Linear Algebra
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Lecture - 4.3
Invertible Linear Transformations

So, we have seen how a linear transformation can be associated to a matrix given a ordered basis in say V and W, and a linear transformation from V to W, we can associate a matrix to this linear transformation, not just arbitrarily, if we can talk about operations of linear transformations, like say addition of linear transformations, or scalar multiplication of linear transformation, or product, or composition of linear transformations, the corresponding operations get carried to the matrices as well. We shall now see that given a matrix can be associate linear transformation to that matrix, and what are the properties that we should be expecting there.

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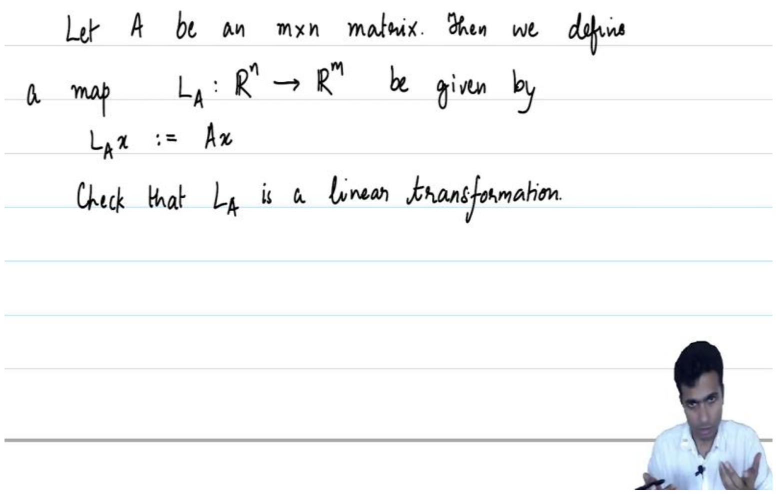

So, let us start with an m cross n matrix, so let A be an m cross n matrix. Then we define a linear transformation corresponding to LA, then we define a map, as of now I am just calling it a map, or a transformation, L subscript A. So, remember this is an m cross n matrix, so the map is defined from Rn to Rm be given by LA x is defined to be Ax.

So, what do I mean by this? So, remember that A is an m cross n matrix and for every x in Rn, we can talk about a column representation of that vector x, Ax is just the matrix multiplication of the matrix A with the column representation.

So, our next statement, next proposition tells us that, if you look at the matrix of LA, then it should be necessarily A, yeah so before that it is a check for you, it is an exercise, check that we have already actually seen this LA, A is a linear transformation, this is not some arbitrary map here, the map that we finally define by matrix multiplication turns out to be a linear transformation, we shall now see the next proposition that the matrix which is associated to LA with respect to the standard bases turns out to be exactly A, no surprises, but let us prove it.

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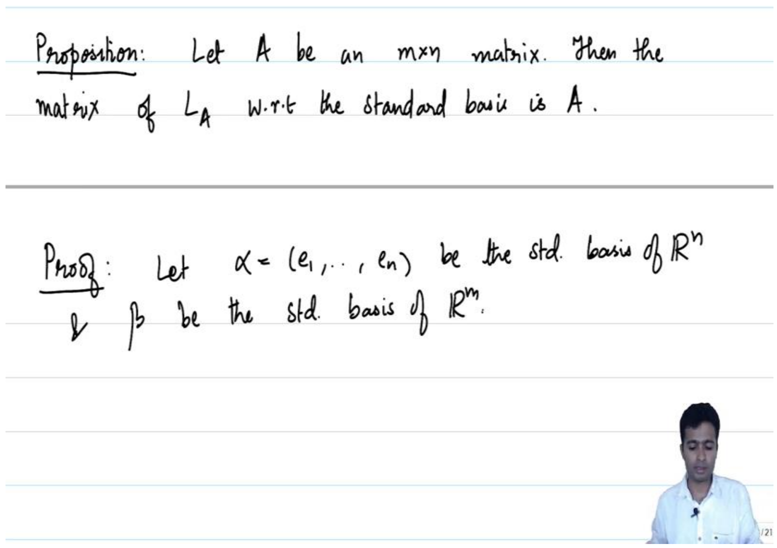

Proposition, which is something which one should expect, so let A be an m cross n, as above, let A be an m cross n matrix with real entries always here (())(3:28) solve all over R. Then the matrix of LA with respect to the standard basis is A. So, let me give a proof of this statement. So, let us fix some notation for standard bases. So, let alpha, which is given by say e 1, to e n, be the standard basis of Rn and beta be the standard basis of Rm, let me now write down explicitly what beta is.

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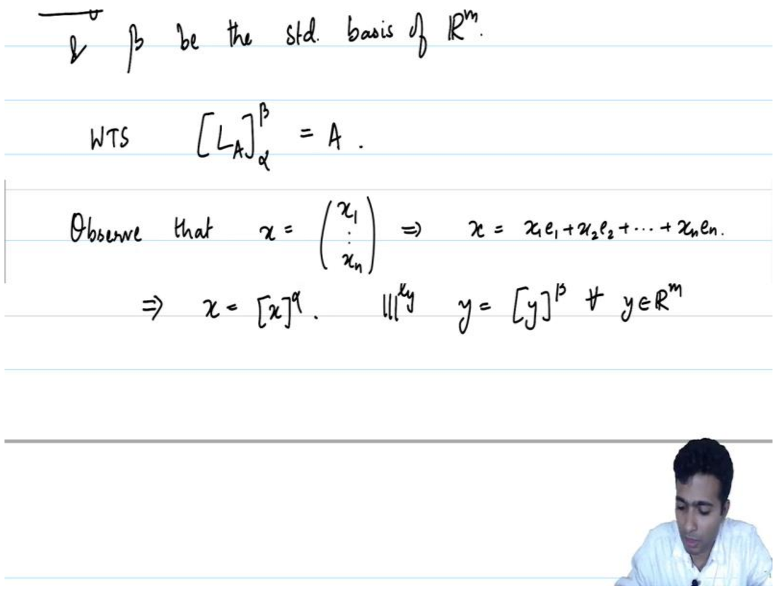

What are we interested in proving? We would like to show that, explicitly write on what you want to show, you want to show that the matrix of LA with respect to alpha and beta is equal to A, WTS stands for we want to show. So, let us do this, so the first observation is that the vector, the column vector representation of a vector is exactly the representation of x with respect to the standard basis.

So, observe that the vector x equal to x 1, x 2, up to xn, when it has the column representation, x 1, x 2, up to xn, this implies that x is equal to x 1 e 1 plus x 2 e 2 plus up to xn en, which implies x is nothing but x alpha. So, our column representation of x is consistent with or is equal to the coordinate representation of x, coordinate vector x, with respect to alpha.

So, this is how the case with every vector y in Rn as well, similarly y is equal to y beta for all y in Rm as well, this is for all x. So, for x in Rn, we can say this, similarly we can say that for y in Rm as well.

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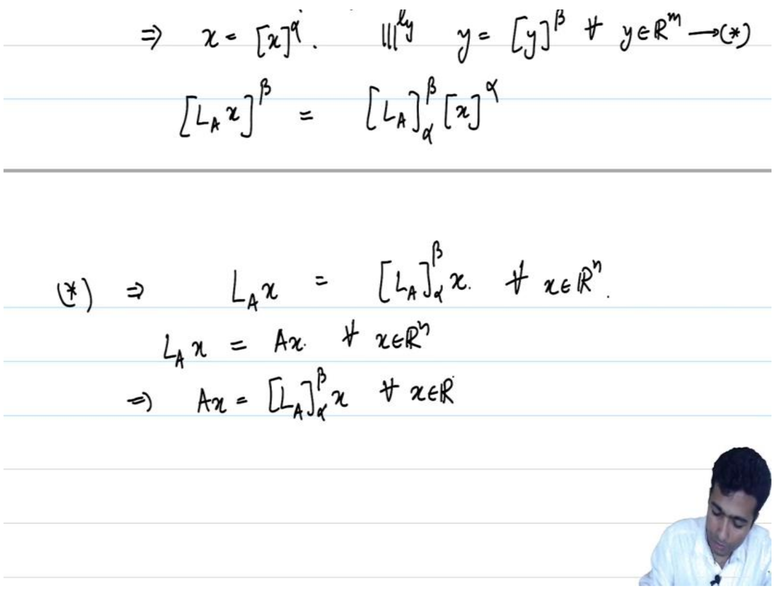

Let us now try to figure out what the matrix of LA with respect to alpha, beta, X, so recall that LA x this is our vector in Rm. So, it is coordinate vector with respect to beta by the very definition of our matrix representation will be the following. Now, note that LA x is a vector in the in Rm, x is a vector in Rn, and we just noted that, this is the same as the vector represented, the column vector representation.

So, this implies or rather what implies, star implies, so let us call this star, so star implies LA x beta is nothing but LA x, and this is equal to LA alpha, beta times x, for all x in Rn, that is what we finally are able to conclude, but we know exactly what LA x is.

LA x is by definition equal to Ax, for all x in Rm, which implies Ax is equal to LA alpha, beta x for all x in Rn, at this point let me leave as an exercise to show that A, the matrix A and the matrix LA alpha beta are the same.

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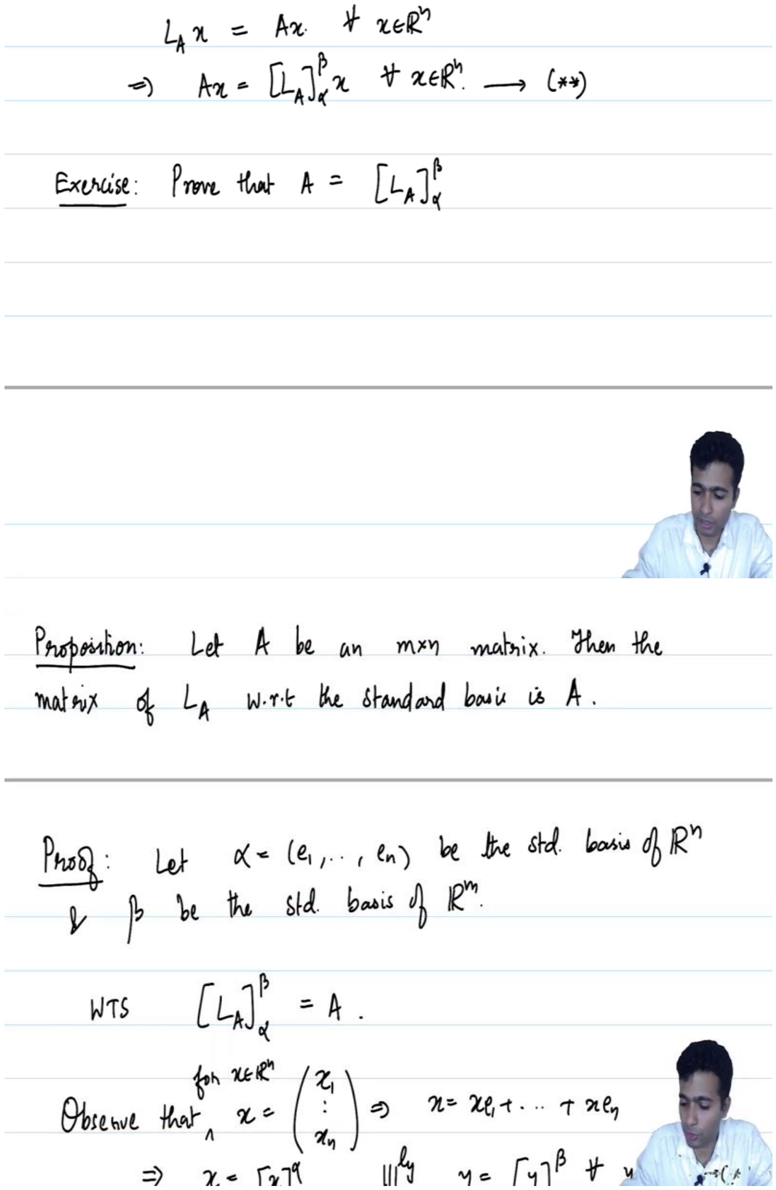

So, exercise, prove that A is equal to LA alpha, beta, so let me give a small hint, remember that the equation star, star is being satisfied for all x in Rn, in particular it is getting satisfied for each of the coordinate vectors in Rn, what can we say about star, star when x is each of the coordinate vectors, then we will be able to prove this exercise, or complete this exercise.

And with this exercise, you would have completed the proposition because the proposition was exactly to show that the matrix of LA with respect to the standard bases is this which we have just shown. Now, this association of a linear transformation LA to A is in some sense the inverse operation of associating a matrix to a linear transformation. So, let us make that precise.

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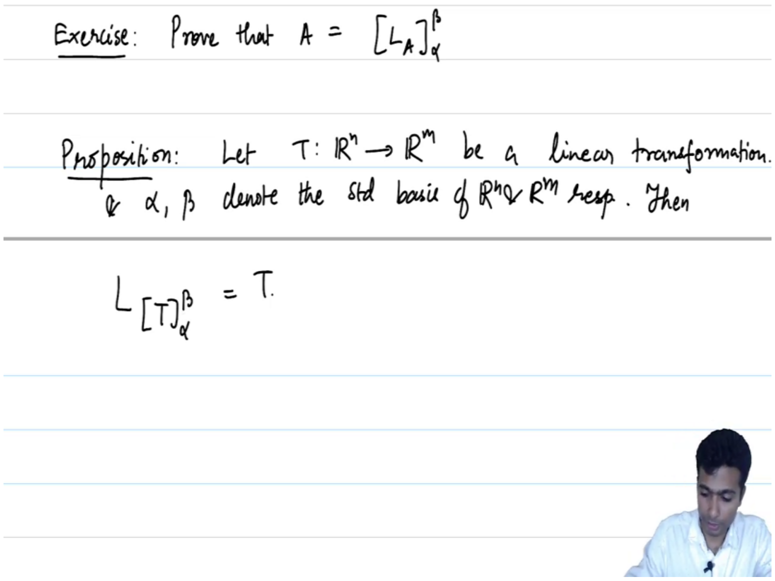

So, again another proposition in some sense this is the converse. So, let T from Rn to Rm be a linear transformation from Rn to Rm and let as usual as above alpha, beta denote, alpha, comma beta denote the standard basis of Rn and Rm respectively.

So, what would be a potential converse to the previous preposition, that would tell us that you look at the matrix associated to T and look at the corresponding linear transformation, linear transformation associated to that matrix, then the linear transformation should be the same as for matrix T.

So, let me just write down the conclusion of the converse, then L T, alpha beta is equal to T, we would like to show that the linear transformation corresponding to the matrix of T is the same as T. So, let us give a proof of this statement next.

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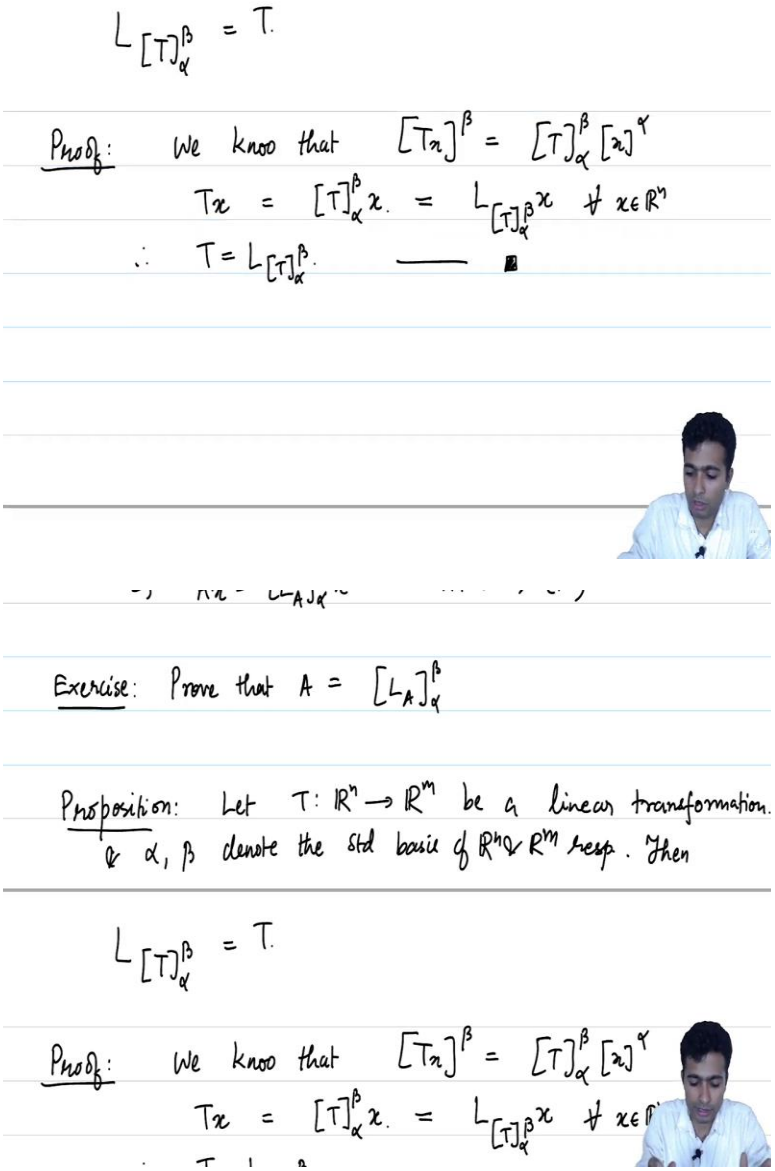

So, what do we know about the relationship between T alpha, beta, Tx and x? We know that Tx beta, this vector is T alpha beta times x alpha, by the very definition of a matrix associated to T, but we just observed earlier that Tx beta is nothing but the coordinate, the column vector representation of Tx and this right hand side is nothing but T alpha beta times x.

What do we do next? We have to somehow bring in our LT alpha beta but that is staring at us, because this by the very definition of LT alpha beta is equal to this, this is for all x in Rn, and therefore, T is equal to LT alpha beta.

So, the proof was quite straight forward, but it said something very crucial, the statement rather, what it tells us is that the idea of associating a matrix to a linear transformation and a linear transformation to a matrix once we fix the basis in the case of Rn and Rm the standard basis, once we do that the notion is exactly the same, one is the inverse of the other. So, in some sense we are identifying L of V comma, L of Rn comma, Rm and all m cross n matrices.

So, we will come to that later in a short while, when we start discussing invertible linear transformations but before that let us explore some more relationships between what happens in this case of, this type of scenario, this is a very powerful result, which we have proved just now. Let us use this result to prove that matrix multiplication is associative, which you might have seen earlier, and which you would have observed is a very tedious process to check.

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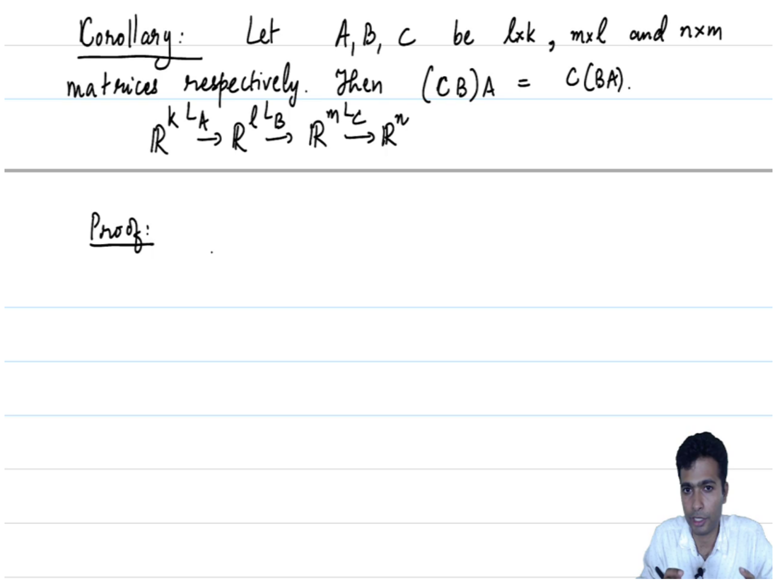

So, let me give a proposition or rather a corollary, so let A, B, C be let me just draw a line, and draw a figure to not make mistakes Rk to Rl to Rm to Rn. So, our A comes here, B comes here, and C comes here, so let A, B, C, be l cross k, m cross l and n cross m matrices respectively.

Then the matrix multiplication is associative, then C B times A is equal to C times BA, it might be a worthwhile exercise to write down the proof of this statement in the classical style by that I mean you write down the notation of the elements of A B C respectively and then by Brute-

Force try to prove that it is associative and see how it is, how tedious it is. We will however, use the machinery which we have developed till now to give a direct proof of this.

So, what we know is that for each of these matrices we have a corresponding linear transformation, so we know, we also know that composition of functions is an associative process.

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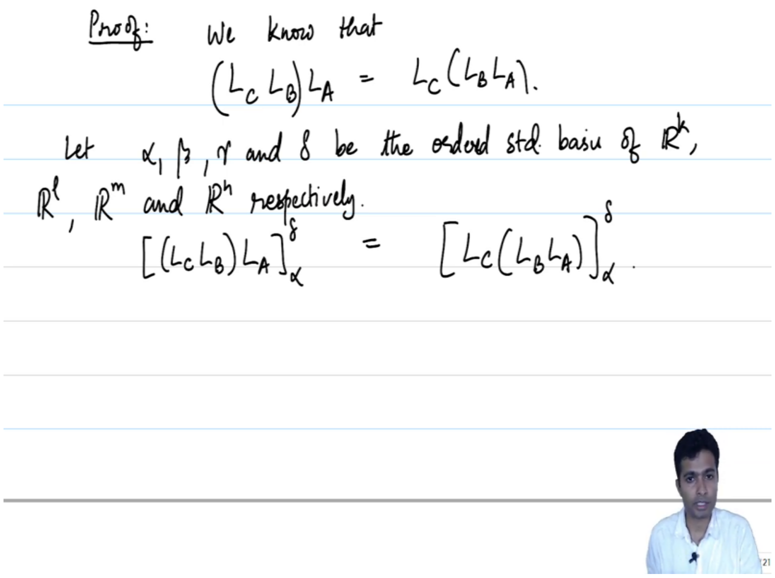

So, we know that LC, LB, LA is equal to LC, LB, LA, this is something which we know from basic functions, knowledge of basic functions. So, to go further, let us fix some notations for the standard basis. So, let alpha, beta, gamma and delta be the ordered or standard basis, ordered standard basis of Rk, Rl, Rm, and Rn respectively. Then if you observe LC, LB, LA this is a map from Rk to Rn.

So, then we know that the matrix of LC LB LA from, so Rk has alpha and it ends with Rn which has delta as its matrix, we know that this is equal to the matrix of LC times LB LA again alpha to delta. We will now proof, we will now use the fact that the matrix of a composition of linear transformation is the product of the matrices.

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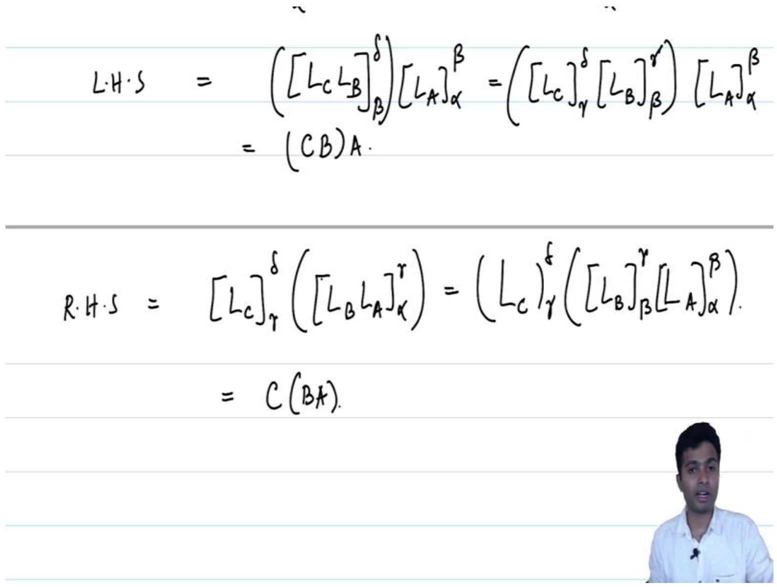

So, if that is to get translated the LHS is equal to LC LB delta and beta and this is LA alpha, beta, LA is a map from, notice that it is a map from Rk to Rl and therefore this will be having alpha to beta Rl has beta as its ordered basis, and similarly RHS and which let us focus on the LHS as of now, this is nothing but again LC gamma delta LB beta gamma times LA alpha beta, but we know all these objects what is our LC gamma delta with respect to the standard bases, LC will have the matrix C. So, this is CB times A.

RHS similarly written out to be equal to LC here, which is gamma to delta times the matrix of LB LA which is from alpha to gamma, which is equal to LC gamma delta, LB beta gamma and LA which is alpha beta, which in turn is equal to C times BA and we are done.

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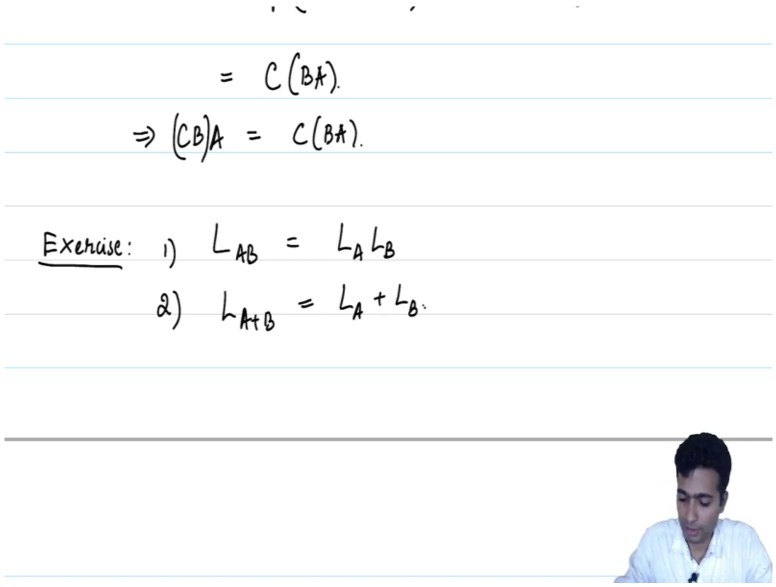

So, as you can see the power of the machinery which we have developed till now helps us very easily in establishing that CB times A is equal to C times BA that matrix multiplication is commutative, there are similar things which we can proof, which may be I will give one as an exercise. It is good exercise to show that LAB is equal to LA LB, similarly, LA plus B is equal to LA plus LB the proof is similar and I would like you to try these exercises.

So, we have seen quite a lot about matrices and linear transformations, we know about a very special class of matrices, which are called the invertible matrices, can we say something about the corresponding linear transformations, we will come to the matrix part later. So, we initially let us focus on what is meant by an invertible linear transformation.

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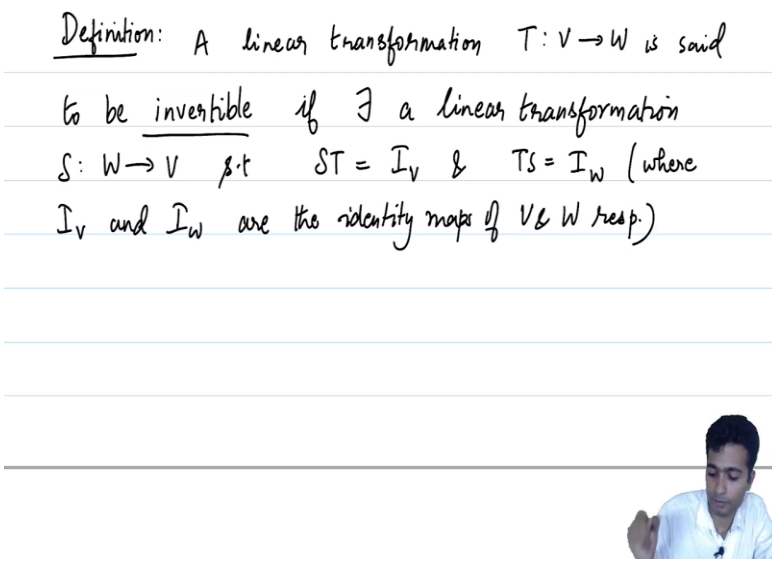

So, I would like to start next with a definition of that of a invertible linear transformation. A linear transformation, notice that till now we were working with finite dimensional vector spaces to go from linear transformations to matrices, but in this definition we are not assuming anything about the finite dimensionality.

This is a linear transformation between two vector spaces V and W overall, this is said to be invertible if there exist a linear transformation S which is from W to V such that S T is equal to the identity of V and TS is equal to the identity of I w, where Iv, and Iw are the identity linear transformations, identity maps of V and W respectively. If there exist, ok.

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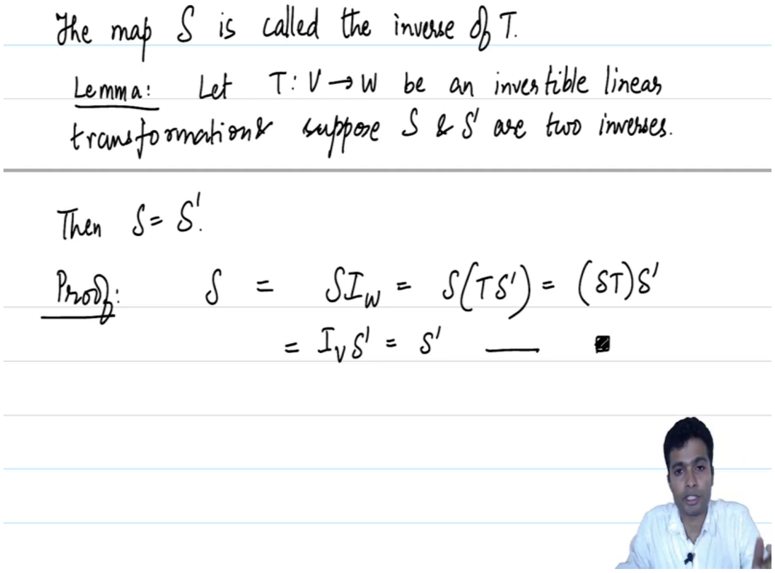

Then the map S is called the inverse of T, and the next proposition will tell us that the inverse is unique and therefore we can talk about the inverse of T, and we will denote it by T inverse. So, let us just quickly prove a Lemma, we have seen this idea before, so let T from V to W be a linear transformation, be an invertible linear transformation and suppose S and S prime are two inverses, then S is necessarily equal to S prime.

So, let us give a express proof of this statement, remember S is a the map from W to V, so S is equal to S times Iw this goes without saying because identity of w takes every vector w to itself and therefore S is just composed with Iw. But we know that S prime is an inverse of T and therefore we can write that this is equal to T times S prime, but then the composition of functions or product of linear transformation that is associative operation.

So, this is equal to ST times S prime which in particular as Iv times S prime, because S is an inverse of T and ST should be Iv by the very definition of Iv, by the very definition of invertiblity, and this is equal to S prime , because Iv is the identity, hence we have (())(24:47) this is a technique which we have proved, which we have seen earlier, we are just repeating repeatedly using the same technique its cut powerful as you can see. So, what we have shown as that every invertible linear transformation has a unique inverse.

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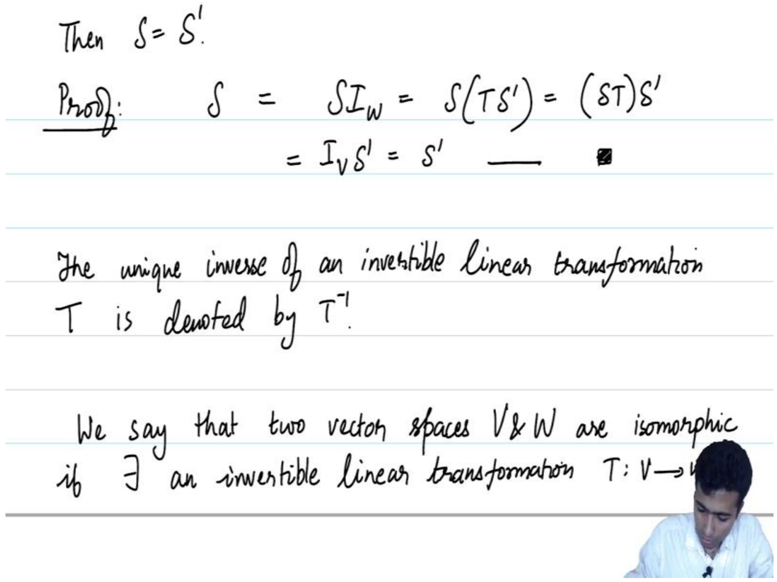

So, we the unique inverse is denoted by of an invertible linear transformation, T is denoted by T inverse. So, we have defined what a linear, invertible transformation is. Now, let us say that there are two vector spaces v and w, such that there is a invertible linear transformation between v and w, we then say that these vector spaces are isomorphic, in a minute we will see that if two vectors are isomorphic, then they share a lot of common properties, effectively we can identify one of the vector spaces with the other.

So, when we give a definition, we say, definition of isomorphism we say that two vector spaces V and W are isomorphic if there exist an invertible linear transformation T from V to W, that is if we have, if we have an inverter linear transformation T from V to W, we also have an invertible linear transformation from W to V, which we obtain from T itself namely the inverse T inverse, T inverse is a invertible linear transformation from W to V so in some sense, whatever we can say for T and T inverse can be said for T inverse and T as well.

So, what are the conclusions we can draw when we know that two vector spaces are isomorphic to each other it is a powerful notion to have as I just mentioned that tell us that many, many properties of both the vector spaces will sound very similar. So, before we get to the properties which are common. Let us observe a couple of things about linear transformations, which are invertible.

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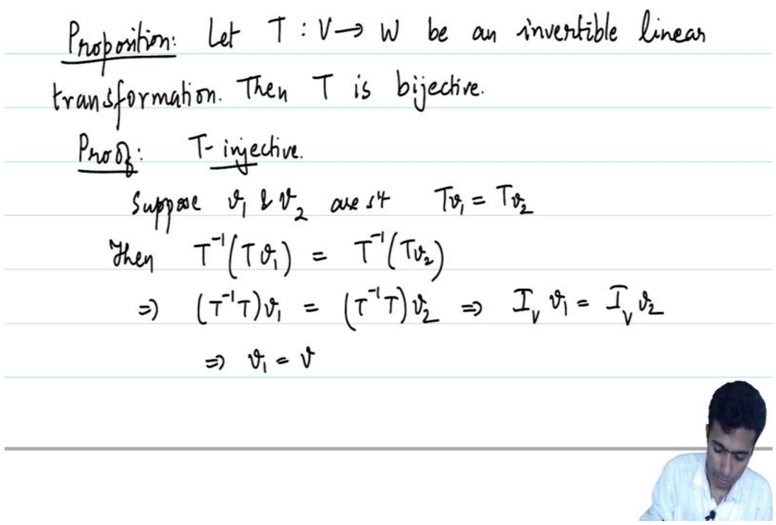

So, proposition, every linear transformation which is invertible should necessarily be injective and surjective. So, let T from V to W be an invertible linear transformation, then T is bijective. So, let us give a proof of this proposition, you would like to show that these both injective and surjective, I think I should leave that as an exercise because we know that if there is an inverse, let me just quickly prove it.

So, T is injective, let us quickly have a look at why T is injective. To show that T is injective, we would like to show that T V 1 be equal to T V 2 for V1 and V2 two vectors in V forces V1 to be equal to V2. Suppose, V1 and V2 are such that T V 1 is equal to T V 2, then what do we know about invertible linear transformation, we know that there is an inverse.

So, let S we have a notation for V inverse, then T inverse of T V 1 is equal to T inverse of T V 2 but what do we know about T inverse T. So, this implies T inverse T V 1 is equal to T inverse T V 2, but T inverse T is nothing but the identity map of V, this implies that Iv of V1 is equal to Iv of V 2 which implies V 1 is equal to V 2, that establishes injectivity.

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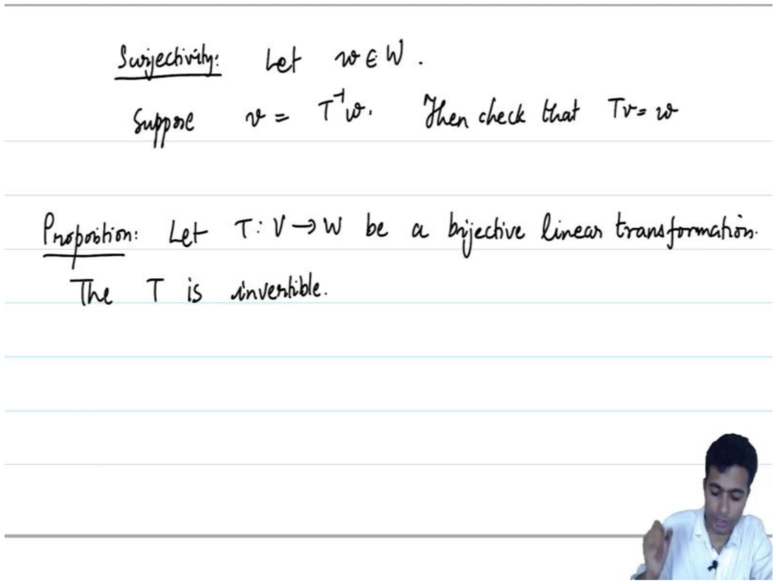

How about surjectivity? Let us look at surjectivity, so let us start with some vector w in capital W, we would like to show that there is a V such that TV is equal to W. So, we have a very immediate candidate so let suppose V is equal to T inverse w then check that I will leave it as an exercise Tv is equal w, and hence it is surjective.

So, we have established our proposition, it says that every invertible linear transformation is a bijective map. The converse is more interesting, the convers tells us that any linear map, which is both injective and surjective is an invertible linear transformation. So, let T from V to W be a bijective linear transformation, then the T is invertible or it is an invertible linear transformation. So, how do we go about proving this proposition?

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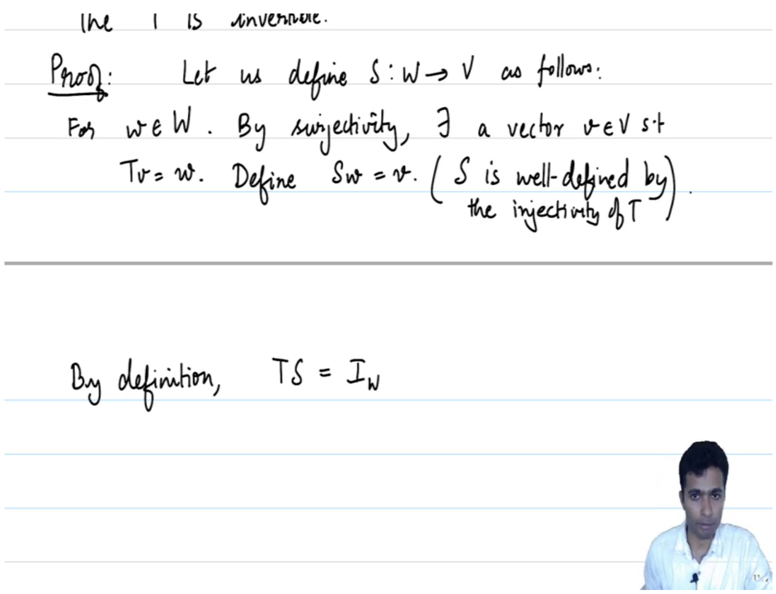

What is the, what is information we have? We have the following information, that it is bijective map. So, in particular let us try to define, or let us try to get hold of an potential inverse map, so let us define, let us define S from W to V as follows. So, let w, for w in capital W, let us pick a vector small w and capital W, then what do we know there exist some V by surjectivity, by a surjectivity, spelling is wrong, by surjectivity, by surjectivity there exist a vector v in capital V such that Tv is equal to w.

Let us define Sw to be equal this v. Define Sw is equal to v, we should be immediately asking a question is this well-defined map at all? The injectivity of T ensures that this is a well-defined map, there exist a unique v which gets map to w by the injectivity of T and therefore, there exist well define map S which takes w to v.

So, let me just write S is well defined by the injectivity of T. So, we now have a candidate, we have a function S from w to v by the very definition of S, we have that, we have St is equal to the identity or yes, I very definition of the map S, ST is equal to the identity of, no, TS is the identity of w.

Let us see why that is the case you take any vector w in capital W, what is a (())(34:34) w, it is a exactly that particular vector, which maps V to W exactly that vector V which maps T to W, that is what S of W is. Then T takes V back to W. So, TS is clearly equal to Iw.

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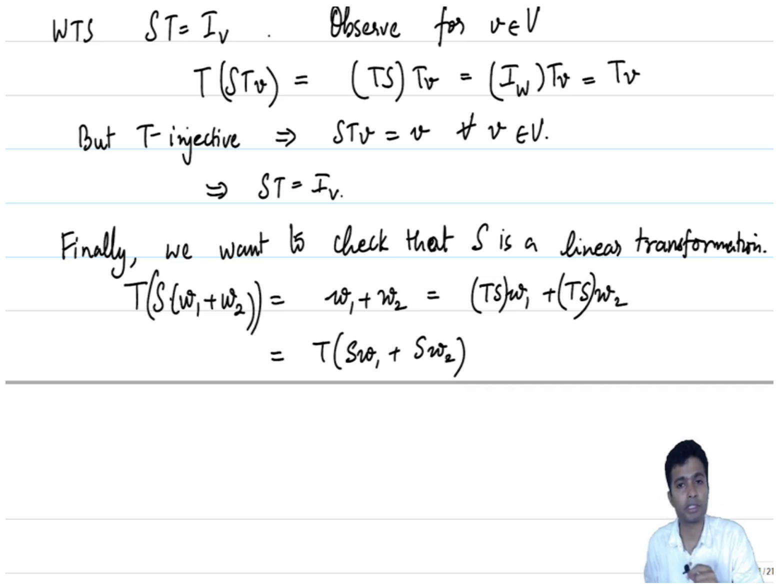

How about the other direction? You also need to establish, there are two things we have to check to show that our map T is invertible. Yes, so what are the things we have to check? We have to first get hold of a S such that TS is Iw, and ST is Iv not just that, you would also like to have the map S to be a linear transformation.

So, we will now established that ST is Iv so let us see, how do we establish that St is Iv, so, we will show that ST is Iv now. So, want to show ST is equal to Iv, to do that observe that if you look at for v in capital V, you look at STv, this T STv is nothing but TS Tv by the associativity of composition, and we already know that TS is Iw. So, this is equal to Iw Tv which is equal to Tv.

But what do we know about RT, we know that T is injective, that is there in the very hypothesis of our proposition, and this implies that STv is equal to v, for all v, our choice of flavors arbitrary after all. And that implies that ST is the identity of v and therefore we are done in establishing that it is a inverse of T that S is an inverse of T.

So, finally we want to check that S is a linear transformation, that also is quite straight forward, we will use a similar technique as above. So, what is S of w 1 plus w 2, this what we would like to study, T, TS of w1 to w2, this just turn out to be equal to TS is identity and therefore this is just going to be w 1 plus w 2, but we know that w 1 plus w 2 is TS w 1 plus TS w 2, because after all TS is the identity map.

And by the property of a linear transformation, this is equal to T of S w 1 plus S w 2, after all T is a linear transformation, we do not know what S but we certainly know that T is a linear transformation. So, what have we obtained here? We have obtained that TS of w1 plus w2 is equal to T of S w 1 plus S w 2 but T is injective.

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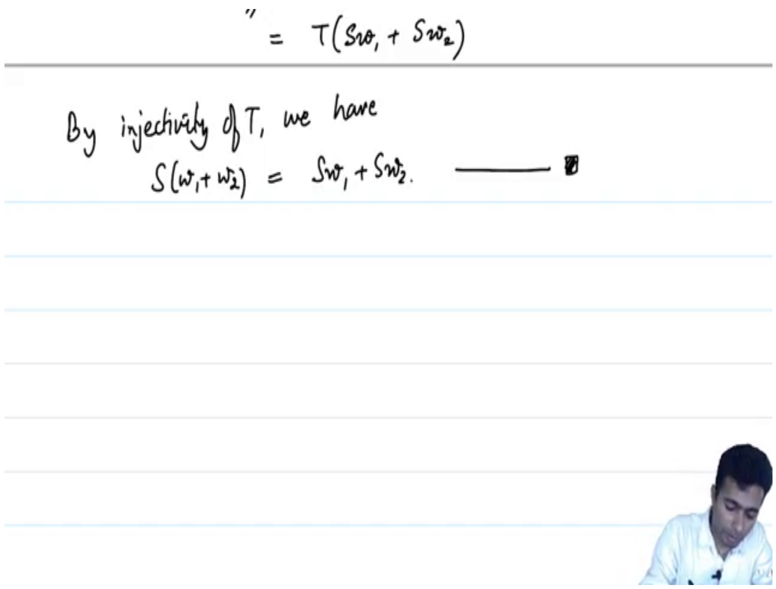

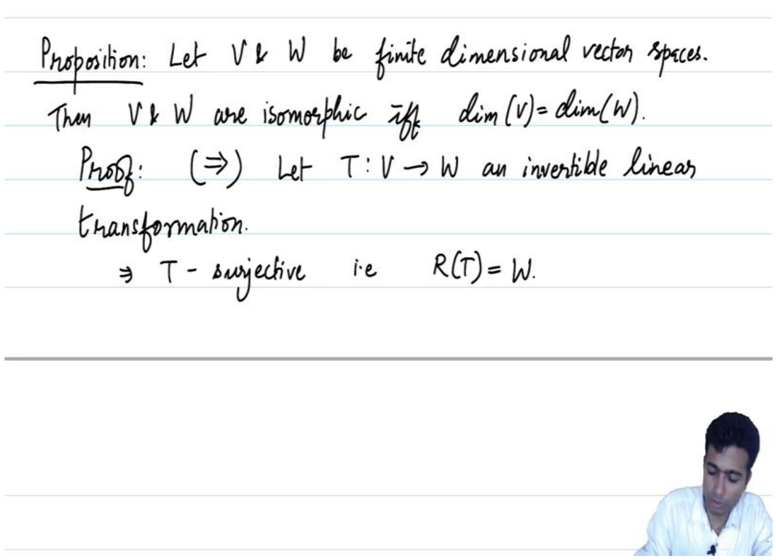

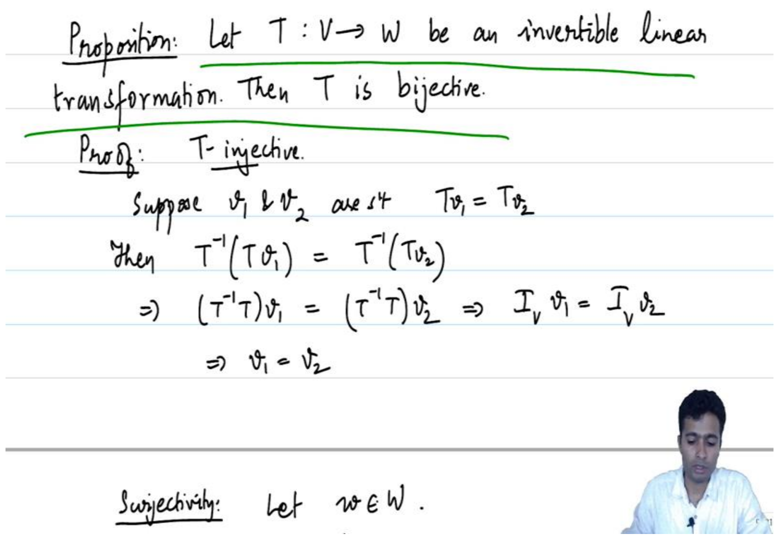

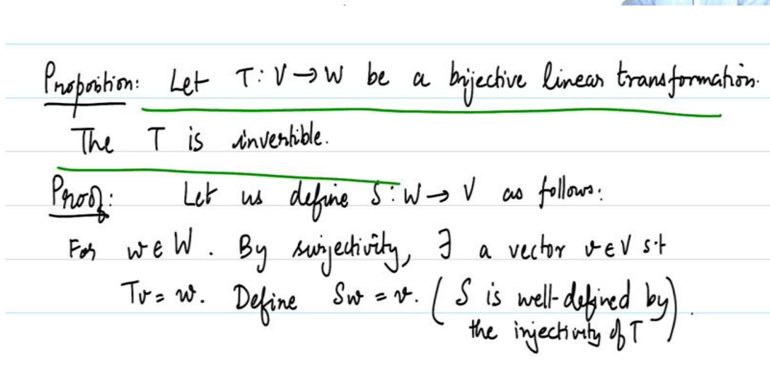

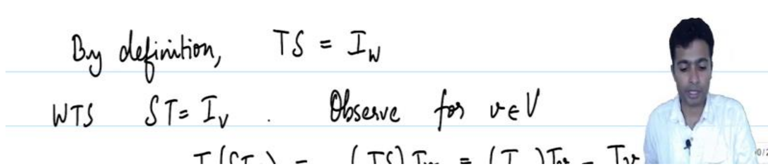

By injectivity of T, we have S of w 1 plus w 2, is equal to S w1 plus S w 2 and we are through. So, let us just try to look back at what we have shown in this proposition the proposition, which I am now underlining by Green, we have shown that in the linear transformation, which is invertible is necessarily a bijective map. And in the next proposition, which I am now going to underline in green again, every bijective linear transformation is necessarily an invertible linear transformation.

So, what we have now shown is that a linear transformation is bijective if and only if it is invertible linear transformation. So, as I was trying to point out earlier, invertible linear transformations are special, let me show that by the following proposition, we will show that two vector spaces are isomorphic or there exist an invertible linear transformation from a vector space V to W if and only if it has the same dimension.

So, let V and W be vector spaces then there exist, maybe we will just restrict ourselves to finite dimensional vector spaces, be finite dimensional vector spaces then V and W are isomorphic, I am using the terminology recall that two vector spaces are isomorphic if there exist a invertible linear transformation from one to the other or the other from the one.

So, if and only if dimension of V is equal to the dimension of W, let us give a proof of this statement, let us try to establish that if V and W is isomorphic, then dimension of V is as same as dimension of W. So, what is a meaning of V and W being isomorphic? V and W are isomorphic means that there exist the T from V to W be an invertible linear transformation. So, there exist in invertible linear transformation from V to W.

What have we just proved about invertible linear transformations, we showed that an invertible linear transformation should necessarily be both injective and surjective. So, this implies T is surjective and what is the meaning of T surjective i.e the range of T is equal to W. That is good.

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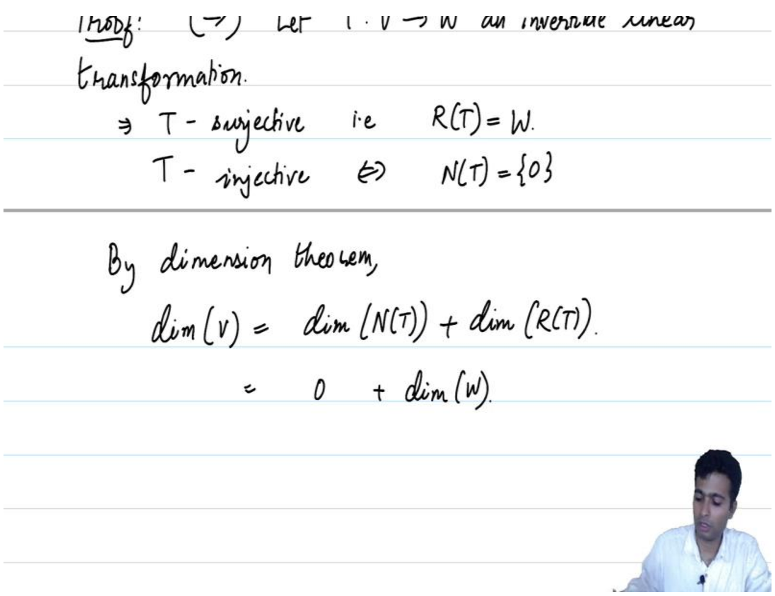

You also know that T is injective, T is injective and we know that T is injective, if and only if the null space of T is the 0 space, 0 vector space. So, we should go back a few lectures and check that these are results which we had proved and in the same lecture we had proved the dimension theorem. By the dimension theorem, let me recall the dimension theorem, for you, dimension of V is equal to the dimension of the null space of T plus the dimension of range space of T.

We now know what our null space of T is, what our range space of T is, our null space of T is just the 0 vector space and we know that the dimension of that is 0 and we know what our range space of T is exactly equal to W, this is just our dimension of W. So, we have proved one side of this, we have establish that if two vector spaces are isomorphic, then they should necessarily have the same dimension. Now, let us try to prove the converse.

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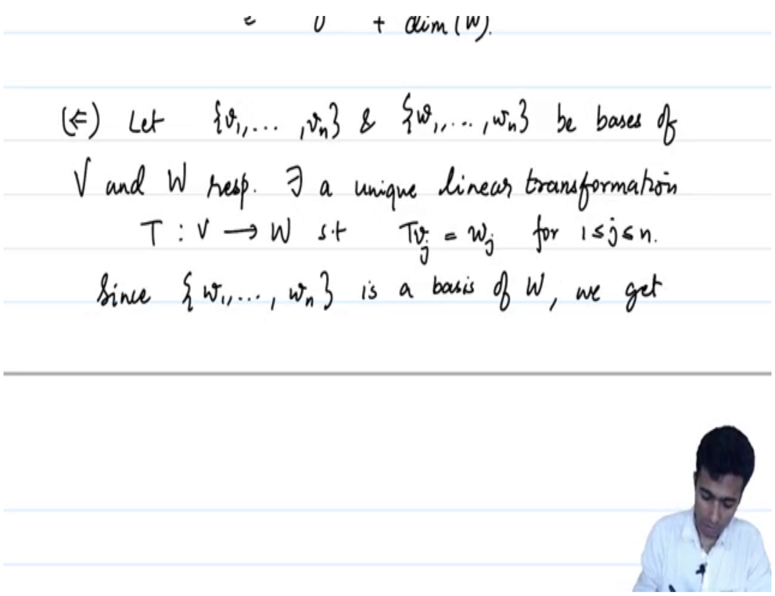

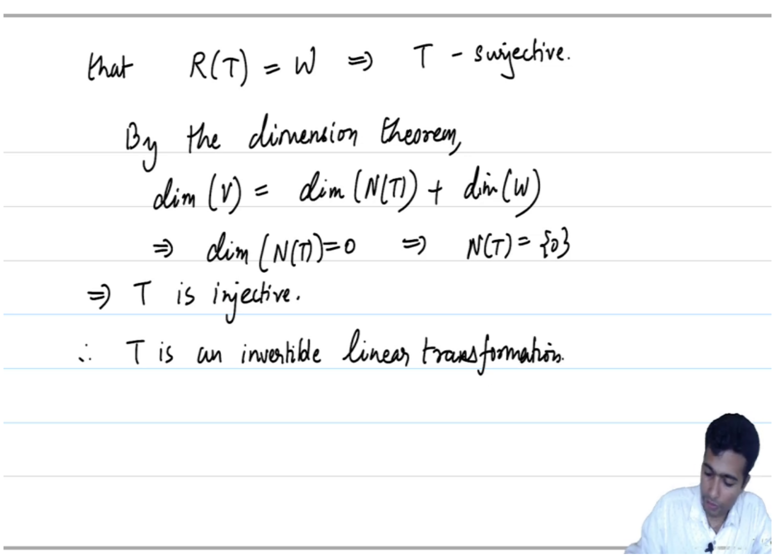

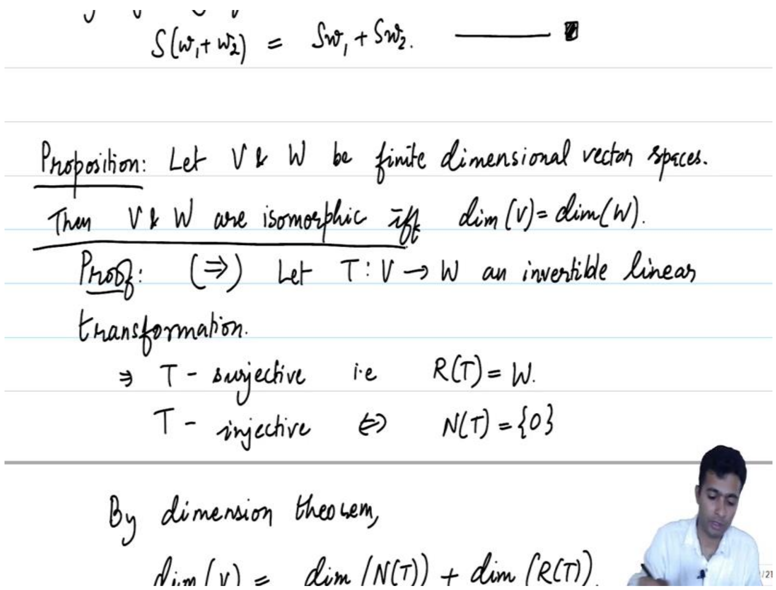

So, let a v 1 to vn and w 1 to wn let these be basis of V and W respectively, so to prove the converse. What is the statement? The statement would be that if dimension of V is equal to dimension of W, then there exist an invertible linear transformation between V and W. So, let us try to construct one.

I think the lectures from the week tells us there exist a unique linear transformation T from V to W such that Tv j is equal to wj for all 1 less than or equal to j less than or equal to n, because Vv 1 to vn is a bases for any vector in fact w 1 to wn there exist is a unique linear transformation such that Tvj is equal to wj.

But we know that w1 to wn is a bases of w and that makes this particularly linear map special. Since, so I will just leave this is as a check, since w1 to wn is the, is a basis of W and should go back and check that Tvj is exactly turn out to be a spanning set of the range space of T, we get that range of T is equal to W, which implies T is surjective.

So, now to show that injective, the injectivity of T we know that T is a linear transformation, let us invoke the dimension theorem again. By the dimension theorem, what do we know about the dimension theorem? Dimension of V is equal to the dimension of the null space of T plus dimension of the range space of T, which we now know is our vector space W, but our hypothesis to begin with is that dimension of V is the same and dimension of W, which implies dimension of null space of T is 0, which implies that the null space of T is just the 0 space, which implies T is injective.

So, what we have established now is that our map T which is a linear transformation is injective and surjective, therefore T is invertible linear transformation not just that we have hence concluded that.

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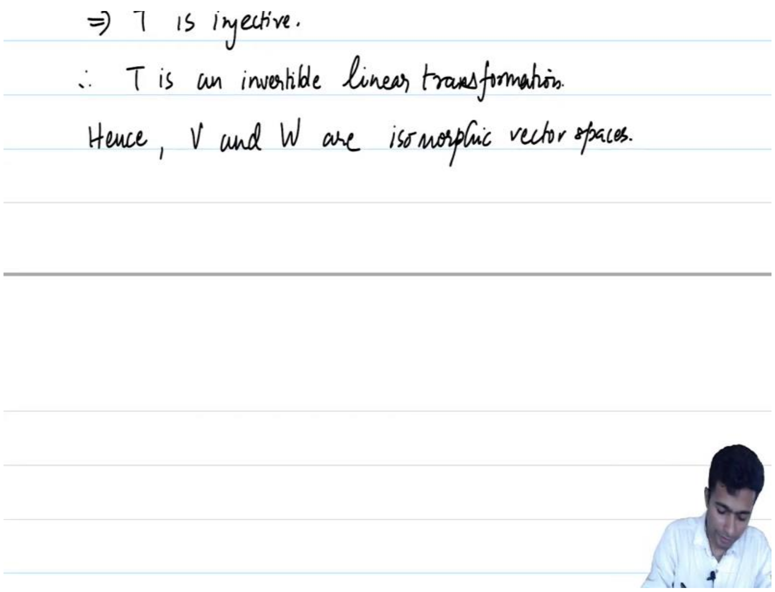

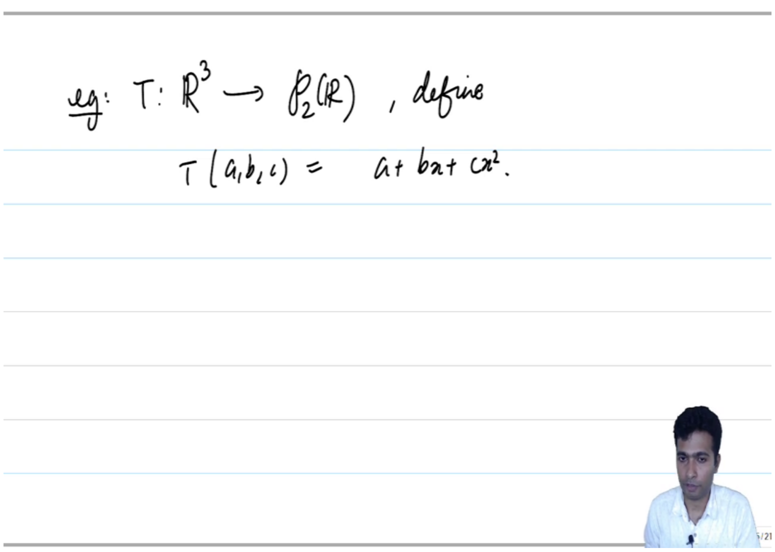

Hence, V and W are isomorphic, even before going to this theorem one could have easily check with an example that say R 3 from R 3 into P 2 of R we define the following map, define T of a, b, c to be equal to a plus bx plus c x square it is easy to check that T is both injective and surjective and therefore, T turns out to be an isomorphism.

And we can therefore for all practical purposes you will see more of it we can identify R 3 and P 2 of R. The reason why we are stressing so much on this is, because we would like to somehow use the language of matrices, which we are able to use in the case of Rn to all finite dimensional vector spaces.

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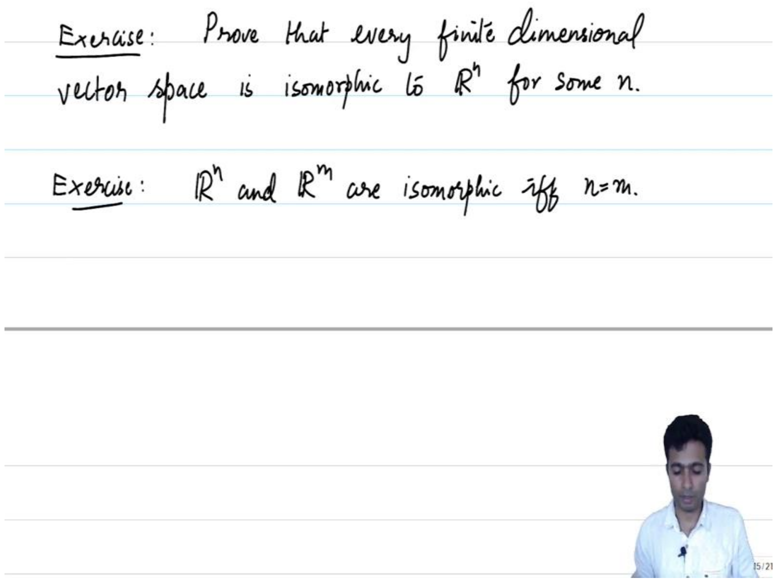

I would you like to give an exercise here at this point before we stop, prove that every finite dimensional vector space, it is a direct consequence of the theorem which we have proven (())(48:45) is isomorphic to Rn for some fixed n. Another exercise is to note that Rn and Rm are isomorphic, if and only if n is equal to m. I will stop here.