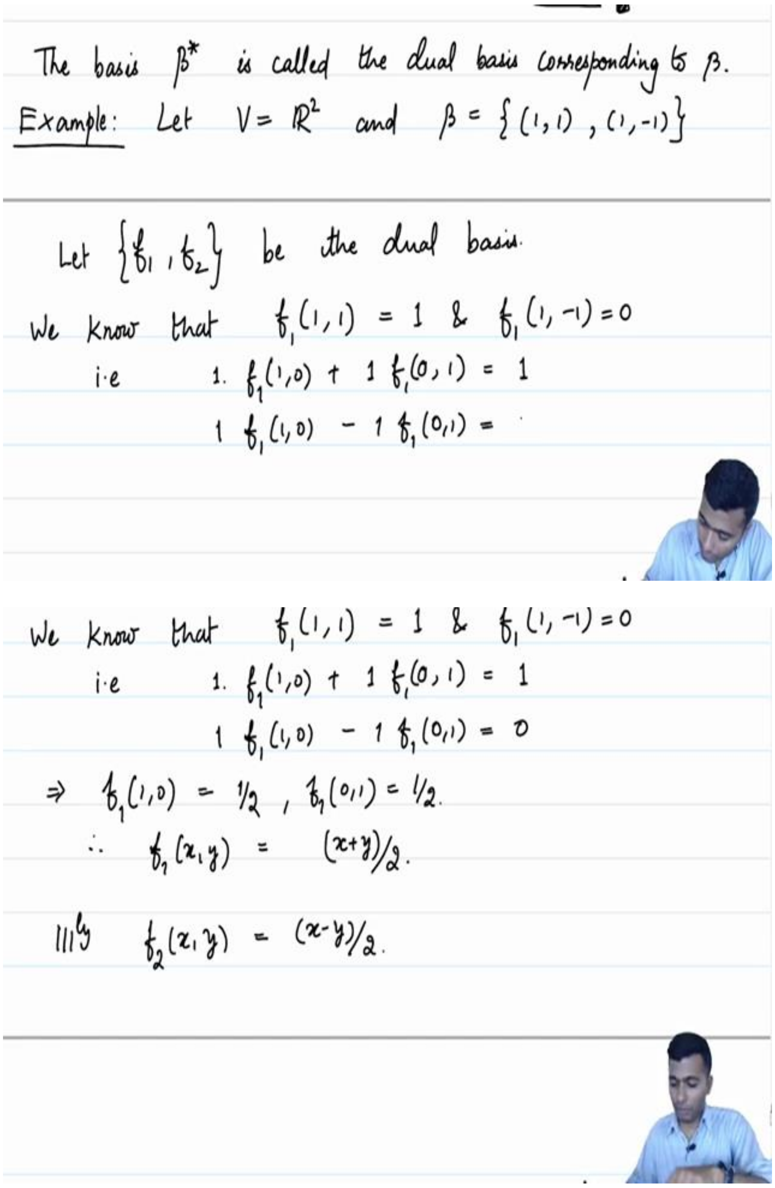




So suppose g is equal to summation f of vi times fi. F is a function, a functional, linear functional on v and therefore summation f of vi times f is a very valid honest element in v star. Let us call it g. We want to show that g is equal to f. What is the meaning of g is equal to f? i.e, g of v is equal to f of v for all v in capital V. This is what we would like to show.

But remember, recall that g and f are both linear maps. Recall that f comma g are linear. Therefore, it is enough to show that g of Vi is equal to f of Vi for every basis vector in beta because if we do that the linearity of f and g will tell us that g of v is equal to f of v for all v. But then what is g of vi? By definition g of vi, this is equal to summation j is equal to 1 to n, f of v j times fj, the whole at vi. This is exactly what g of vi is, but by definition again this is equal to summation j is equal to 1 to n, f of vj fj of vi.

(Refer Slide Time 22:30)



Let us now look at an example. So an example, let us consider the simple case of v being R2. So let v be equal to R2 and suppose beta is the ordered basis given by 1 comma 1, and 1 comma minus 1. v1 v2. So let f1 comma f2 be the dual basis. We would like to explicitly find out what f1 and f2 are.

So what we know is that f1 of 1 comma 1 is 0 and we know that f1 of 1 comma minus 1 is sorry, f1 of 1 comma 1 is 1, and f1 of 1 comma minus 1 is 0. Using this, let us try to see whether we can find out f1 entirely. So we know that f1 of 1 comma 1 is equal to 1, but f1 is a linear function, so in particular i.e, 1 times f1 of 1 comma 0, plus 1 times f1 of 1 comma, 1 0 comma 1 is equal to 1.

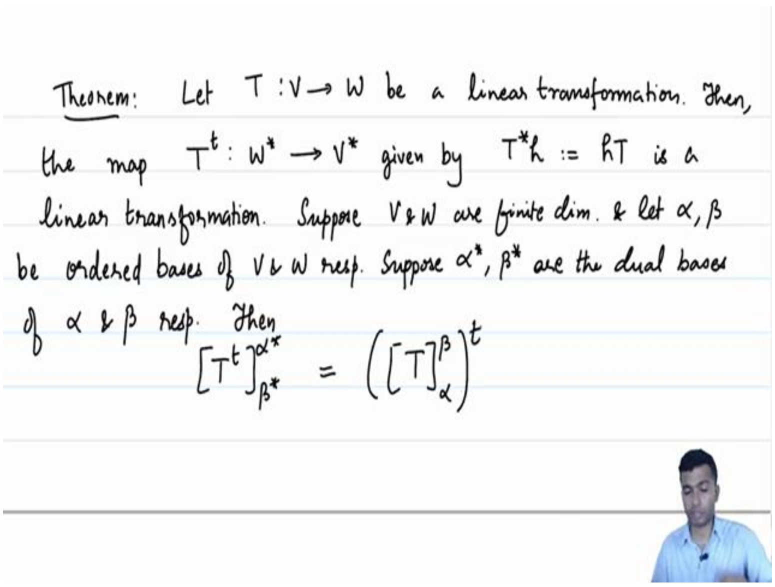
We also know that f1 of 1 comma minus 1 is 0, right? And that is telling us that 1 times f1 of 1 comma 0 minus 1 times f1 of 0 comma 1 is equal to 0. These two equations together tell us that f1 of 1 comma 0 is equal to half, f1 of 0 comma 1 is also equal to half. Similarly, you can check. Before we go into what f2 is, we now know what to do with f1 of x comma y.

f1 of x comma y is x times f1 of 1 comma 0 plus y times f1 of 0 comma 1, which is equal to x by 2 plus y by 2 which is equal to x plus y by 2.Similarly, it is an easy check to see that f2 of x comma y is x minus y by 2. You have seen something similar already and you see that many of these things are related. That is why when we consider 1, 1 and 1 minus 1 we are getting f1 and f2 like this.

So we have explicitly computed what is the dual basis corresponding to 1, 1 and 1 minus 1. Maybe in fact, it is a good exercise for you to check what the dual basis corresponding to the standard basis is. It will be a good exercise to get hold of concrete basis of R2 and see what the dual basis is. Now, let us consider a linear map from V into w. Let us call it T. Let T be a linear map from V into w.

We now know that corresponding to v, we have a dual basis v star; corresponding to w, there is a linear dual vector space w star. Can we somehow using t obtain a natural linear transformation between these two? That is going to be the content of the next theorem.

(Refer Slide Time 26:30)



So let us have a look at what the next theorem is. So let T from v to w be a linear transformation. So let us assume that okay, let us not assume as of now that it is finite dimensional. We do not need finite dimensionality yet. Then the map T transpose which is from w star to v star. Observe that T transpose is a map from w star to v star, not from v star to w star.

And what is this map? The map given by T star of let us say h, defined to be h composed with T is a linear transformation. Not that we have not assumed any finite dimensionality of V or w, but now let us consider the case when V and w are finite dimensional. Suppose, V and w are finite dimensional and suppose alpha and beta and let alpha comma beta be ordered basis of v and w respectively.

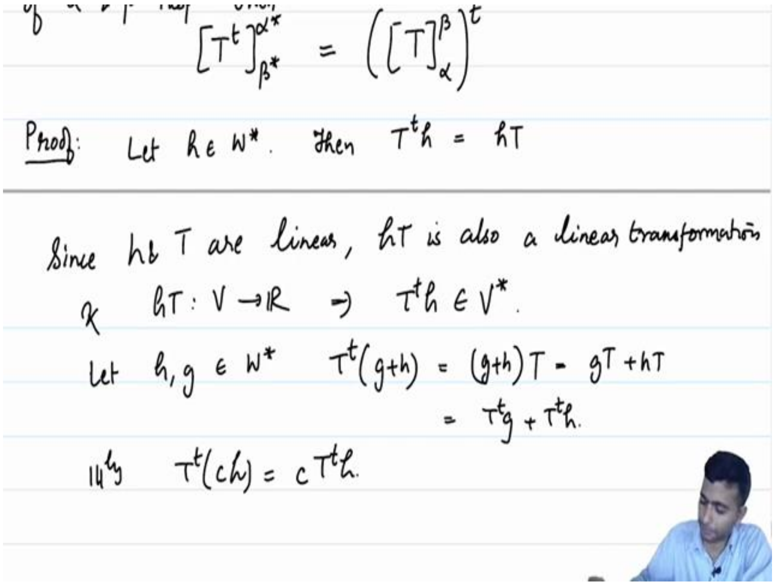
Now, given any ordered basis of a vector space, there is a corresponding dual basis as well in the dual space. So alpha star and beta star is the dual basis of alpha and beta. Suppose, alpha star and beta star or the dual basis of alpha and beta respectively, then the matrix of T transpose, so T transpose is from w star to v star. The dual basis in w star is beta star and the dual basis in v star is alpha star.

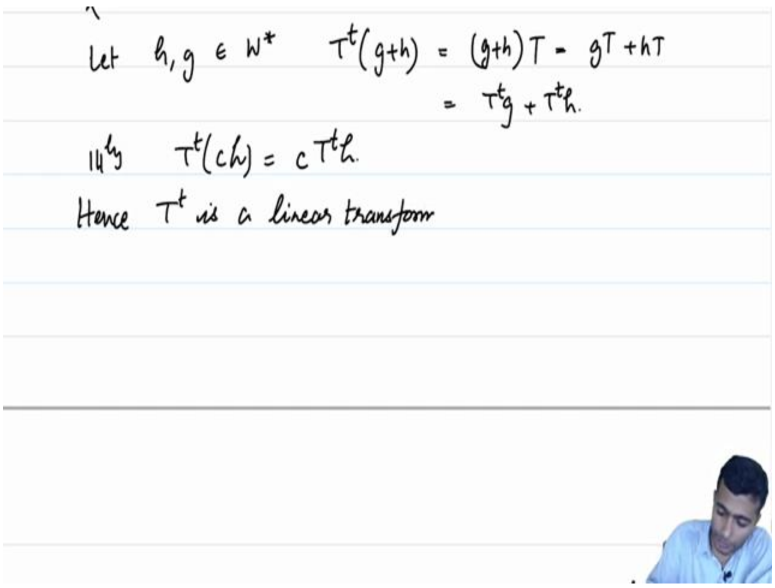
So this is a matrix corresponding to these basis. Now this, how is this related to our matrix of T? The relationship is quite simple, it turns out to be T transpose, transpose of the matrix and that is precisely why the map is being denoted by T transpose.

So let us give a proof of the statement. So let me show you the statement once more. The first part said that if you define T transpose as T transpose of h is equal to h composed with T, then it is a linear transformation from w star to v star. In fact, it is not even clear yet that T star maps w star to v star, but that will be clear in a moment, the moment we start writing down the proof.

And then further it says that if you fix two ordered basis, alpha, beta and let alpha star, beta star be the corresponding dual basis, then the matrix of T transpose with respect to beta star and alpha star will be the transpose of the matrix of T with respect to alpha and beta.

(Refer Slide Time 30:31)





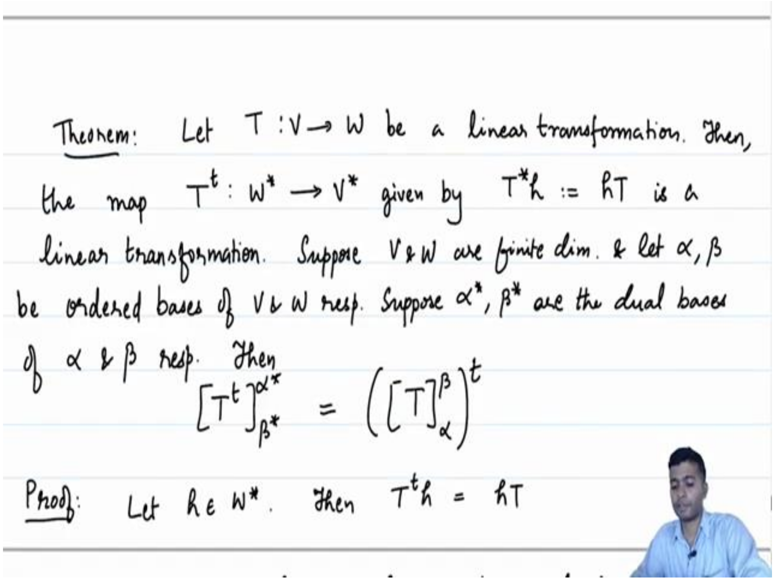
So let us first prove the first part of the theorem. So let h be an element of w star, then T transpose h is equal to h times h transpose T. It is actually h times T, it is a linear transformation, both are linear and since, h and T are linear, ht also linear. Composition of linear transformations is a linear transformation. Further ht maps w into R.

From linear transformation and ht is a map from w into R. This implies that T transpose of h belongs to w star. w into sorry, it is from V into R, therefore this belongs to, so note that t is a map from v to w and h is a map from w to R. So ht is a map from v into R. It goes through w though. T transpose h is an element of v star hence.

Let us now check that it is a linear transformation. We checked that T transpose h is a linear transformation, but why is T transpose a linear transmission from w star to v star? So that is actually a simple check. So let h comma g be in w star. Let us see what T transpose of g plus h is. In fact see g plus h is. I would have done it in one go. I have just proved the additive part. You can show the scalar multiplication also is linear.

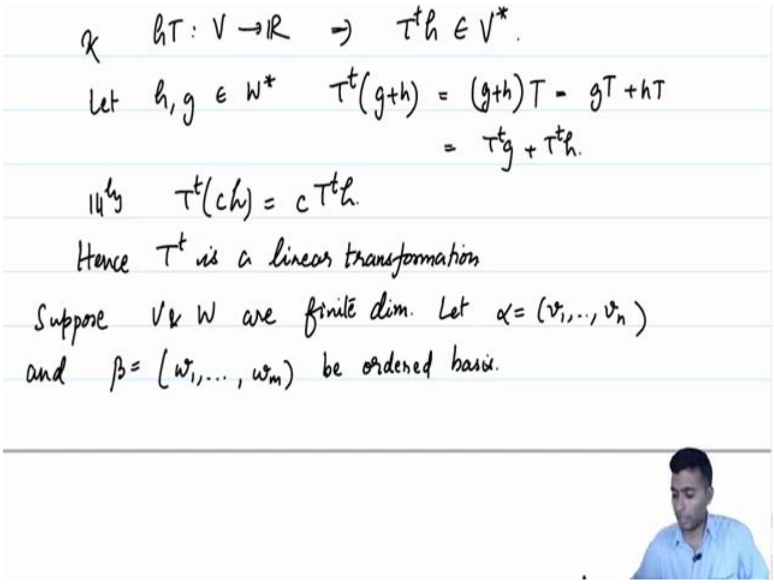
So this by definition is g plus ht but what is the meaning of g plus hT? It says gT plus hT, but gT is precisely equal to T transpose g and hT is precisely equal to T transpose h and that is precisely what we wanted to show. Similarly, T transpose of C times h is equal to c times T transpose h. So that tells us that T transpose is a linear transformation. So we have proved the first part of the theorem.

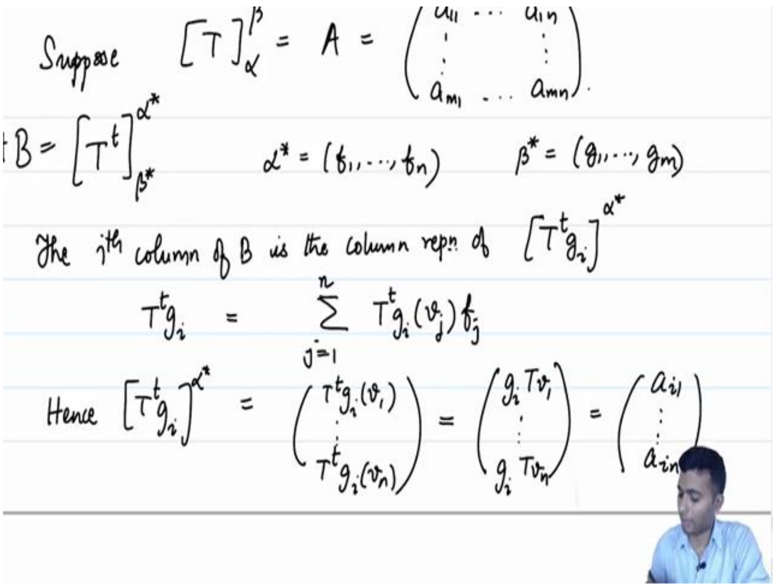
(Refer Slide Time 33:30)



Let me show you the statement of the theorem wants to recall what we were proving. We were proving that given any linear transformation T from v to w, T transpose from w star to v star is a linear transformation. So what we just showed was that T transpose indeed maps w star into v star. Not just that, we further showed that T star is indeed a linear map. Therefore, the first part, the part till here has been proved. Now, let us move on to the case when both are finite dimensional.

(Refer Slide Time 34:05)





So suppose, v and w are finite dimensional and let alpha comma beta be two basis, two order basis. Let alpha equal to say v1 to vn and beta be equal to W1 two wm be ordered basis and let us now consider the corresponding dual basis. So let alpha star be equal to f1 to it will be n of them because v is n dimensional by assumption and beta star be equal to g1 to gm.

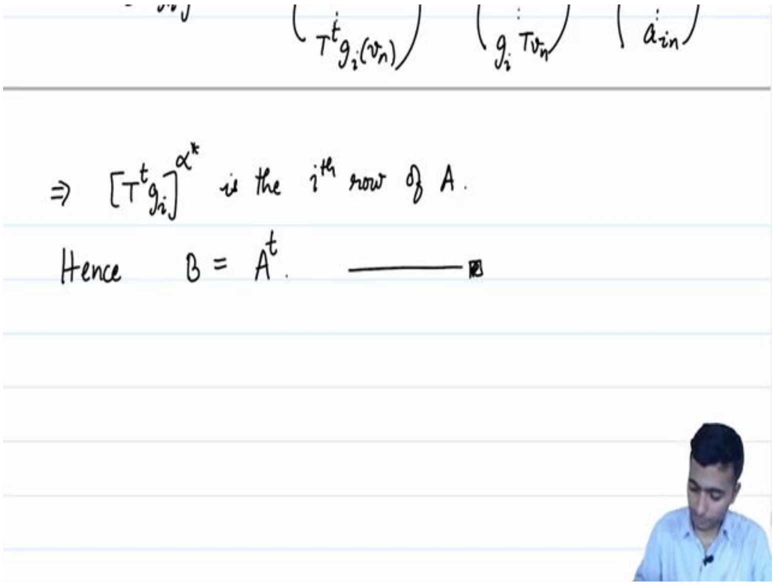
And suppose the matrix of T corresponding to alpha, beta be equal to a matrix A which is given by a11 to a1n, am1 to amn. So we would like to see what is, so we want to calculate what is T transpose beta star alpha star. This is what we would like to calculate, but what is our beta star and alpha beta star? Beta star is g1 to gm and alpha star is f1 to fm. So this is going to be an n cross m matrix for sure.

T transpose with respect to this has to be an n cross m matrix and the ith column of, so let us call it B, of B is the column presentation of T transpose gi with respect to alpha star. Remember, so let me just write it down here alpha star is f1 to fn and beta star is equal to g1 to gm. Now, what is T transpose of gi?

That is precisely what we would like to see. T transpose of gi we know from the previous theorem. This is equal to summation j is going from 1 to n, T transpose gi of vj and then fj the n vectors and this is, so hence what is T transpose gi with respect to alpha star? That is nothing but T transpose gi of v1, T transpose gi of vn. But by definition what is T transpose gi? It is giT.

This is equal to gi Tv1 up to gi Tvn but again what is gi? gi's are the dual basis corresponding to beta. So hence it is the ith coordinate with respect to w1, w2 up to wn. So this is going to give you the IC ith coordinate. So this is going to be your, ai1 up to ain. But what is ai1, ai2 up to ain?

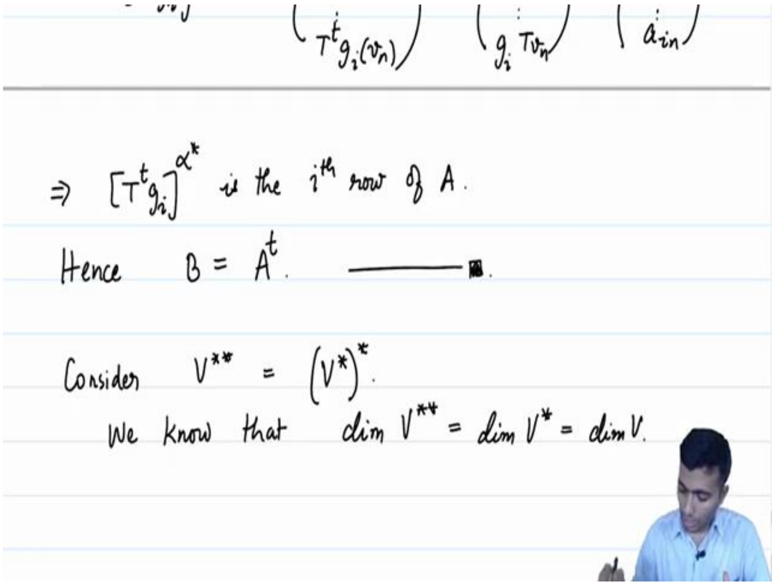
(Refer Slide Time 38:59)



That is the is the ith row of A, which implies T transpose gi alpha star is the ith row of A. Hence, the ith column of B is the ith row of A and the dimensions match. Hence, B is equal to A transpose. That is precisely what we were trying to prove. So, we have taken a vector space v, looked at its dual v star, and for every ordered basis beta, we gave a corresponding dual basis and there is a natural correspondence once we fix a given basis.

However, the correspondence between v and the dual of its dual is far more natural. So let us consider what I just said.

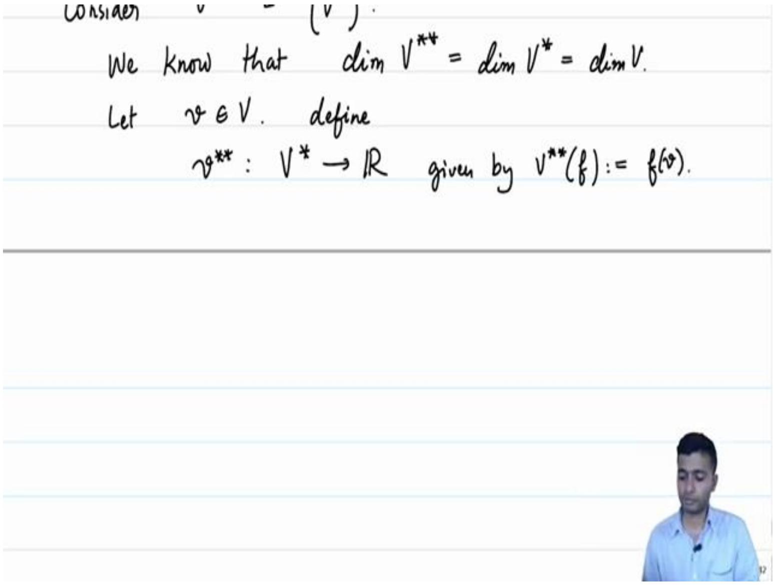
(Refer Slide Time 40:18)

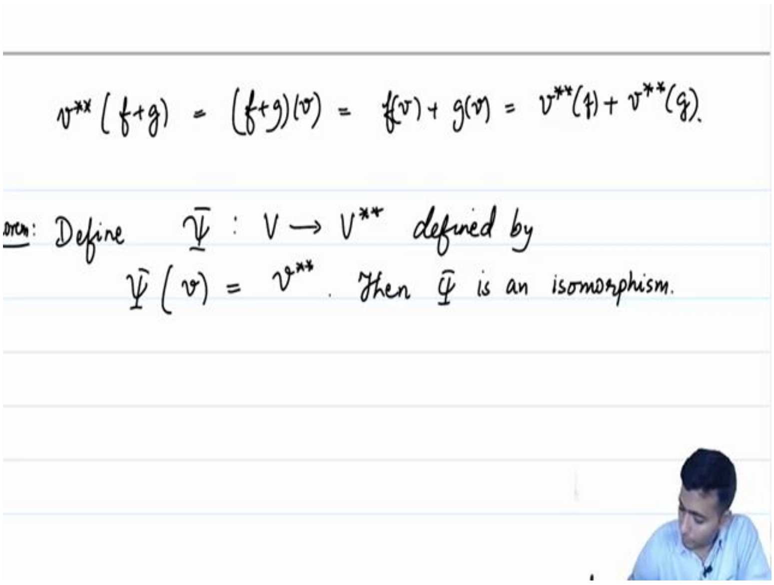


So consider, v star. Consider v star star in fact, which is the dual space of the dual space of v, which is the v star is the dual space and you look at the dual space of v. So in other words it is all the linear functionals on v star. So, what we will do is, so we know that dimension of V star is n and the dual of dimension, the dimension of the dual of v star will also be n. So we know that dimension of v star star is equal to the dimension of v star by what we have done.

And this is nothing but the dimension of B. Hence, we know that v star star and v are isomorphic to each other. The fact is that this isomorphism is far more natural than we would expect. In fact, we can talk about an isomorphism here without going down to a basis. Without referring to any basis we can talk about an isomorphism here.

(Refer Slide Time 41:35)

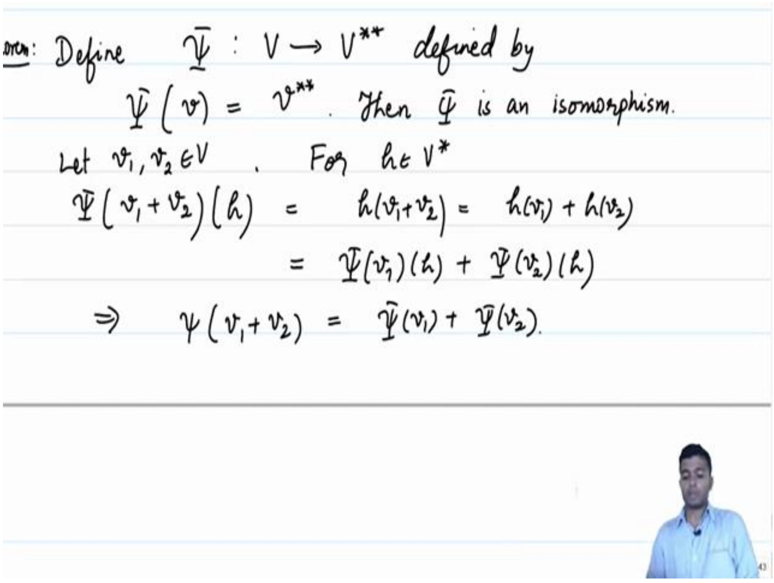




So let us start the groundwork for doing that. So let v be an element of capital V. Define, v star star, small v star star are mapped from v star into R given by v star star of f is equal to by definition, or is defined to be f of v. It is an easy check to see that v star star is a linear functional on v star. Let us just have a quick look at it, V star star of say f plus g is just going to be equal to f plus g of V, which by definition is equal to f of v plus g of v, which is equal to v star star of f plus v star star.

And similarly, v star star of cf is equal to c times v star. c times v star star of f. So let us now define a map. So define let us call it psi, from v into v star star defined by psi of v to be equal to v star star. Then so let me just call it a theorem. As you can see, to talk about psi, we do not even have to worry about whether, we do not have to refer to any ordered basis of v, directly from v, we are able to get hold of a, for every element v, we can talk about a corresponding element in the double dual, or v star star. So the theorem says that psi is an isomorphism. So to show that psi is an isomorphism first we need to check that psi is a linear map, so how to do that?

(Refer Slide Time 44:14)

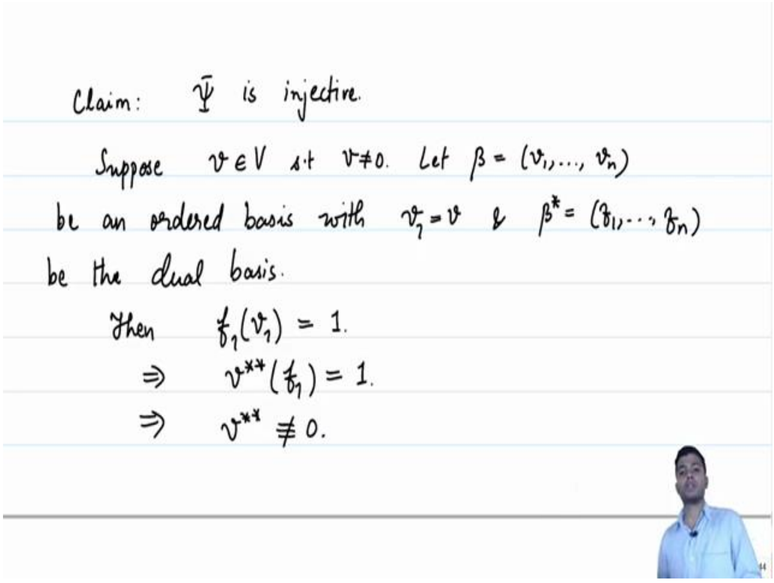


Let us consider v1 and v2, two vectors in capital V. Let us look at psi of v1 plus v2. What is this? This is basically, so this will be vector in v star star. So in other words, this will act on some element, h. So for h in v star, we would like to see what is psi v1 plus v2 of h. So this is by definition equal to v1. This is going to v1 plus v2 star star of h, which is just h of v1 plus v2, but h is a linear map, h is an element in v star and therefore, h of v1 plus v2 is h of v 1 plus v2.

But that is v1 star star of h1, sorry, v1 star star of h plus v2 star star of h which is just psi, v1 of h psi v2 of h, which hence implies that psi of v1 plus v2, yes, this is equal to psi v1 plus psi v2 of h. So, this is equal to psi v1 plus psi v2 of h. So psi hence is a linear map. You can check that the scalar properties also satisfied in a very similar manner, but we have only obtained linear map from v to v star star.

But notice that v and v star star, both have dimension equal to the dimension of V, let us say n. So if dimension of v is n, v and v star star both have dimension n, and we already know that if you have an injective map from vector space V to another vector space w of the same dimension, then by the dimension theorem, we can conclude that this is a surjective map. So let us see, let us try to conclude that psi is an injective map.

(Refer Slide Time 46:55)

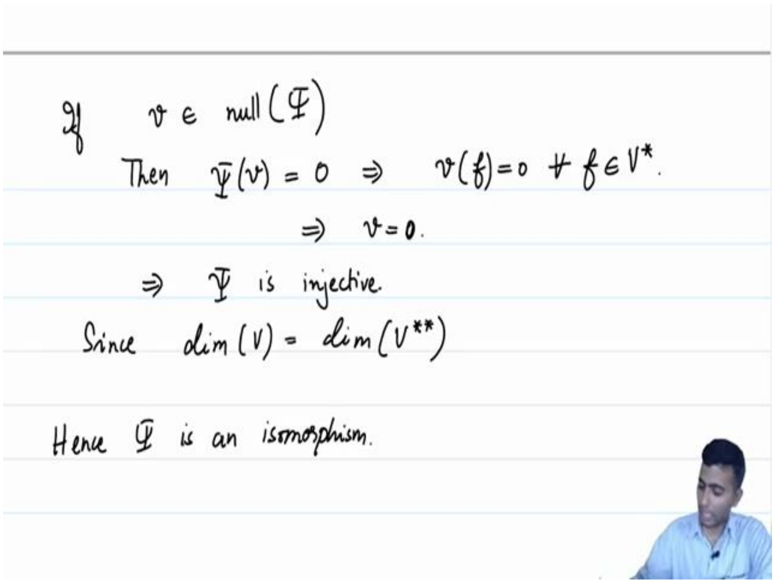


So claim, psi is injective, if we show this, notice that it is enough to show that psi is injective to show that psi is subjective as well, and therefore it turns out to be an isomorphism. Why is psi injective? Let us consider some element in the null space. So let psi of v be equal to 0, so you will show that that forces v to be equal to 0.

So suppose, v is an element such that v is not equal to 0. Suppose v is a non-zero element. And then this is the first time we will be going down to our basis in this proof. Let beta equal to v1 to vn be an ordered basis with v1 equal to v, extend v to a basis of capital B and beta star be equal to f1 to fn be the dual basis corresponding to beta.

But what is the property of the dual basis? If you consider f1, f1 satisfies some nice properties. Then f1 of v1 is equal to 1, but our v1 is V and f1 of v1 or f1 of V is precisely equal to v star star of f. This implies v star star of f1 is equal to 1, and what does this imply? The meaning of this is that this implies v star star is not equal to the 0 element, but what do we want? Why are we considering this? Why are we doing this?

(Refer Slide Time 49:01)



So if v belongs to the null space of psi, we want to show that psi is injective. So suppose v belongs to the null space of psi, then what do we have? Then psi of v is equal to, identically equal to or is equal to the 0 element of v star star, which implies that v of f is equal to 0 for all f in v star, but we just checked that if v is non-zero, there exist at least one element in v star where v of f is equal to 1, but that will not happen if, in fact, if v of f is equal to zero for all f, then it forces because of the observation we just did, this forces our v to be identically equal to or v to equal to the 0 vector.

This implies that psi is injective because null space being the 0 vector space (())(50:09) psi is injective, since dimension of v is equal to dimension of v star star. Argument using the dimension theorem which we have already proved shows that the image of psi or the range of psi has dimension n and v star star also has dimension n, therefore the range has to be equal to v star star and hence subjective.

What do we know about injective and subjective or bijective linear transformation? We know that it is invertible and hence an isomorphism. Hence psi is an isomorphism. So indeed, when proving that psi is injective we did use a basis, an ordered basis and considered its dual, but if you observe carefully, this correspondence between v and v star star does not refer to any ordered basis. There is a very natural correspondence between v and v star star. So this is a notion which is quite essential when we do more analysis of function spaces.