Linear Algebra
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Lecture 4.4
Invertible Linear Transformations and Matrices

So, we have already seen what it means for 2 vector spaces V and W to be isomorphic to each other, we have developed the notion of an invertible linear transformation. In this video we will explore how a linear transformation being invertible reflects upon the matrix associated to it. So, we begin by recalling the definition of an invertible matrix.

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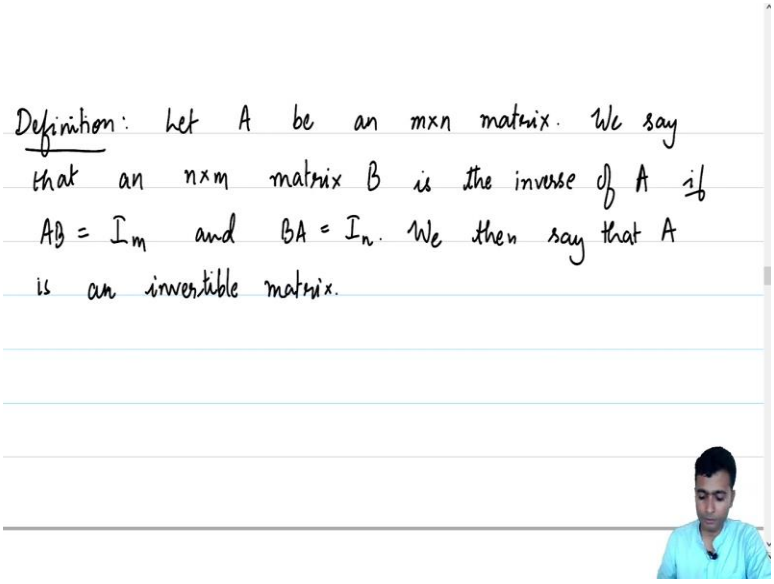

Let us start by recollection, definition so that A be an m cross n matrix, we say that a matrix B matrix or rather an n cross m matrix B is the inverse of A if AB is the identity matrix of size m and BA is the identity matrix of size n. So, a is an m cross n matrix, B is an n cross m matrix, if you look at the product of these matrices AB it will just turn out to be an m cross n matrix and we demand that that matrix maybe identity matrix. Similarly, BA is demanded to be the identity matrix of size n, so we say that the matrix B is then an inverse is the inverse of the matrix A. So, we then say that A is an invertible matrix.

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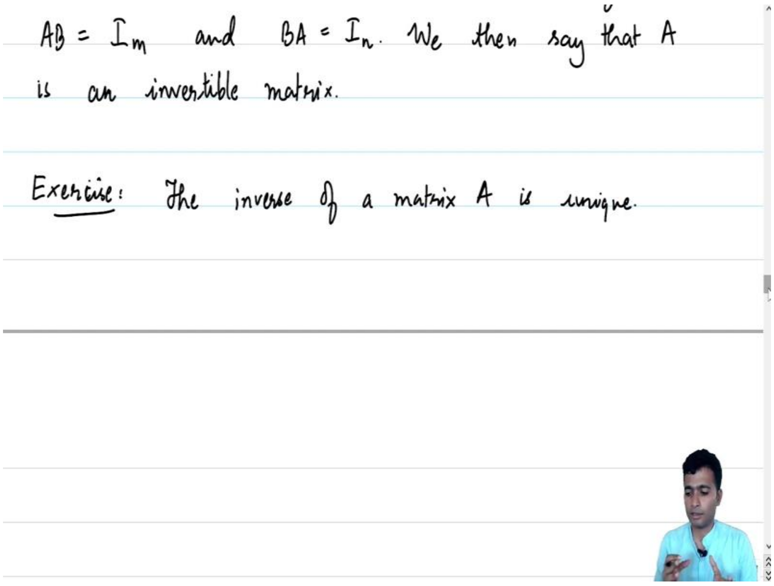

So, an immediate exercise which will follow this is to prove that the matrix A has a unique inverse, the techniques to prove have been developed and used many times. So, you should also use the theorem which was put in the last video wherein we showed that matrix multiplication was or is associated. So, the exercise is to show that the inverse of a matrix A is unique.

So, start off with 2 potential inverses, B and B prime and prove that b is equal to be b prime. Alright, so why did we talk about an invertible matrix to begin with, so as it is to be expected, our goal would be to show that an invertible linear transformation as an invertible matrix associated to it.

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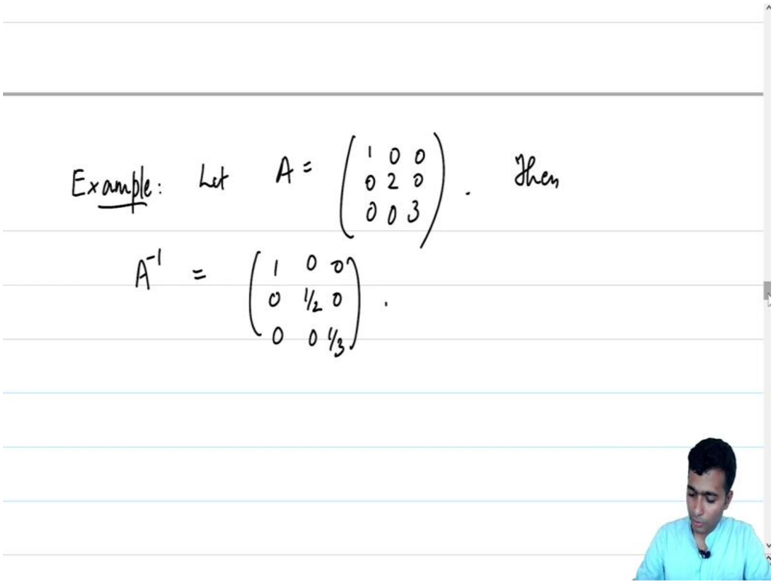

So, we will come to that before that, let me just for the sake of completion, let me give an example. So, let A be a matrix given by 123 in the diagonals let me make my life easy here. Then A inverse is equal to 1 0 0 0 1 by 2 0 0 0 1 by 3. So this is a familiar concept for you most probably from your high school so need not spend too many.

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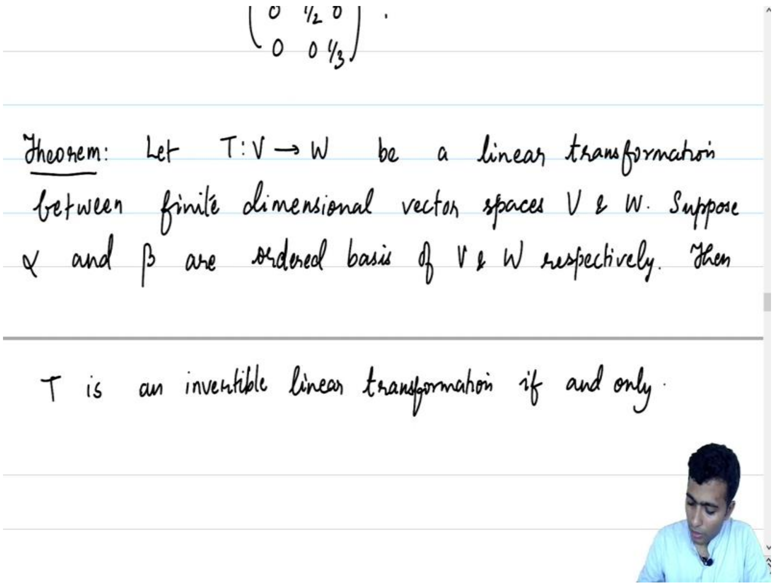

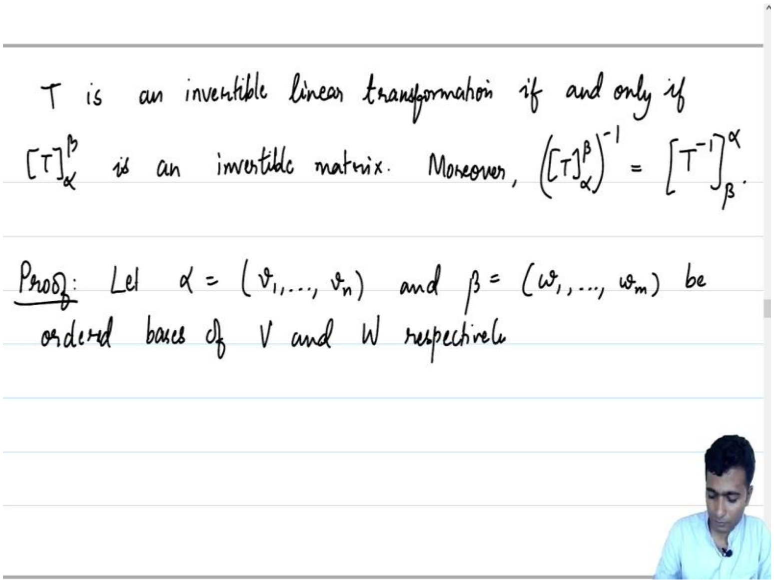

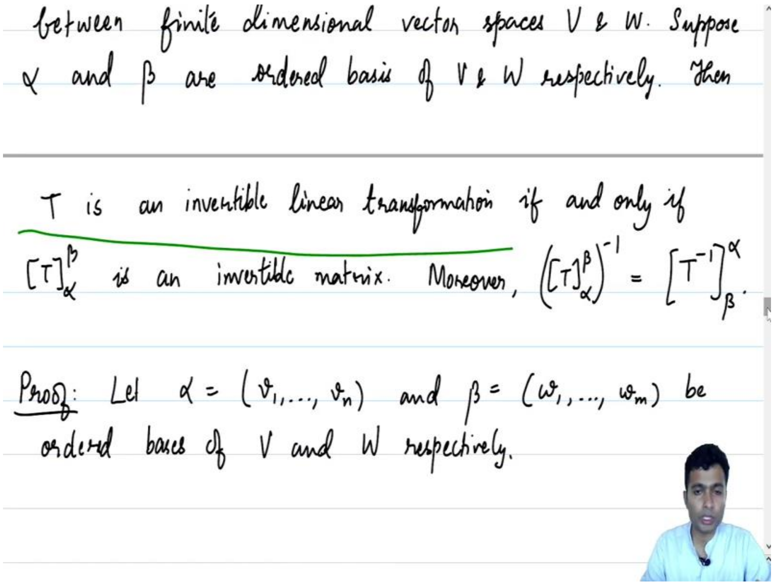

Let us just spend time on too many examples, let me directly jump into the theorem here, which essentially captures the relationship between invertible linear transformations and invertible matrices. So, let T from V to W, be a linear transformation between finite dimensional vector spaces, V and W. Let us fix in ordered basis alpha and beta of V and W. Suppose alpha and beta are finite that is already known to be finite ordered basis of V and W respectively.

Then then theorem states that T is an invertible linear transformation if and only if the matrix associated to T with respect to beta, alpha and beta is an invertible matrix if and only if the matrix T so alpha beta is an invertible matrix, not just this we exactly know what the inverse of this matrix is.

Moreover, the inverse of alpha beta, inverse of this matrix, this turns out to be the matrix associated to the inverse of T, inverse linear transformation of T. This is just the matrix of T inverse not, recall that T inverse will now be a map from W to V, therefore, this will be beta alpha. So, we exactly know what inverse of our matrix corresponding to this linear, this particular linear transformations. So, let us give a proof of this statement. So, we are given alpha and beta.

So, let us once and for all in the context of this proof, let alpha be equal to say, we want to vn and beta be equal to w1 to wm be ordered basis of V and W respectively. Alright, so let us see what our goal is. So, let us first assume that T is an invertible linear transformation. What I am underlining in green and let us prove that the matrix associated to T is an invertible matrix.

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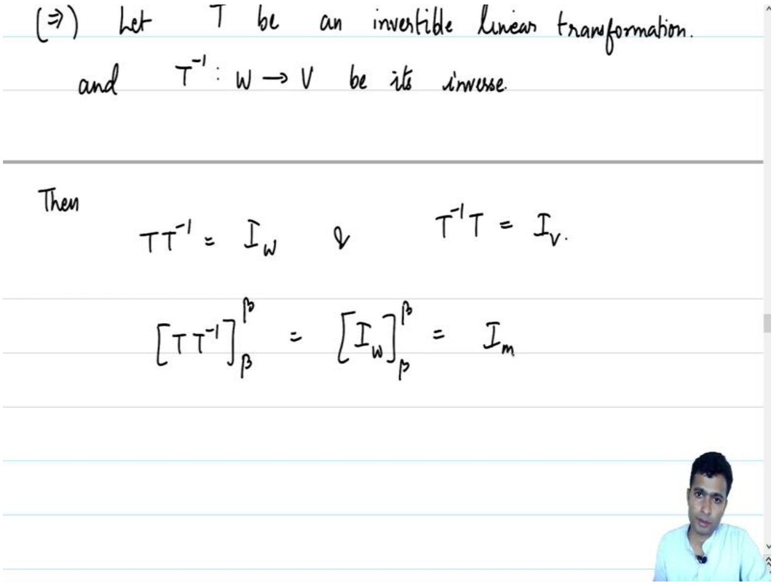

So, let T be an invertible linear transformation. What does it mean to say that a linear transformation is inevitable, it means that then there exist an inverse. And we know that the inverse is unique. And suppose T inverse from W to V be its inverse. What do we know about this inverse when composed with T it gives us identity on both sides. So, if you look at TT inverse, then TT inverse is just the identity map of W.

And similarly, T inverse T is just the identity of V by the very definition. We also know that the identity map of W so we know, so let us see what is TT inverse from beta alpha. So, recall that TT inverse is a map from W to itself and sorry so it will not be beta alpha, it will be beta beta.

This is just the matrix of the identity map with respect to the bases beta. And you know what that particular matrix is, the matrix of the identity with respect to beta will just turned out to be the m cross m identity matrix. We also have already seen how the matrix corresponding to the linear transformations behave under composition.

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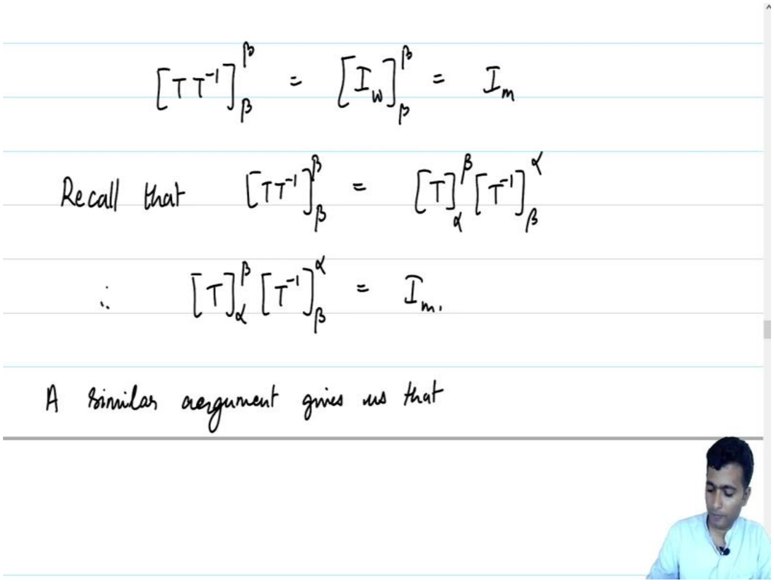

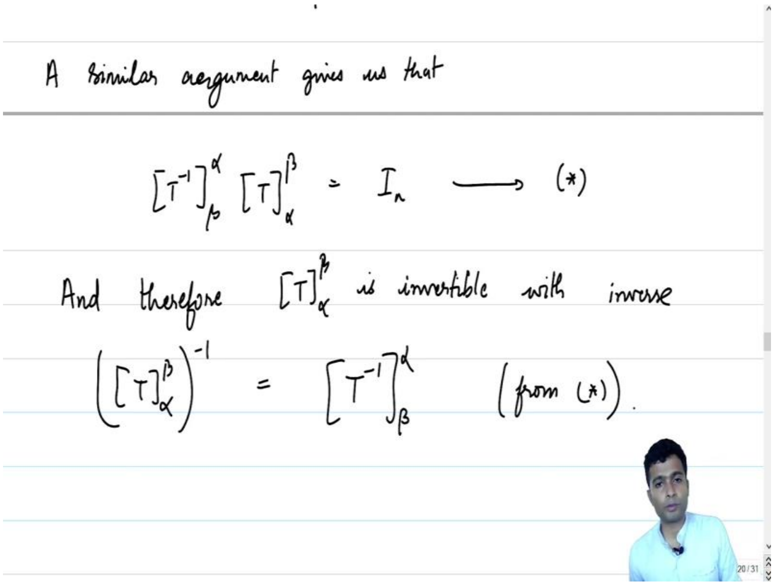

So, let me just note that down, recall that, TT inverse beta beta is equal to T inverse from beta to alpha and T from alpha to beta. And therefore, we have T alpha beta times T inverse beta alpha is equal to the identity matrix, identity matrix of size m. A similar argument is noted a similar argument tells us that T inverse beta alpha, T alpha beta is the identity matrix of size n because T inverse T will just turn out to be the identity map of the vector space V and this will hence tell us.

And therefore, matrix T alpha beta is invertible, what is the inverse, with inverse T alpha beta inverse being equal to the inverse of the matrix of the inverse from star, this is from star, alright so, we have proved one half of the theorem which states that if T is an invertible linear transformation, then the matrix associated to T with respect to alpha beta is invertible, moreover, if T alpha beta is the matrix, the inverse of that matrix will be the matrix of T inverse corresponding to beta alpha.

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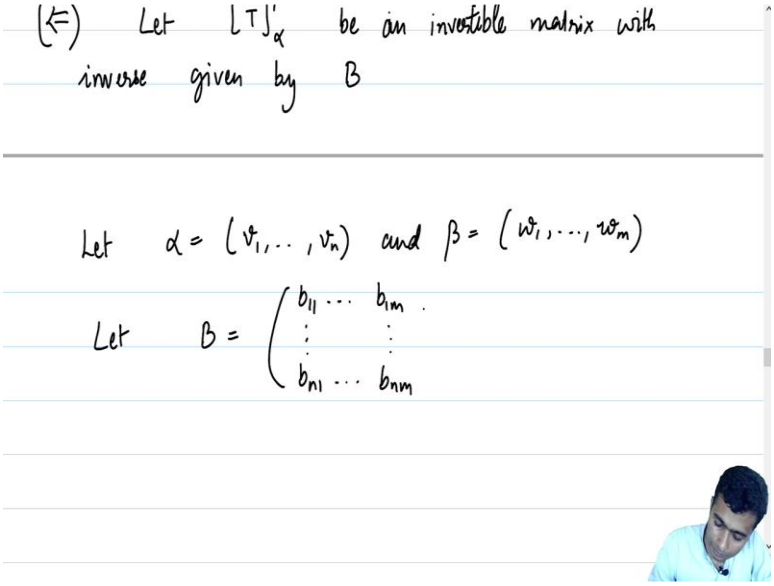

So, now let us prove the other direction. So, what is the other direction, the other direction demands that if you start off with an invertible m cross n matrix corresponding to a linear transformation, then the matrix the linear transformation is also an invertible linear transformation. So, let T alpha beta be the matrix of T sorry, of course, this is the matrix of T with respect to alpha beta, let me try that again and again. Let this matrix be an invertible matrix with inverse given by B.

So, let us give some name to vectors, let us list down the vectors in the ordered basis alpha and beta. So, let alpha B or call that alpha is an ordered basis of V, which has dimension n and let v 1 v 2 up to v n correspond to the vectors in the ordered basis alpha and beta similarly, be some w 1 to, recall that W is of dimension m, so, this will be some list of m vectors so let alpha and beta be ordered basis of v and w consisting of v1 to vn as the vectors in alpha and w1 to wm as the vectors in beta.

And we know that there is an inverse to the given matrix associated to T corresponding to these basis. So, let us see what the entries of this matrix is. So, let B be equal to something like b11 so, where is what is b? b is the inverse of the matrix associated to t which is going to be an m cross n matrix. So, this is going to be an n cross m matrix. This is going to be n 1 n m.

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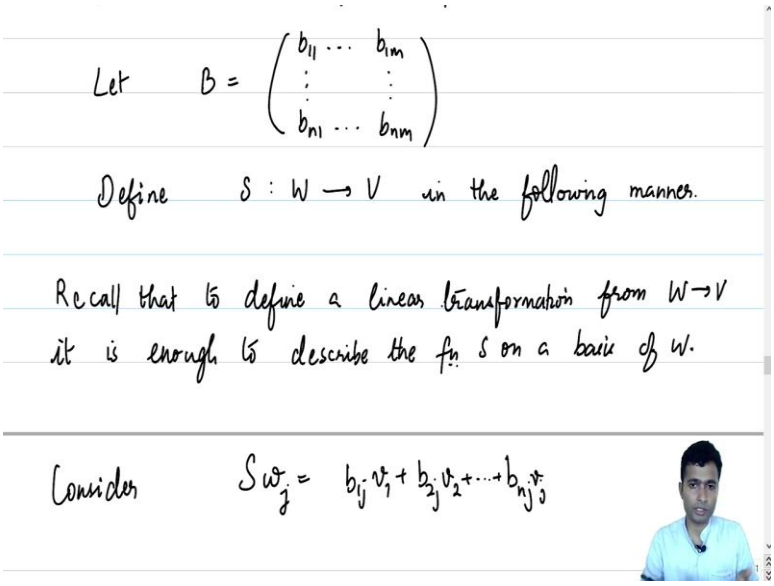

Now, let us define a function. So define function S from W to V in the following manner we will try to use, we will invoke one of the theorems we have proved in one of the previous videos namely which says that if you are given basis of the domain in this case W and say basis consisting of say w 1 w 2 up to w m and suppose you are given vectors say x 1 x 2 W, v is already used u1 u2 to um, then there exists a unique linear transformation which maps Wi TUi. So, let us use that to define this particular map s by defining what S is on a basis of w.

So, recall that to define linear transformation on w linear transformation from W to V. It is enough to describe the function S on a basis of W but we are already given a basis of W. So, consider S of wj, this is how we would like to go about defining our linear map s, but what would Swj be, when we know that the matrix of s should B, so that is our goal, so we would like to somehow construct an inverse of T, what is the information we have?

The information we have is the inverse of the matrix corresponding to T and we are already familiar with what the matrix of this inverse should be, it will be the inverse of the matrix. So our expectation should be that the matrix of S should be B corresponding to beta alpha. If that is to be the case, let us see what Swj should be. Let us let me show you what B is on the top. And let me define Swj by keeping B above, so this will just turn out to be equal to b1j, v1 plus b2j v2 plus bnj vj which let us call it as uj.

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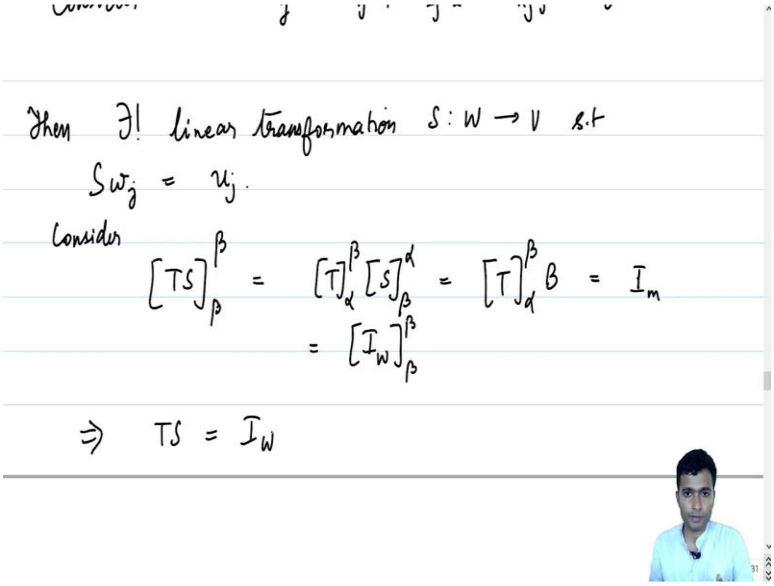

By one of those theorems which we proved earlier then their excess unique linear transformation S from W to V. So, let us call this uj such that Swj is equal to uj. My claim now is that this linear transformation S is the inverse of T. So, how do we go about proving that?

So, let us see what the matrix corresponding to T S is. So, consider T s with respect to beta alpha, so, we know that T S is our map linear transformation from W 2 itself. So this is not beta alpha, this is beta beta. And we know that this is the composition of 2 linear transformation.

So, this will be S beta alpha to the right and the T alpha beta. And we know that this is equal to T alpha beta times B, which you should go back and check by the very definition of our S here the matrix of S turns out to be B. So, this is equal to T alpha beta times B but what was B, B was exactly the inverse of our matrix, T alpha beta which is just equal to the identity map of size m.

But we know some linear transformation which gives us this particular matrix which is just going to be equal to the identity map from W to itself corresponding to beta. So, from here, it is straight forward to check that TS is equal to IW. So, it is an exercise for you to see that this is equal on each of the basis vectors. Therefore, by the, or rather just straight forward I would not want to call it an exercise but nevertheless you should check this particular arrow carefully.

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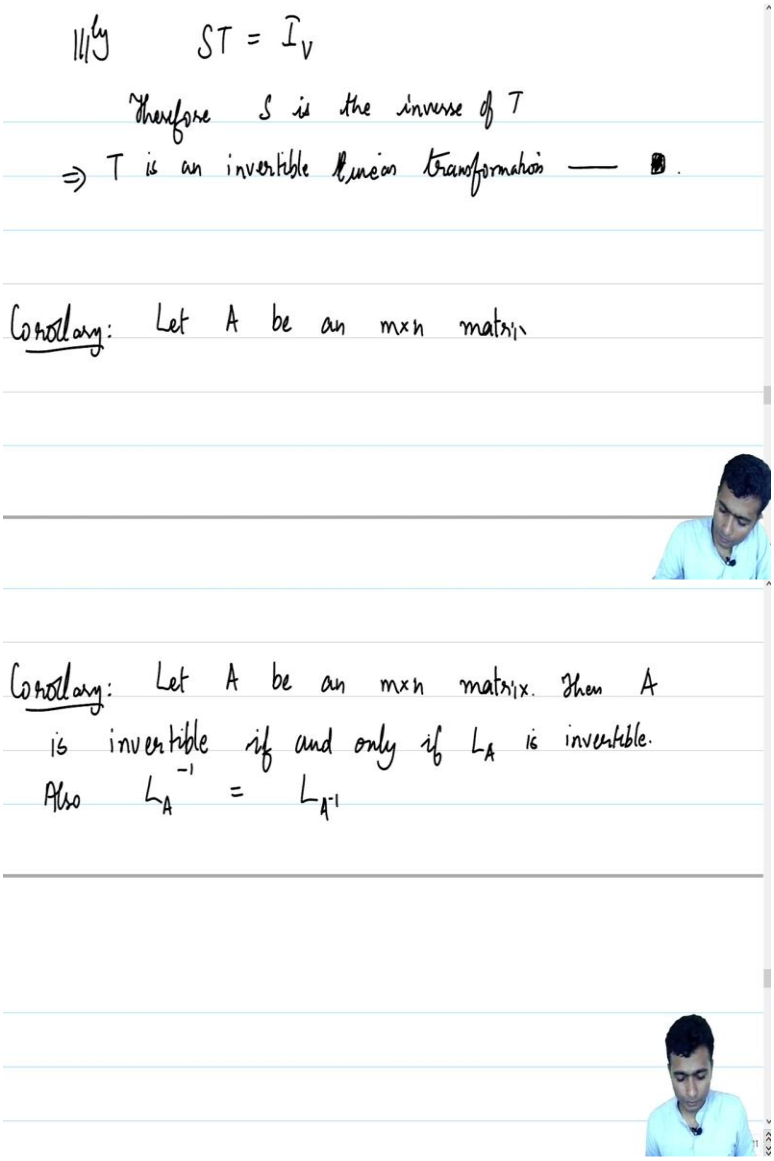

Similarly, you will be able to conclude that ST is equal to IV and therefore, S is the inverse of T which gives T is an invertible linear transformation that completes our proof. Alright so, now given a matrix A that is a very natural linear transformation which can be associated to it namely LA. So, this theorem tells us that A is an inverted, so let me write it down as a corollary.

So, let A be and an m cross n matrix then A is invertible if and only if LA is invertible not just that we know more, we know exactly how or what LA is. Also LA inverse the inverse of linear transformation LA will be the linear transformation corresponding to a inverse. So, the prove is just a direct application it is a corollary, it is a direct application of the previous theorem.

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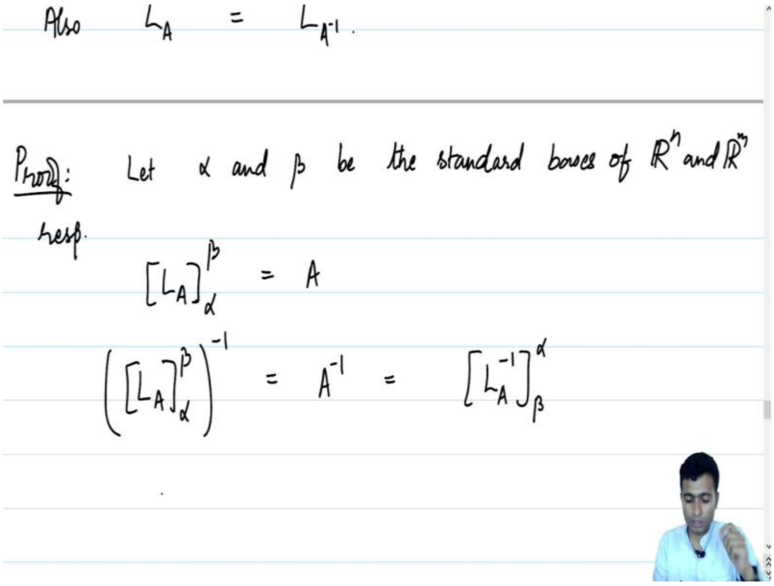

So, the first part is fairly straight forward. Notice that with respect, so it is just finally boiling down to the choice of the right basis. So, let alpha and beta be the standard bases of Rn and Rm respectively and just boils down to taking that with respect to the standard bases the matrix of LA is just A and the previous theorem tells us that LA inverse LA alpha beta the whole inverse the matrix inverse which is equal to A inverse, this is equal to LA inverse beta alpha by our theorem which we have just proved.

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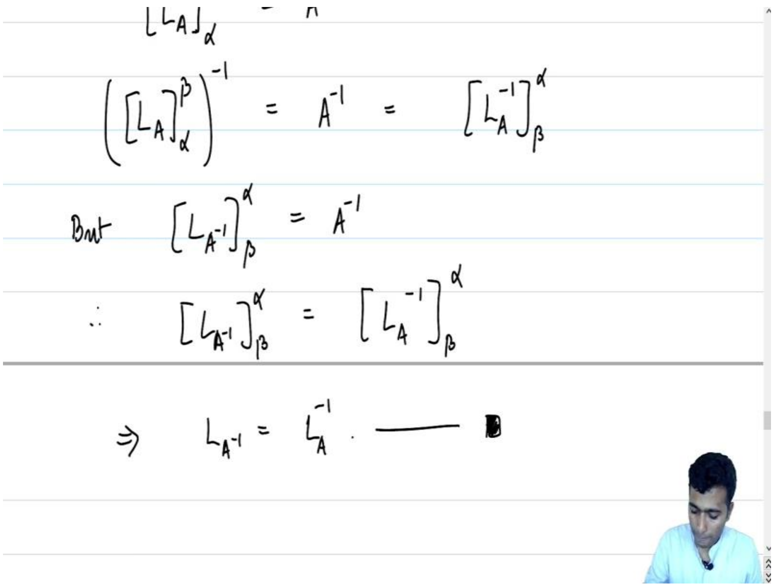

But we also know that A inverse is the matrix of the linear transformation corresponding to A inverse but let me write it down LA inverse with respect to say beta alpha is nothing but A inverse. And therefore, LA inverse beta alpha is equal to the matrix of the inverse of the linear transformation LA with respect to beta alpha and from there again, on the basis it is equal and therefore, it will be equal everywhere. And that establishes the corollary.

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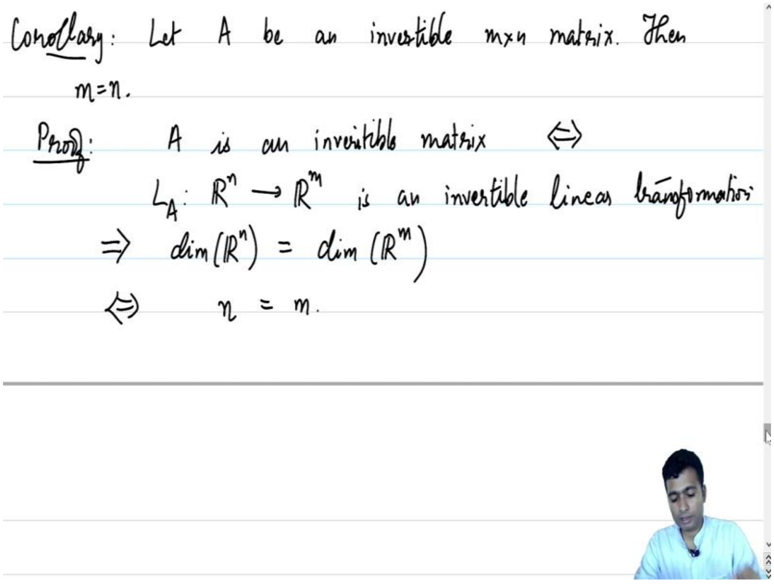

Another corollary is that so let me just note it down another corollary states that if A is an m cross n, it is a corollary to this corollary which states that if A is invertible m cross n matrix and m is necessarily equal to n. So, let us see how that is. So, let A be an invertible m cross n matrix then m is equal to n. So, by the previous theorem. So, let us look at a proof, by the previous theorem A is invertible matrix if and only if LA, so, where is LA from? LA is a map from Rn to Rm. So, LA is an invertible linear transformation.

Now, what do we know about isomorphic vector spaces so, we know that two vector spaces are isomorphic or in other words there exist an invertible linear transformation between them, if and only if their dimensions are the same. So, which we have proved in one of the videos in this week, this is if and only if the dimension of Rn which is equal to n.

So, dimension of Rn is equal to the dimension of Rm but what is dimension of Rn that is n and what is dimension of Rm that is m and therefore, this is a if and only if n is equal to m. So, A is an invertible linear transformation. So, there is a slight error which one should be very careful about so this will tell us that dimension of Rn is equal to dimension of Rm.

It is not an if and only if statement. You could have LA which is not necessarily an invertible linear transformation, even though the domain and the range are Rn. So, for example, look at the map LA corresponding to the 0 matrix that will not be an invertible transformation. So, this direction of the implication here which I am now circling in green one should be very careful I had written and if and only if there which is not the case. So, this implies that dimension of Rn is equal to dimensional Rm, which is if and only if and n is equal to m.

So, we have effectively shown that if A is invertible m cross n matrix then n is necessarily equal to m. Alright so, in the proof of the theorem which we proved just now, the theorem which stated that if T or rather T is an invertible linear transformation if and only if the matrix associated to T is invertible.

We used an argument to construct a specific inverse linear transformation of T, we shall use that style of construction to prove that the vector space of all linear transformations between v and w which we had seen earlier and we had given a name L of V, comma W that is isomorphic to all m cross n matrices over R, so, that is the next theorem that we will be proving.

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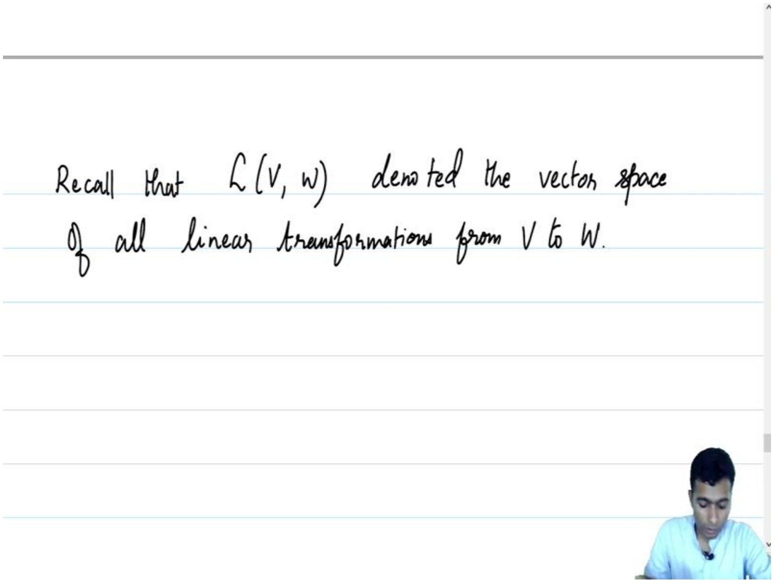

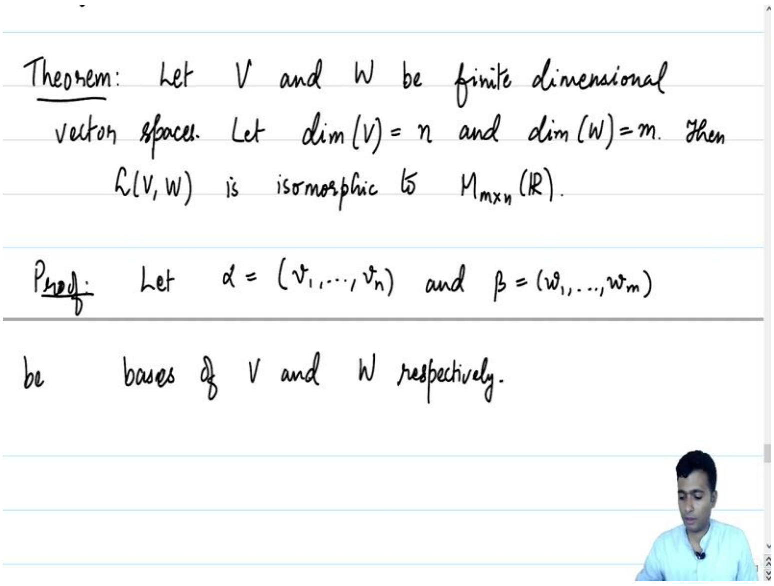

So, recall that L of v comma w denoted the vector space of all linear transformations from V to W. So, this theorem tells us that L of V W is isomorphic to the matrices of size m cross n when V and W are respectively of dimension m and n. So, let us see what the statement is. So, let V and W be finite dimensional vector spaces.

Suppose, the dimension of V is equal to n and the dimension of W is equal to m. So, let the dimension of V is equal to n and dimension of W be equal to m, then the theorem states that L of V comma W is isomorphic to the matrices of R of size m cross n, let us give a proof of this, the proof is quite straightforward.

We have seen the idea that is being used in this proof, just a few minutes back, so it is just going to be a imitation of that. So, let us start by fixing an ordered basis. So, let alpha be equal to, v1 to vn and beta equal to w1 to wm be bases of v and w respectively.

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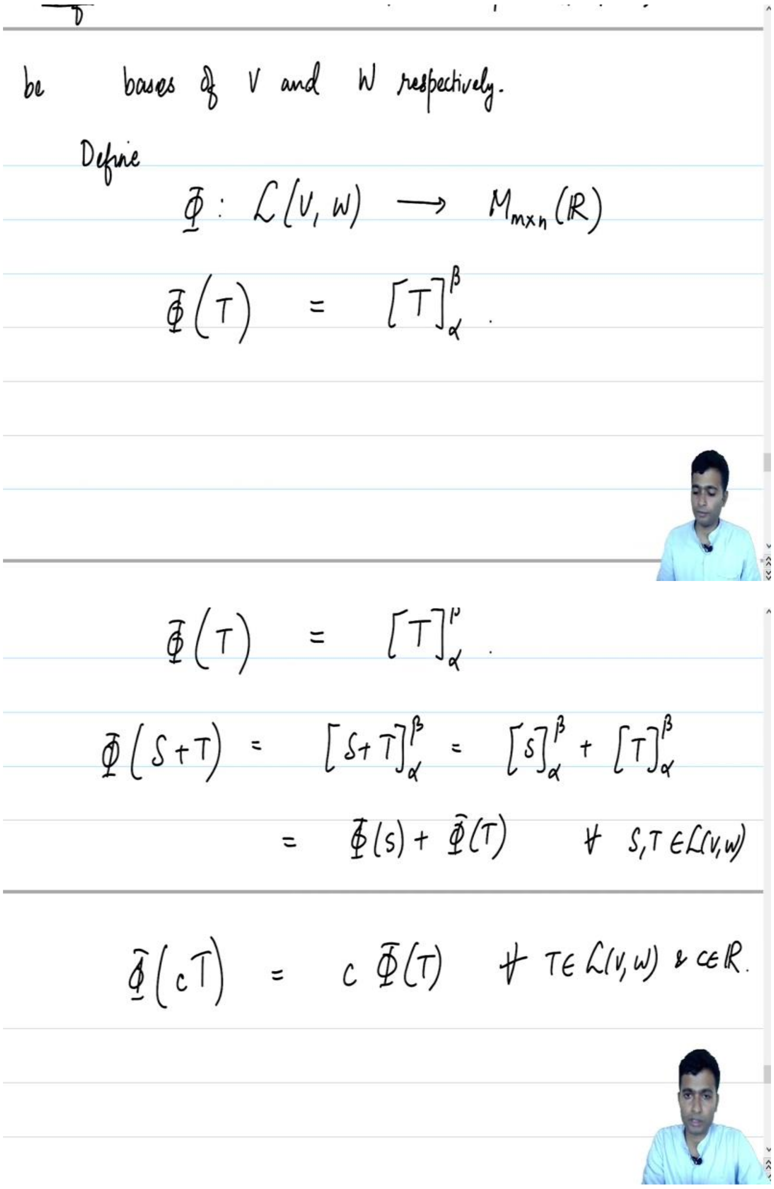

Let us now define a map phi so define capital phi from L of v, w into the m cross n matrices over R. So, recall that m cross n matrices over R was a vector space consisting of all m cross n matrices and it was having a basis with all the matrices with 1 say ij and 0 elsewhere so there will be mn of them.

So, the dimension of the vector space on the right is actually mn. So let us define a map phi from L of v, w into m cross m over R by what you should be expecting. Phi of a linear transformation T is just going to be the matrix of T responding to alpha beta.

So, we have seen that phi is linear transformation in another guys, we have already seen it, let us check that phi is linear transformation, so to see that phi of S plus T is going to be equal to the matrix of S plus T alpha beta. And we have already seen that this is equal to S alpha beta plus T alpha beta, which is equal to phi of S plus phi of T.

And similarly, the scalar multiple, so, this is for all S and T in L of v, w and similarly, phi of c times T is nothing but c times phi of T for all T in L of v, w and C in the scalars. So, what is phi of cT it is going to be the matrix of cT with respect to alpha and beta which is going to be c time the matrix of T with respect to alpha and beta, which is nothing but c times V T.

So, you have seen these two in the first video of this week. So, we have what we have done just now is to establish that our map phi which we just defined is a linear transformation from L of v, w into the m cross n matrices over R. So, what remains to show is that it is an isomorphism.

So, invoking another result, which we have already shown, we will prove that phi is both injective and surjective. And thereby proving that an injective and surjective linear transformation is an invertible linear transformation, you use that research to prove that phi is then an invertible linear transformation.

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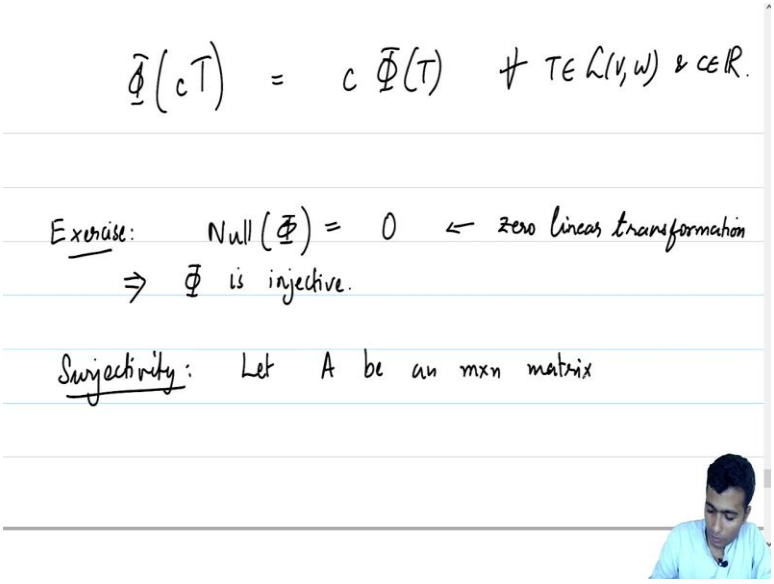

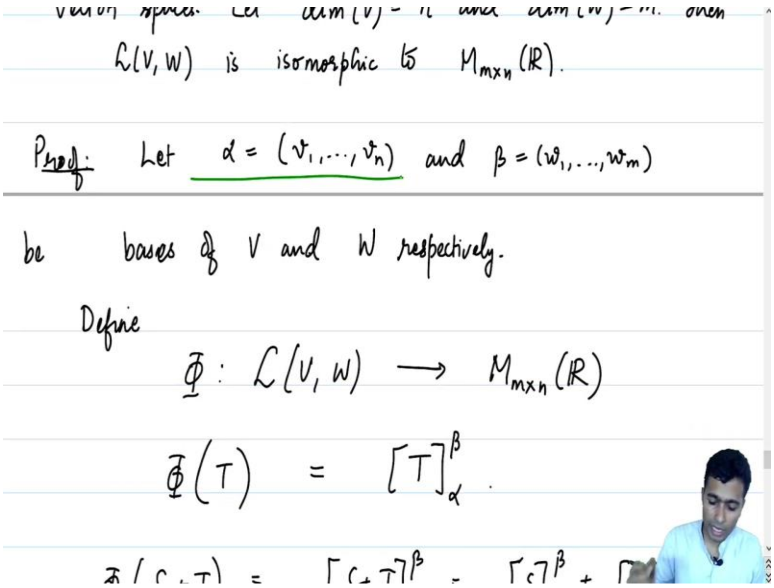

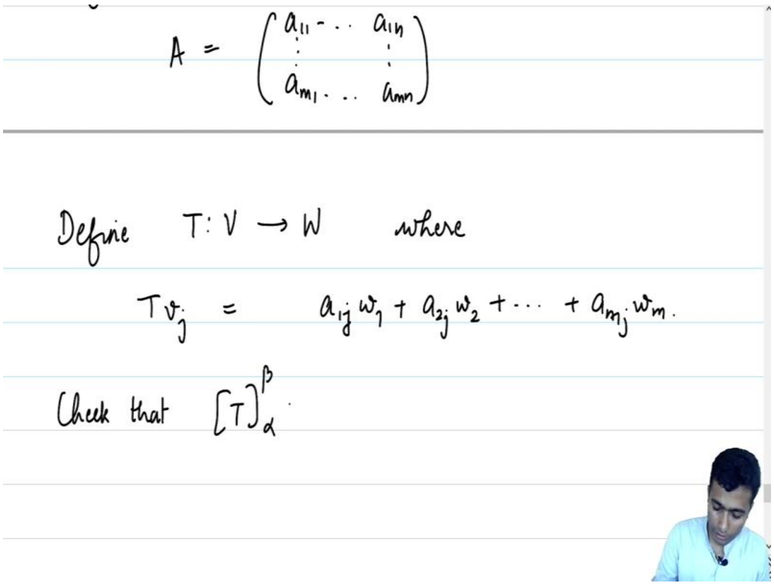

So, the injectivity comes by noting that so this I will leave it as an exercise for you to check that the null space of phi is necessarily the 0 linear transformation is the 0. So, this is the 0 linear transformation. Just notice what the matrix corresponding to phi will turn out to be if the linear transformation is being mapped to the 0 vector, 0 matrix. It will just be the matrix with 0 entries. Which means that every bases vector is being send to the 0 vector and therefore, the linear transformation is the 0 linear transformation.

So, this particular exercise establishes that phi is injective. Notice that we have already shown that phi is a linear transformation and this tells us that phi is injected and how about surjectivity. So, let surjectivity that is non- exclusive. So, let us take some arbitrary m cross n matrix so let A be an m cross n matrix. We have seen this technique, this is what I was talking about a few minutes back. So, let A be the matrix given by these entries and did we give what the vectors? Yes.

So, v1 to vn and w1 to wm are the vectors in the ordered basis corresponding to alpha and beta. So, define, let us define a map T, V to W, where Tvj will just turn out to be equal to a1j w1 plus a2j, w2 plus anj or rather mj wm. So, check that the matrix of T corresponding to alpha beta, sorry.

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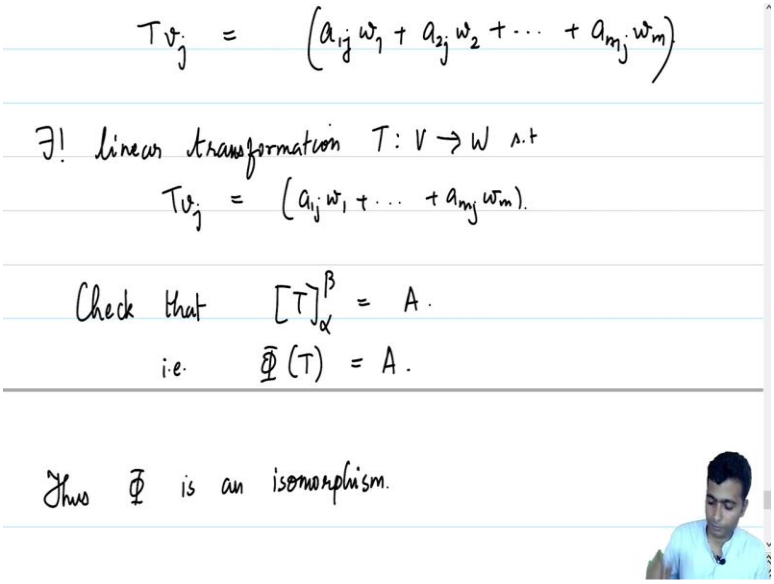

Before that I just defined what T is on the basis vectors alpha. And by 1 of the theorems you have proved earlier that it is a unique linear transformation, which maps each of the vj to the vector which I just put in the bracket. It is called Tuj where it is a unique linear transformation there exists a unique linear transformation by one of the theorems proved earlier liner transformation T which maps, so what is that?

Here I have only defined T for the basis elements, v1 v2 up to vm and with that there exists a unique linear transformation T from V to W such that T of vj is equal to a1j w1 plus amj wm and what is the matrix of T with respect to alpha beta, check that this is equal to A. But what does it mean to say that the matrix of T with respect to alpha beta is a, it means that i.e phi of T is equal to A, that shows that our map phi is surjective. So, we have shown both injectivity and surjectivity. Thus, phi is an isomorphism.

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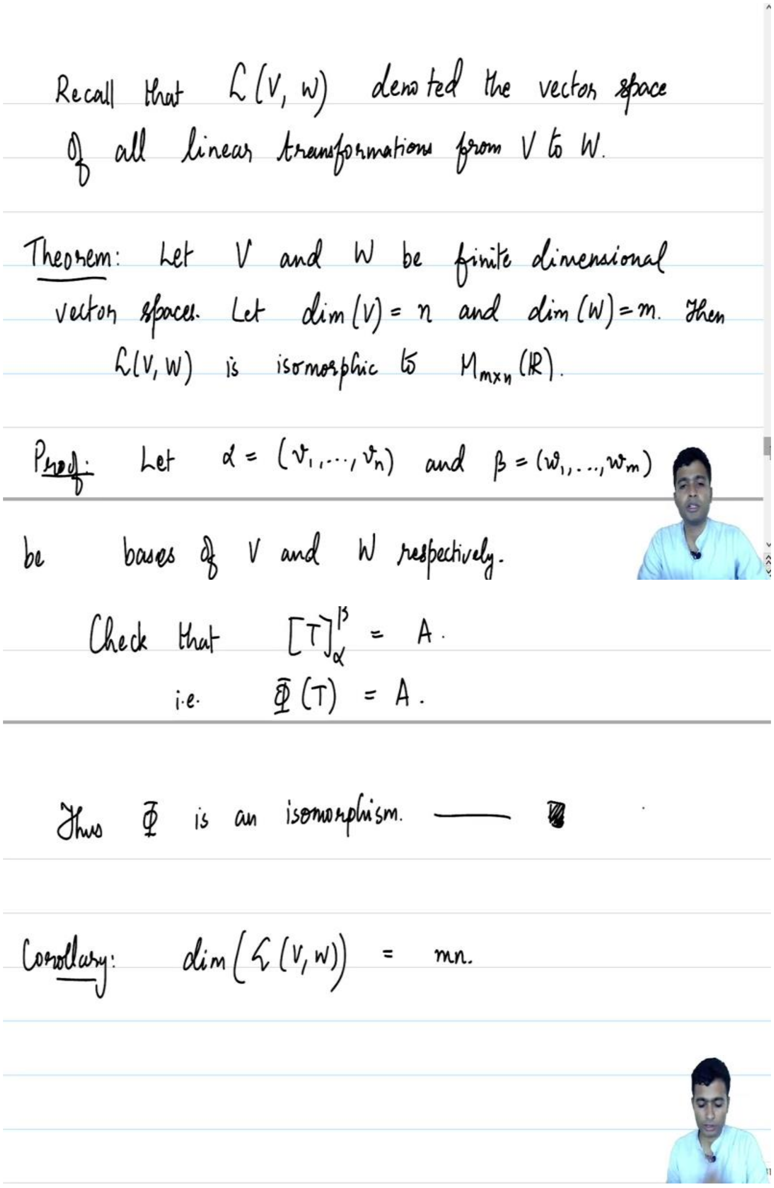

So, what we have essentially shown it is a very powerful result. Let me just show you the result once more. This theorem tells us that the vector space of all linear transformations from V to W is isomorphic to the m cross n matrices over R. And as a corollary, so, let us stop with the corollary which is a direct consequence of the theorem, it tells us what the dimension of L of v, w is.

Dimension of L of v, comma w remember that two matrices are isomorphic if and only if their dimensions are the same, so this is nothing but dimension of the m cross n matrices over R which is equal to m times n, so this has already proved the result corollary states that the dimension of L of v, w is (())(39:33). Let me stop here.