Linear Algebra
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Lecture - 5.1
Change of Basis

So, in the last week we discussed how given a vector space v and an ordered basis of v we could associate column corresponding to every vector. We also saw that if v and w are two vector spaces and if we fixed two basis of v and w respectively, then every linear transformation can be associate to a matrix. So, we begin this week by discussing how this association depends on the basis in other words we would like how the column changes, the column representation changes when we change the ordered basis with respect to which we are looking at it or how the matrix of the transformation changes when we change this basis.

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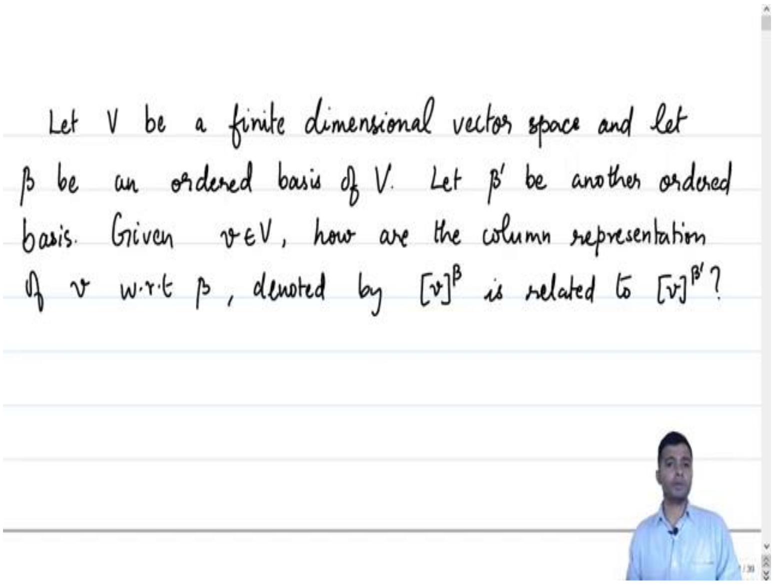

So, let us begin by considering a vector space v so, let v be a finite dimensional vector space and let beta be an ordered basis, beta be an ordered basis of v. Let us, now consider a new basis beta prime, let beta prime be another ordered basis. So, let us ask the right question given a vector v in capital V we would like to see how are the column representations of v with respect to beta which we already denoted by v beta is related to the column representation of v with respect to beta prime this is what we would like to first explore.

So, in order to study the relationship between v beta and v beta prime what we will do is we will consider the most natural linear transformation we can think of which is the identity linear transformation.

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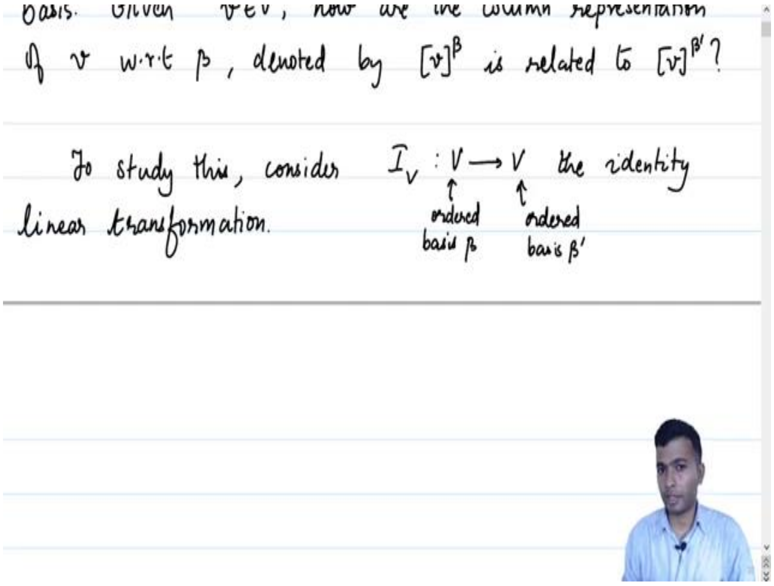

So, to study the above, to study this consider Iv which is from v to itself the identity linear transformation, identity linear transformation. So, what we will do is we will consider the domain v with the basis beta, so here, let us consider v with ordered basis beta and this one is the ordered basis beta prime, it is considered with the ordered basis beta prime.

So, remember v and w we have to fix a basis for v, we have to fix a basis for w and then we would like to locate the matrix corresponding to these two basis and that precisely what we will do here, we would like to see how the matrix of identity with respect to beta and beta prime look like.

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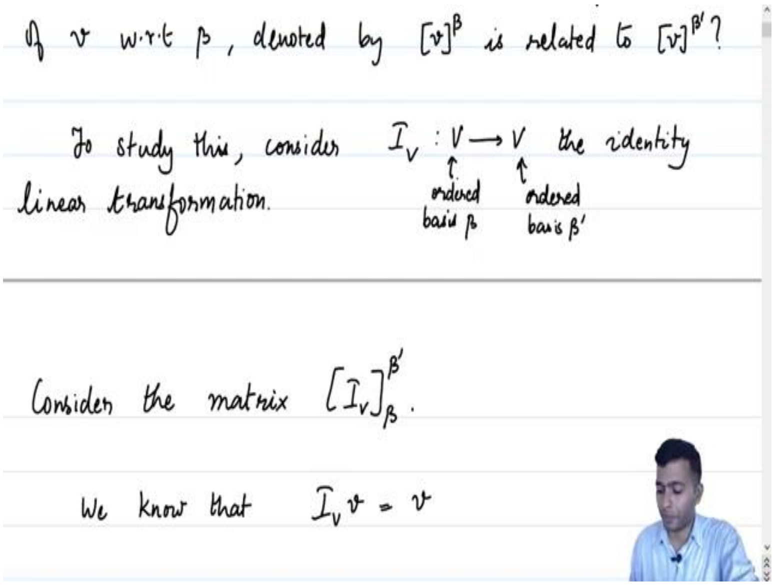


Consider the matrix $[I_V]_\beta$.

We know that $I_V v = v$

Then $[I_V v]^{\beta'} = [I_V]_\beta^{\beta'} [v]^\beta$.

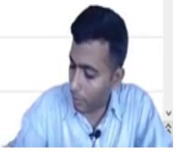
$\Rightarrow \quad [v]^{\beta'} = [I_V]_\beta^{\beta'} [v]^\beta$.

The matrix $[I_V]_\beta^{\beta'}$ is called.

The matrix $[I_V]_\beta^{\beta'}$ is called the change of basis matrix from β to β'.

Proposition: $[I_V]_\beta^{\beta'}$ is invertible.

Proof: The matrix corresponding



Proof: The matrix corresponding to an invertible linear transformation is invertible. I_V is an inv. lin. trans.

Thus $[I_V]_\beta^{\beta'}$ is hence an invertible.

* If $\dim V = n$, then $[I_V]_\beta^{\beta'}$ is an $n \times n$ matrix

So, consider the matrix Iv with respect to beta and beta prime. So, let me just give the definition for it or maybe I will just tell you how this matrix is important and then let me give you the name of this matrix. So, we know that Iv of v is equal to v after all Iv is the identity transformation but, think of Iv as a linear transformation from v to itself in the above set up, where the domain v is with the basis beta and the co-domain or the target v is with the basis beta prime.

Then, whatever the theory we have developed in the last week tells us that Iv of v with respect to beta prime, remember this is in the co-domain, this is in the range this is equal to Iv beta, beta prime times v beta, Iv is after all a linear transformation but, what is Iv of we? This gives v beta prime is equal to Iv beta, beta prime times v beta.

We have found out explicitly what the relationship between v beta and v beta primes, it is related by the matrix multiplication of the matrix Iv with respect to beta and beta prime this matrix, the matrix Iv beta, beta prime is called the change of basis matrix, the change of basis matrix from beta to beta prime.

So, immediately let me give you an proposition Iv beta, beta prime is invertible it is an invertible matrix, why is this invertible? we have already seen that if we have a linear transformation which is invertible the matrix corresponding to the linear transformation should also be necessarily be invertible and we know that identity is a invertible linear transformation it is the inverse of itself.

So, the matrix the reason is that matrix corresponding to invertible linear transformation with respect to any basis is invertible you fix any two basis and if you look at the matrix of invertible linear transformation, we have proved that it is invertible and identity is an invertible linear transformation and thus Iv from beta to beta prime is hence an invertible matrix, what else can we observe?

Observe at Iv beta, beta prime is an n cross n matrix where, n is the dimension of v if dimension of v is equal to and, then Iv beta, beta prime is an n cross n matrix, we should be able to say more, I leave it as an exercise for you to check may be I will just do it, I will leave it as an exercise for you to check that the change of basis matrix from beta prime to beta is exactly v inverse of the change of basis matrix from beta to beta prime.

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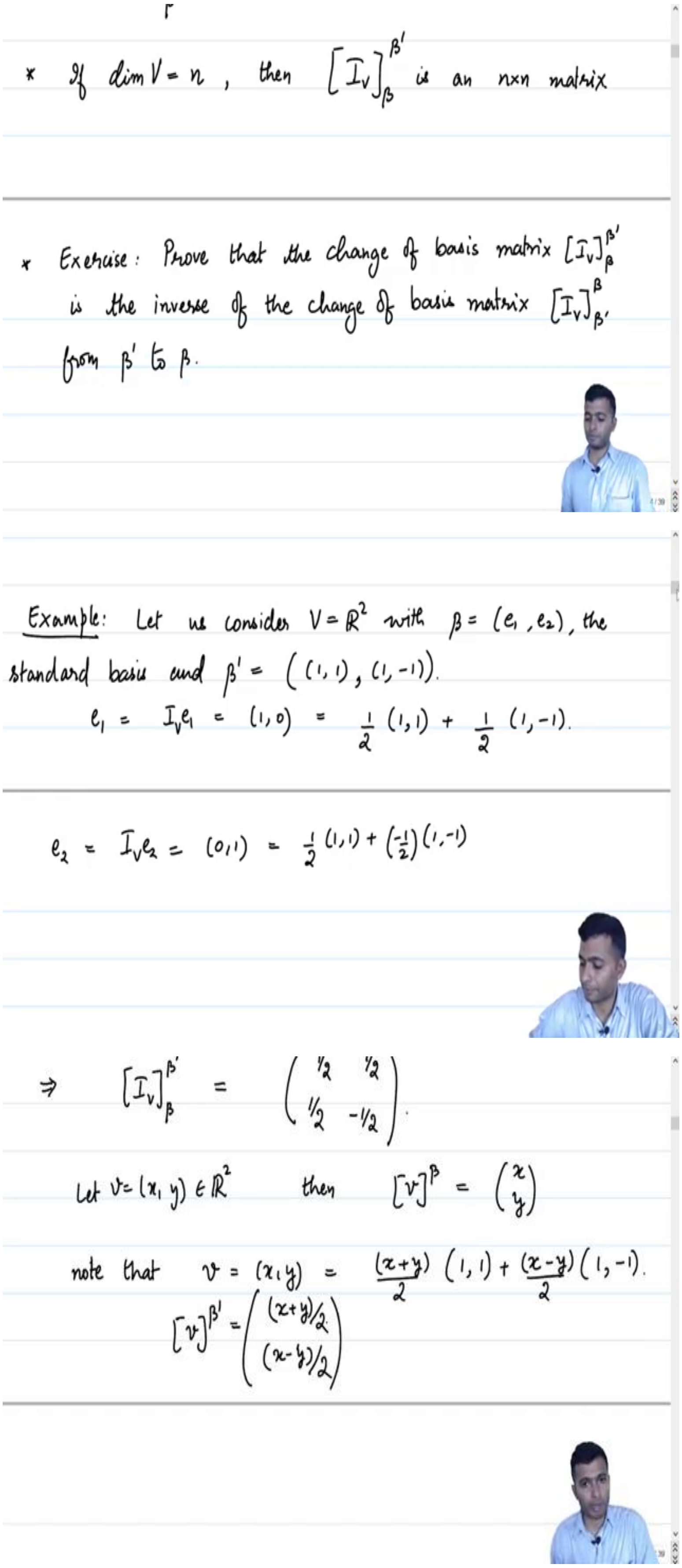

Moments thought should be view that the change of basis matrix Iv form beta to beta prime is the inverse of the change of basis matrix Iv beta prime to beta from beta prime to beta. So, we have explored how the change of basis matrix gives us the relationship between the column vector representation of a vector with respect to beta and the column representation of a vector with respect to beta prime.

Let us look at an example, so an example let us consider the simplest example, may not the simplest but a simple example, let us consider v to be just R2 with beta the standard basis e1 which is 1, 0 and e2 which is 0, 1 the standard basis and beta prime be the vectors may be 1, 1 and 1, minus 1. Let us try to see what the change of basis matrix is. The change of basis matrix is calculated by evaluating what the identity matrix is or at e1 and at e2, they will give the columns with respect to 1, 1 and 1, minus 1.

So, observe that e1 which is equal to Iv of e1 is equal to 1, 0 which is equal to half of 1, 1 plus half of 1, minus 1. So, the first column will just turn out to be half and half e2 on the other hand which is equal to Iv of e2 which is 0, 1 this is equal to half of 1, 1 plus minus half of 1, minus 1. So, these two calculations tell us that Iv of beta of v of identity with respect to beta and beta prime is nothing but 1 by 2, 1 by 2, 1 by 2, minus 1 by 2 that is good.

So, let us now see given an arbitrary vector, let us see by brute force how Iv beta is and what v beta prime is and also let us see whether they are related in the manner we have just seen, so of course be the same not able look at it in brute force. So, let x, y be some vector in say R2 this is clearly equal to then there is no confusion on what v beta is, then v beta by very definition is x, y this is after all x times e1 plus y times e2.

So, note that v which is equal to x, y this is equal to x plus y by 2 times 1, 1 plus x minus y by 2, 1, minus 1 (())(14:36)this is what our x, y is with respect to 1, 1. So, v beta prime is x plus y by 2 x minus y by 2. So, 1 is x, y so if say v is 2, 3 v beta will be 2, 3 here and if v is 2, 3 then v beta prime will be 5 by 2 and minus 1 by 2.

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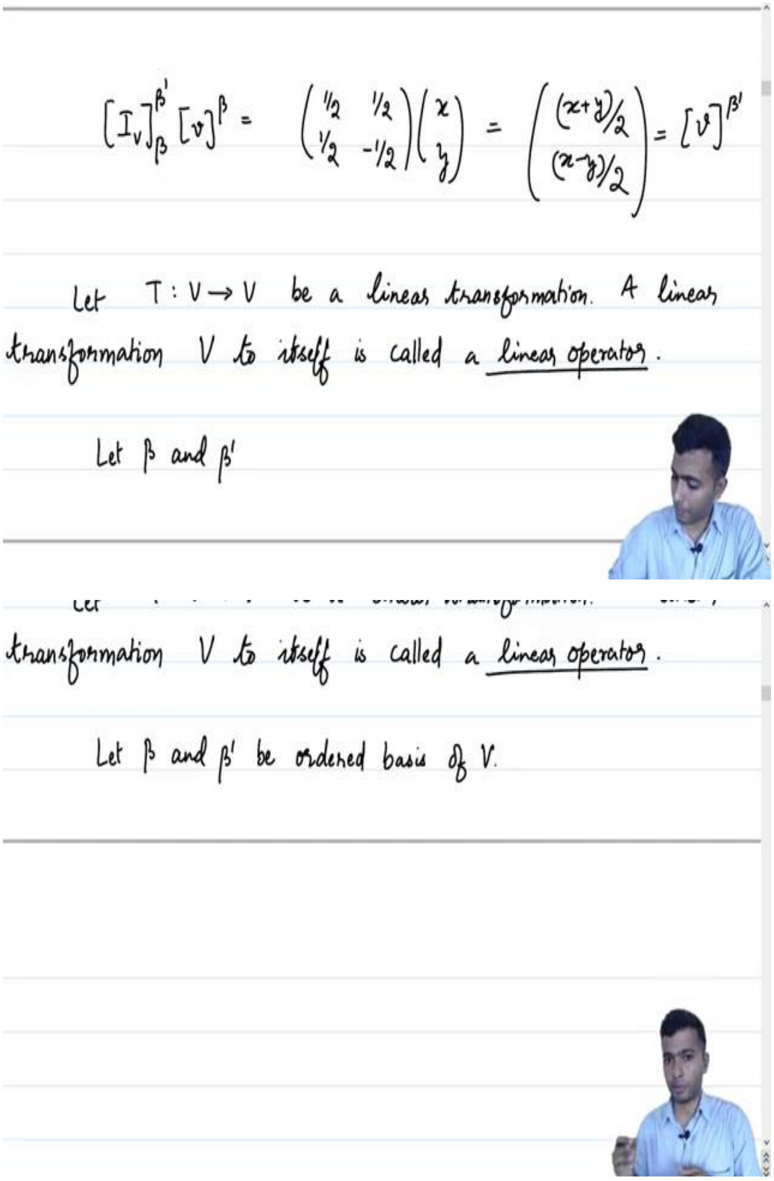

Then if you observe what is Iv beta, beta prime times v beta this is nothing but recall what Iv beta prime is half, half, half, minus half this is going to be half, half, half, minus half, and what was our v beta? v beta was x, y this is equal to you do the calculations this will turn out to be x plus y by 2 and x minus y by 2 which precisely is our v beta prime and that is precisely what we have just shown.

So, this is exactly how the change of basis matrix comes into the picture when we would like to look at how the column vector of v with respect to beta is related to the column vector of v with respect to the beta basis beta prime. Now, let us consider how the matrix of an linear transformation changes when we change the basis of the given vector space so, for simplicity let us do the study for linear transformations from v to itself.

So, change of basis matrix and the linear transformations so, not to do that let T, not to study that consider a vector space linear transformation from the vector space v to itself be a linear transformation, such a linear transformations are called linear operator as well so, I will just note that as well linear transformation. A linear transformation v to itself is called linear operator.

So, let us look at how the matrix of a linear operator changes when we change the basis ordered basis with respect to which we are looking at the matrix. So, again let us fix a basis let beta and another basis let beta and beta prime be ordered basis of v. So, consider two different ordered basis of we, then we have a matrix of T with respect to beta and beta and there is also a matrix of T with respect to beta prime and itself.

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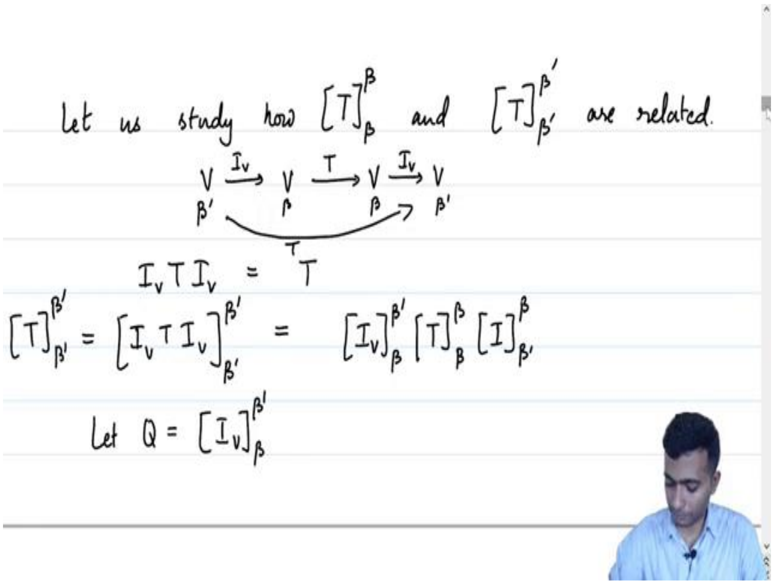

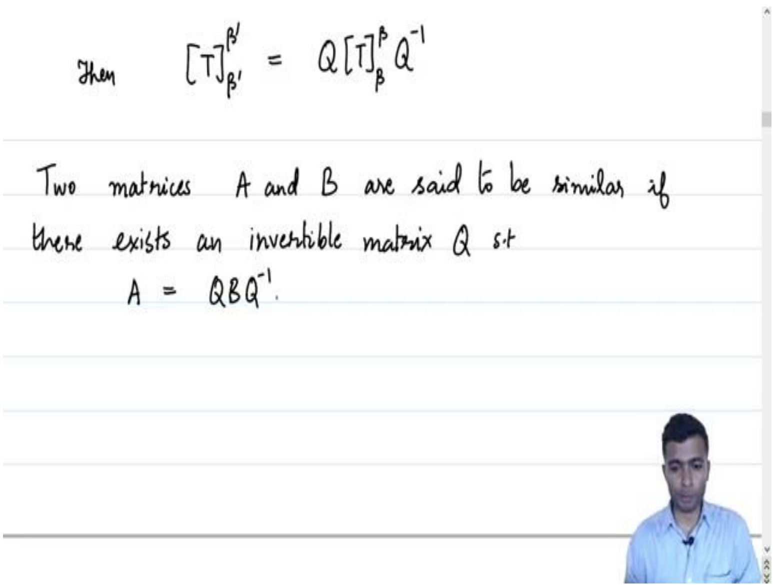

So, consider so let us study how T beta, beta and T beta prime, beta prime are related. So, the idea is almost the same but we will do is we will consider the following. So, let us consider T is a map from v to itself again this is with respect to the basis, say beta and this is with respect to the basis beta that is what will give you T beta, beta and then let us consider the matrix to be the linear transformation identity, however the range let us consider with respect to beta prime and here again let us consider the identity matrix here with respect to beta prime.

So, what this chain tells us is that Iv T Iv is equal to identity composed with T composed with identity it is nothing but our T. So, this map that is also T but, we have developed some notion of how the matrix corresponding to linear transformations behave when we consider compositions, they will just turn out to be the product.

So, let us write that down so, this Iv T Iv with respect to beta prime, beta prime this by what we have seen earlier is nothing but Iv from beta to beta prime here from here times T beta to beta from here and the times Iv from beta prime to beta but this left hand side is nothing but T because Iv T Iv is just T and this is with respect to beta prime and beta prime.

So, let Q be the matrix Iv from beta to beta prime, then T beta prime, beta prime is equal to Q T beta, beta Q inverse we have already checked but at least it was given as an exercise that the change of basis matrix from beta to beta prime is the inverse of the change of basis matrix from beta prime to beta and of that matrix is Q, then the matrix of T with respect to beta prime, beta prime is equal to Q times T matrix with respect to beta and beta times Q inverse.

So, that is precisely how the matrix of a linear operator or a linear transformation from v to itself behaves when we change the basis from beta to beta prime, such a relationship is given a name it is called similar, similarity two matrices A and B are said to be similar if there exists an invertible matrix Q, such that A is equal to Q B Q inverse. We will see later that such matrices, similar matrices share a lot of common properties therefore, this particular relation that we have just explore is of extreme importance when we study linear algebra further.

So, we will see more examples of how linear transformations are the matrix of linear transformations are related to each other through further concrete examples and problems in the problem section.