Linear Algebra
Professor Pranav Haridas
Kerala School of Mathematics, Kozhikode
Lecture 5.2
Product of Vector Spaces

So, given two sets x and y we are familiar with the notion of the Cartesian product of x and y. It is the collection of all tuples x, y where x is in capital X and y is in capital Y. So, we would like to ask this question, if we start off with vector spaces v and w instead of arbitrary sets, we can still take the Cartesian product of V and W, can we associate a vector space structure to V cross W which directly or which is dependent on the vector space structure of V and the vector space structure of W respectively.

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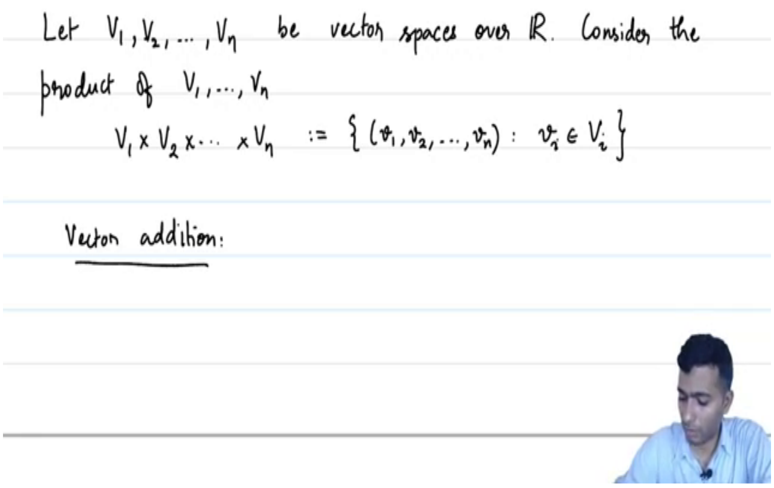

So, let us begin by associating some such thing so, do that let us begin by considering vector spaces V1, V2, up to Vn. Let V1, V2, up to say Vn be vector spaces over R all of these are vector spaces over R, we are not demanding any extra restrictions on these V1, V2, up to, Vn could be any vector space. Consider, the Cartesian product of V1 to Vn it is just the product of sets as of now given by V1 cross V2 cross Vn which by definition is the set of all in tuples V1, V2, up to Vn where vi belongs to capital Vi.

So, the ith component in this tuple is a vector from Vi. So, we would like to ask, we would like to ask whether we can associate addition and scalar multiplication to this Cartesian product and give it a vector space structure. So, we know that each of these Vi's are the vector spaces and that there is a scalar multiplication, there is an addition operation, there is a scalar multiplication in each of these Vi's. We are going to use that information to get hold of new operation on V1 cross up to Vn and give it a vector space structure.

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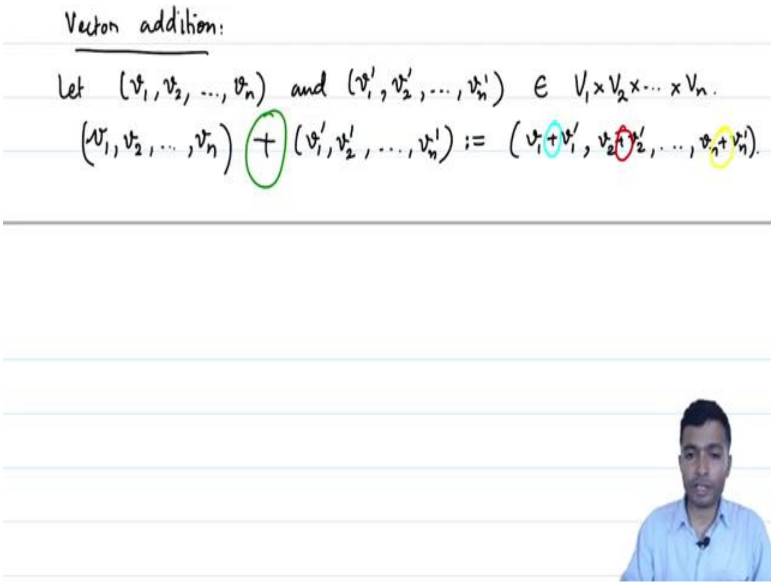

So, let us try to define the addition, vector addition. So, let us take two arbitrary vectors so, let V1, V2, up to, Vn and V1 prime, V2 prime, up to Vn prime be two different vectors, be vectors or elements in as of now there are no vectors, because we have not given any vector space structure to it. So, be two elements so, let me just use the shorthand notation be elements of V1 cross V2 cross Vn our goal is to somehow give an addition operation on V1 cross V2 cross up to cross Vn.

So, not up to do that we should be able to say what is these vector sum of this vector V1 to Vn and the vector V1 prime to Vn prime. The addition operation is the natural addition operation that comes to our mind initially which is namely this plus this V1 prime to Vn prime, this is being defined that to be V1 plus V1 prime, component wise V2 plus V2 prime, up to Vn plus Vn prime. So, I would like to draw your attention to a few things.

So, let me use some colors the green which is being circled is the operation which we are defining, the blue circle is the operation which we know from capital V1, the red is the operation in capital V2, the yellow is the operation in capital Vn. So, as you can see we are using the same plus notation everywhere the use of notation is a quite evident however, the situation is clarifying which operation is being used where the fact that this operation, this yellow one which I will just underlined is the operation that is being defined is quite clear from the context and we will be making use of this clarity in order to define such things.

Alright so, we have defined V1, V2, up to, Vn plus V1 prime, V2 prime, up to, Vn prime to be component wise.

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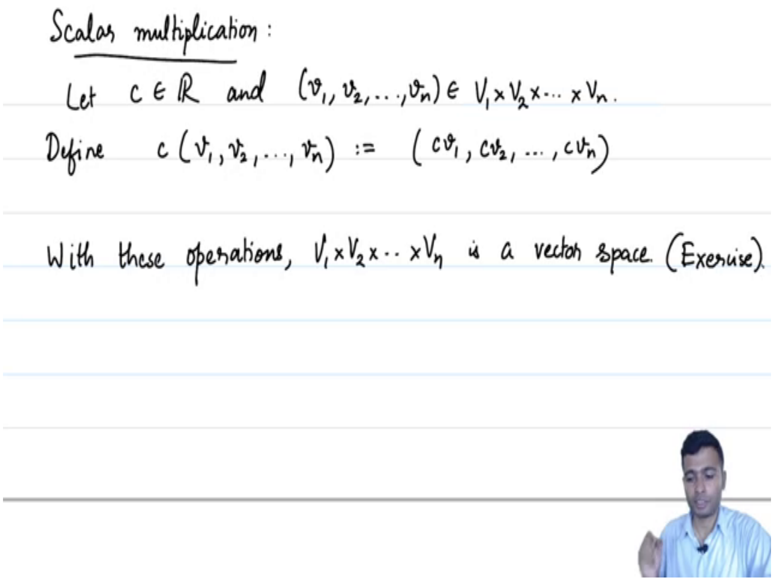

Similarly, we will define scalar multiplication also component wise, scalar multiplication, scalar multiplication needless to say these are closed operations both but, it is something which we should certainly check V1 plus V1 prime is going to be an element of V1 again because, V1 is closed in vector addition and similarly for all components.

So, now let us define scalar multiplication again this is an operation which will take a scalar, let c be an element in R and V1, V2, up to, Vn be an element in we would like to define c times V1, V2 up to Vn define c times V1, V2 up to Vn to be this is by definition equal to or rather we are defining this to be equal to cV1, cV2, up to cVn again I would like to draw your attention to the various operations that are involved in this, the operation here is the operation we are defining the operation here, here and here are operations which we already know respectively in V1, the scalar multiplication in V2, and the scalar multiplication in Vn.

So, we are using that to define the new operation on V1 cross V2 cross V3, up to Vn and we have defined the scalar multiplication and again a similar argument as above tells us that this operation is closed or to put it no, let me make it precise V1 cross V2 up to, Vn is closed under the scalar multiplication which we have just defined with these operations V1 cross V2 cross up to Vn is a vector space.

So, I will leave that as an exercise for you to check the various properties of the vector space the definition of a vector space and convince yourself that this a vector space at this stage I will leave this as an exercise, this is an exercise however you should certainly sit and work this exercise out to convince yourself that this a way.. There are two things however which I would like to draw your attention to, what will be the identity, additive identity of this vector space? The additive identity of V1 cross V2 up to Vn should be V tuple 0000 where each of the zeroes are the identity in the corresponding vector space.

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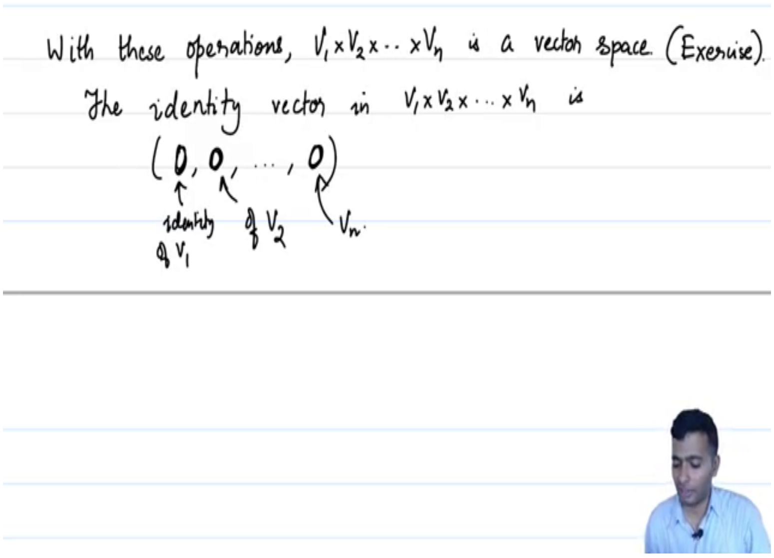

So, the identity operation not identity operation, identity vector the additive identity vector this additive identity that we are discussing in V1 cross V2 cross this is additive identity by the way Vn is 0, 0, 0 I would like to stress upon the fact that this is the additive identity of V1, this is of V2 and this is additive identity of Vn you should check that certainly this works. Similarly, one more thing I would like to draw your attention to is the inverse.

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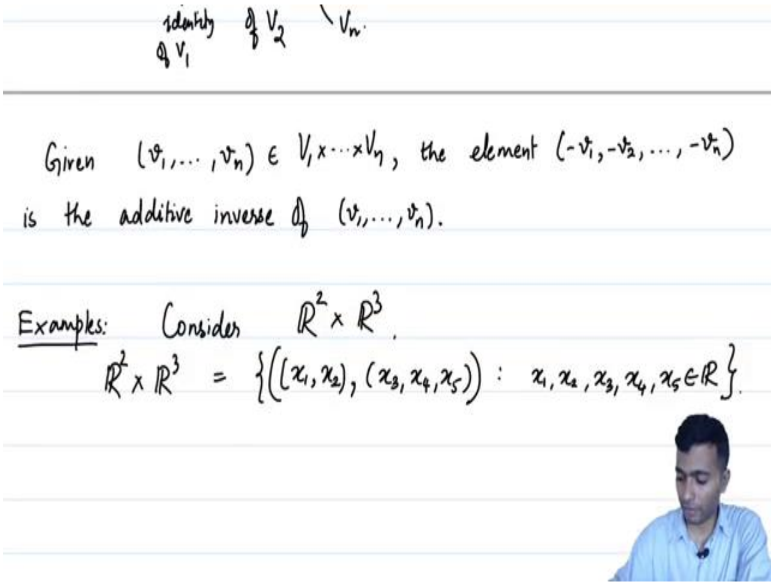

So, given say v1 to vn in capital V1 cross up to capital Vn, recall that there exists an additive inverse of V1, V2, up to Vn if it all it has to be a vector space, my claim is that the vector of the element this is all be vector only once it is a vector space, so as of now we are still in the process of establishing that this is a vector space even though I have left it as an exercise. The element minus V1 minus V2 up to minus Vn is the additive inverse of V1 to Vn, note that minus of v1 is the additive inverse of v1 in capital V1 similarly, minus v2 is the additive inverse of v2 in capital V2.

So, as you can see we are using the vector space operations in V1, V2 up to Vn respectively, extensively to talk about the vector space structure on V1 cross V2 up to Vn. Alright so, let us look at may be a couple of examples, well consider R2 cross R3 first two examples that came to my mind. So, let V1 we have to and V2 to be R3, let us just consider product of two vector spaces.

So, by what we have just described, what will be this set R2 cross R3 will just be equal to the set of all x1, x2 and then x3, x4, x5 this is an typical element in the tuple of such elements where x1, x2 belongs to R2 which is same as demanding that x1, x2, x3, x4 and x5 are all in R.

One will be tempted to tell that R2 cross R3 which we have just defined is nothing but R5 but that is not technically correct R2 cross R3 can certainly be identified with R5 by or through the notion of an isomorphism which we have defined, but it is incorrect to say that R2 cross R3 is equal to R5, this is after all a tuple of vectors in R2 and vectors in R3, it is not the same as giving a vector in R5 we will make that precise.

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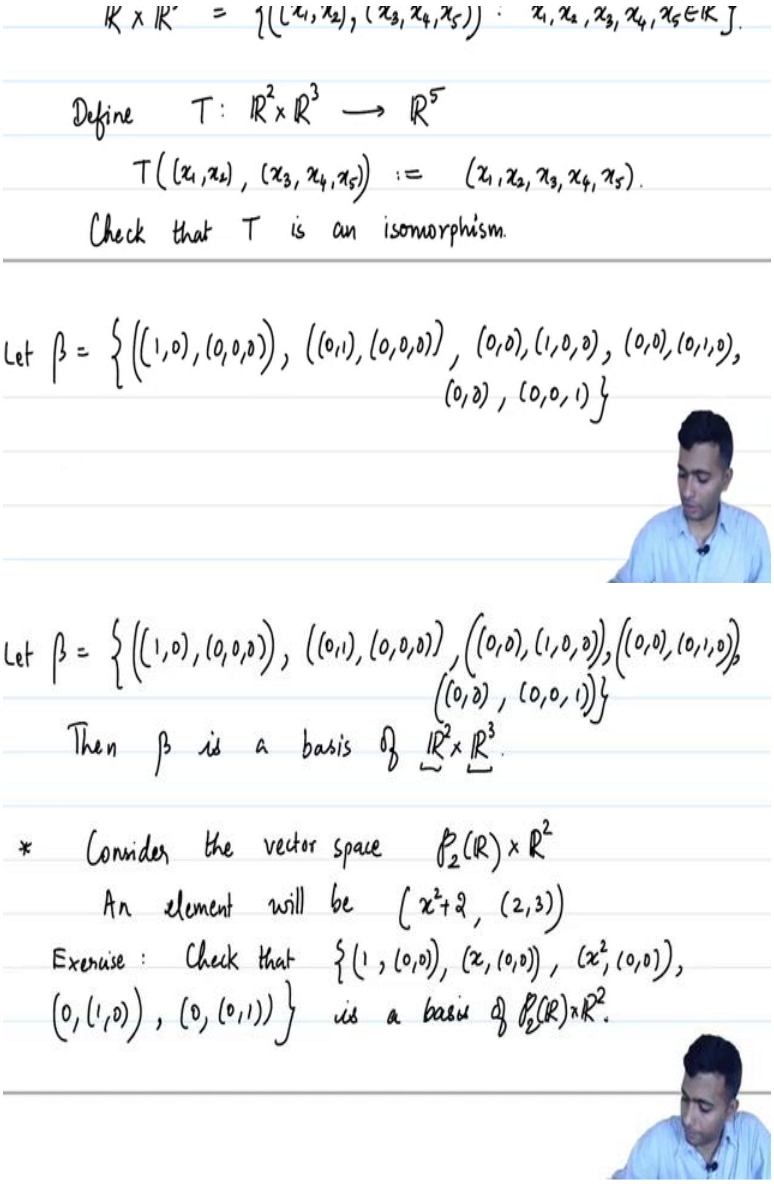

Define T from R2 cross R3 to R5 the first natural map that comes to our head which is T of x1, x2 and x3, x4, x5 to be equal to, this is defined to be x1, x2, x3, x4 and x5 define T to be this particular map from R2 cross R3 into R5, it is very easily checked that T is a linear transformation and that it is both injective and surjective therefore, it is a isomorphism.

So, I will just leave that as an exercise, check that T is an isomorphism. So, as already notated an isomorphism is something which essentially helps us and identifying to different vector spaces. So, in particular one of the results from the previous week tells us that the dimension should be the same and that is to be expected and suppose, so if e1, e2 is the standards basis or may be let us write down explicitly what the basis here will be so, the first one will be 1, 0 0, 0, 0 then 0, 1, 0, 0, 0 and then 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1 this is essentially obtain by pulling back the basis from R5 to R2 cross R3 through the linear transformation T this, let us call it something beta, let beta be this set, then beta is a basis of R2 cross R3.

If you carefully observe, we have just pulled back the standard basis of R5 which contains five vectors I have not put brackets in certain places, let me do that otherwise it makes no sense. So, there are five elements here and these are essentially obtained by pulling back the standard basis elements.

Let us observe that the dimension of R2 cross R3 is 5 here that is also the dimension of R2 plus the dimension of R3 the vectors 1, 0, 0, 0, 0 and 0, 1, 0, 0, 0, are obtain from the basis in R2 and the remaining 3 are obtain from the basis in R3, we will make that precise in a moment it is just an observation however, let us look at one more example while just considered R2 and R3.

Consider, the instead of R2 let us consider may be consider the vector space P2 of R instead of R3, let us put P2 of R cross R2. So, what will be a typical element? An element will be something of this type a tuple of this type, may be x square plus 2 and 2, 3 this is how a typical element in P2 of R cross R2 will look like, again I will leave as an exercise for you to check that the following will turn out to be a basis of P2 of R cross R2. Check that 1, 0, 0 x, 0, 0 x square 0, 0 0, 10 and 0, 0, 1 is a basis of P2 of R cross R2.

No surprise is here, we know that 1 x, x square is a basis of P2 of R and also know that 1, 0 and 0, 1 is a basis of R2 and we have use that information to conjecture that this will be a basis, off course it is an exercise for you to check that it is a indeed a basis it thus, span and its indeed linearly independent.

Again if you closely observe dimension of P2 of R as 3 we have used the basis of P2 of R and the basis of R2 to obtain a new basis. So, that leads us to conjecture the following proposition namely that if you have V1, V2 up to Vn.

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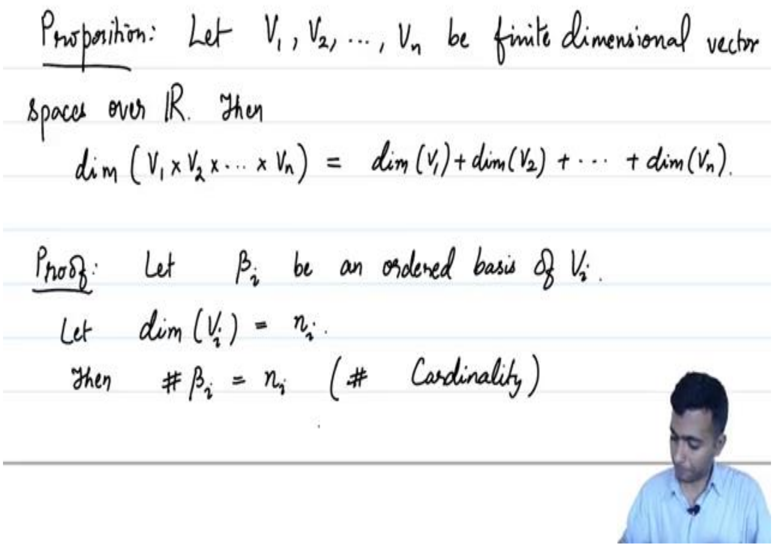

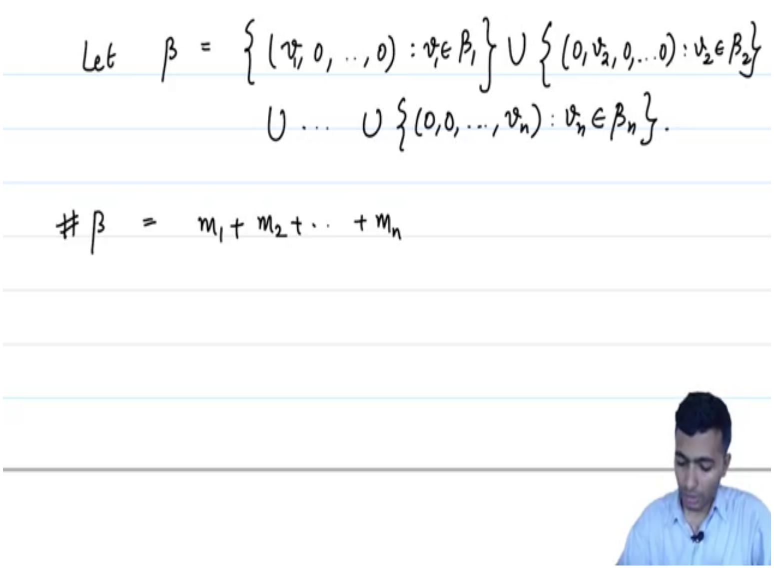

So proposition, let V1, V2 up to Vn be finite dimensional vector spaces. I would like to draw your attention here to the fact that when we defined the product, we were not demanding anything about finite dimensionality we could have started off with infinite dimensional vector spaces, talked about vector addition in this manner, scalar multiplication in this manner and we would have still got a vector space V1 cross V2 up to Vn.

So, to talk about product of vector spaces, we really do not need anything about finite dimensionality. However, in order to talk about dimension, we need to have a finite if it is infinite dimensional, then we not interested in talking about the dimension per say. So, we will restrict our attention to finite dimensional vector spaces in order to talk about dimension this proposition exactly talks about the dimension of the product of V1 cross V2 cross up to Vn when each of these Vi is a finite dimensional.

So, then dimension of V1 cross V2 cross Vn is equal to dimension of V1 plus dimension of V2 plus dimension of Vn, basically the dimensions add up. Let us give quick proof of this, so we know that V1, V2 up to Vn are finite dimensional, so let beta i, be an ordered basis of Vi. So, we know that beta i has size dimension of Vi.

So, let us give it a name let dimension of Vi be equal to ni, then the size of cardinality of beta i, is equal to ni. So, by hash I have just used it to capture cardinality of the set beta i, or the size of the set beta i, the cardinality of beta i, is equal to ni just give it a name, we will define a basis beta of V1 cross V2 cross up to Vn, you seeing these basis vectors beta i.

Let us see how we do that, let beta be all the sets of this type v cross v, 0 up to 0 union such that v belongs to, so let me call it v1, v1 belongs to beta 1 union 0, v2, 0, 0 such that v2 belongs to beta 2, union 0, 0 up to say vn where vn belongs beta n. So, notice that each of these are a copy of beta i, with the 0 vector in the remaining components and because of that cardinality of beta is nothing but cardinality of beta 1 which is m1 plus the cardinality of beta 2 which is m2 up to the cardinality of beta n which is mn and if we manage to proof that this beta is indeed a basis of v1 cross v2 cross up to vn.

Then we would established that the dimension of v1 cross v2 cross up to cross vn is dimension of v1 plus up to vn, that is precisely m1 plus m2 plus up to mn.

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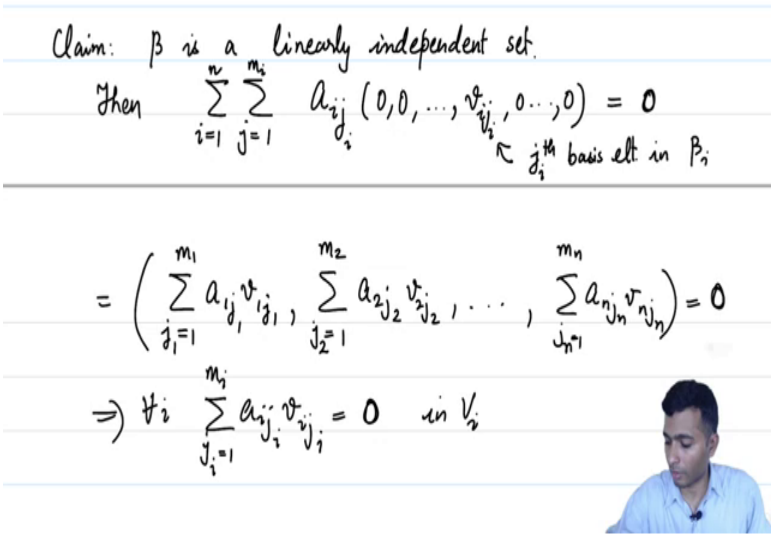

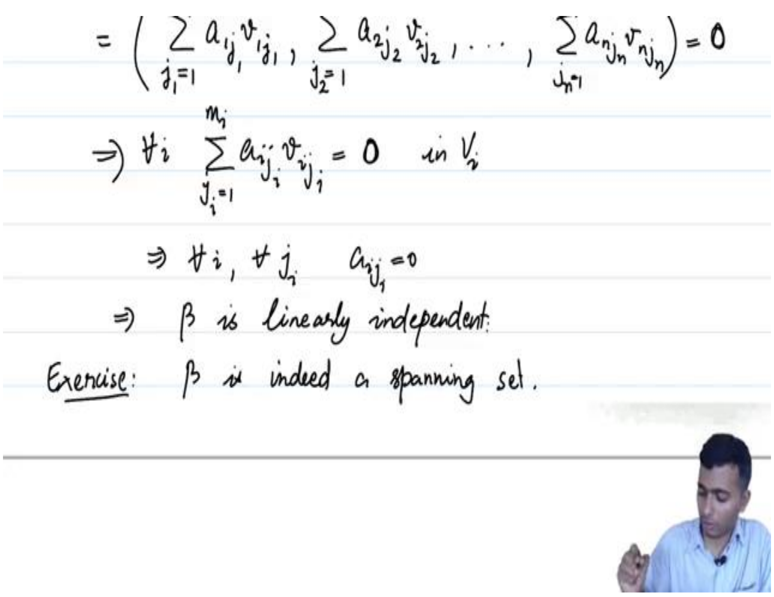

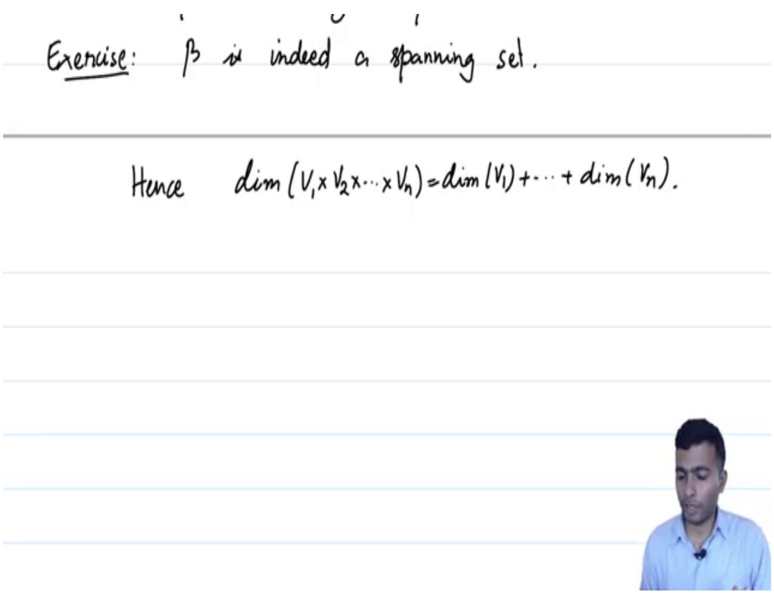

So, how do we see that beta is a linearly independent sets, so let us starts with that claim beta is a linearly independent set. So, suppose there is a linear combination which is equal to 0, then what would happen is we will be having a relation of the following type, then summation ai j I will not may be the indices should be a bit more carefully chosen, so let me carefully look at to the choices here, then we can write it like this summation i, equal to 1 to n, j equal to 1 to mi, ai ji of let me call the vector, the jth we write it like this v may be i, ji this captures all the basis elements.

So, this what is this? vi ji it is the jith basis element in beta i. Alright, but then the scalar multiplication and the vector addition is define in such a manner that this above this is equal to 0. Let us consider some such relation which is equal to 0. our goal is to show that each of these ai ji' s are 0 but, these above can be rewritten using the properties of the vector addition or the product space which we just defined this is equal to summation j1 going from 1 to m1 a1 j1 of v1 j1 summation j2 is going from 1 to m2 of a2 j2 v2 j2 and so on.

Summation an jn where jn is going from 1 to mn this times vn jn but, this is equal to 0 implies that each of the things inside, each of the components are individually 0, this implies that for all i ai ji where ji is going from 1 to mi vi ji is equal to the 0 vector in vi and we know that beta i is linearly independent set which implies for all i, for all ji, ai ji is equal to 0 by the linear independence of beta i, and this implies that beta is linearly independent.

The spanning property of beta can be proved very similarly the indices have to be kept track of very carefully, let me just me leave as an exercise for you to check that beta is indeed a spanning set as well, beta is indeed a spanning set and with this dimension of v will turn out to be the size of beta which is equal to what is v here by the way it is V1 cross V2 cross up to Vn is equal to the dimension of V1 plus dimension of V2 to dimension of Vn.

So, in the next video we will discuss how more vector spaces can be obtained from existing vector spaces by considering another notion what is called as the Quotient of vector spaces. So let me stop here.