Linear Algebra
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Lecture 4.2
Linear Transformations and Matrices Continued

So, in the last video we saw what is meant by a matrix associated to a linear transformation, once we are given an ordered basis of the domain and the (())(0:23). So, in other words if T is a linear map from V to W and if alpha is an ordered basis of V and beta is an ordered basis of W, we defined the notion of a matrix associated to T with respect to alpha and beta.

Now, we are familiar with matrices, we know that matrices can be added, we can talk about scalar multiplication of a scalar of the metrics. We can also talk about product of matrices. So, the goal of this lecture is to explore if something similar can be done with linear transformations and if something similar can indeed be done, is it compatible or in sync with what we know about matrices because after all, we are able to associate a matrix to given linear transformation. So, let us begin.

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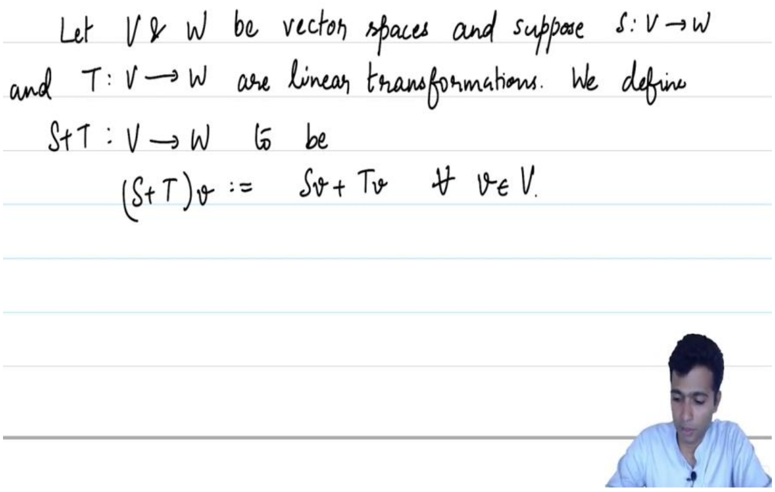

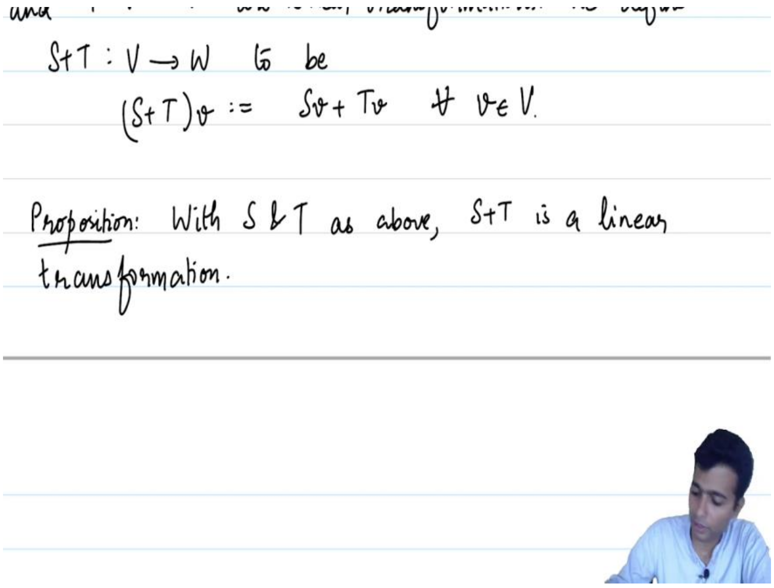

So, let V and W be vector spaces. So, our first question is can we talk about the sum of or addition of 2 linear transformations S and T between 2 vector spaces V and W. So, let V and W be vector spaces and suppose S from V to W and T from V to W are linear transformations.

We define S plus T, a function transformation from V to W to be S plus T of v is equal to by definition, Sv plus Tv for all v in capital V. Note that Sv and Tv both are vectors in W and therefore, we can talk about addition of these vectors in W. Moreover, it is also important to note that as of now, we do not know whether S plus T indeed happens to be a linear map, as of now, it is just a transformation. It is just a map.

And yes, so the next proposition will tell us that S plus T is always a linear transformation with S and T as above, S plus T is a linear transformation. So, let us give a quick proof of it.

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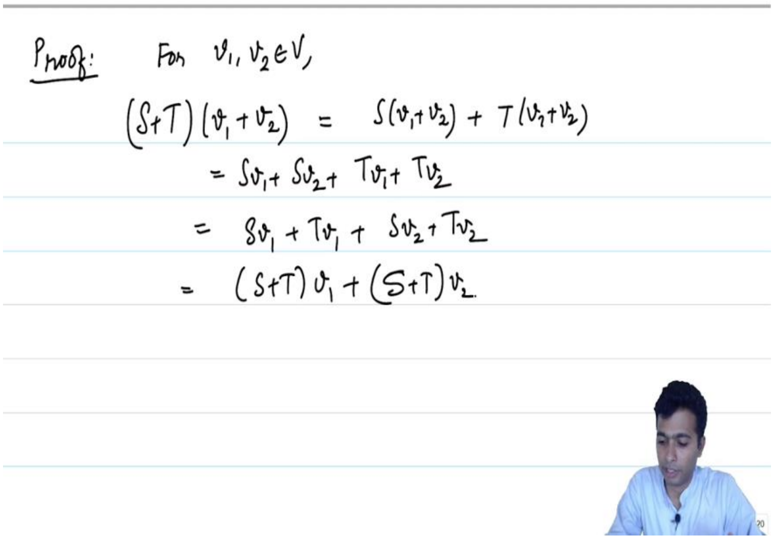

I will just indicate one part of the proof to check that S plus T is a linear transformation, we need to take that S plus T preserves both the addition and least scalar multiplication. Let us just check the addition part. So, for v 1, comma v 2 in capital V. Let us ask what is S plus T of v 1 plus v 2, we would like to show that S plus T of v 1 plus v 2 is S plus T of v 1 plus S plus T of the v 2.

But this by definition, turns out to be S of v 1 plus v 2 plus T of v 1 plus v 2 which is equal to Sv 1 plus Sv 2 by the linearity of S plus Tv 1 plus Tv 2, which by the vector space axioms will be Sv 1 plus Tv 1 plus Sv 2 plus Tv 2, which is equal to S plus T of v 1 plus S plus T of v 2. We have just checked that S plus T indeed preserves the additive structure.

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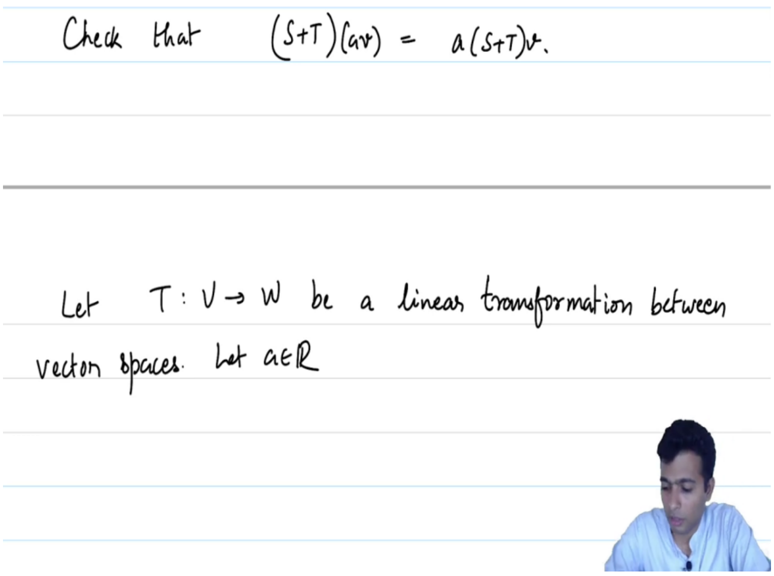

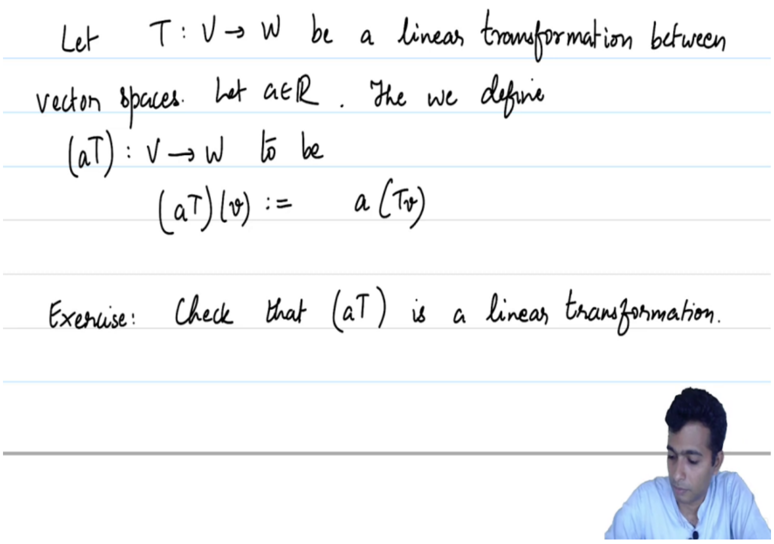

I will leave it as a check that S plus T of av is equal to a times S plus T of v. And with this we would have established that if you are given 2 linear transformations, S and T, we can talk about the sum S plus T. We would also like to now see, if we can talk about scalar multiplication, as to be expected, it will be the straightforward definition that we can give just like in the case of addition.

So, let a T from V to W be a linear transformation between vector spaces. And as usual our vector spaces are always real vector spaces here in this course be linear transformation between vector spaces. Now, let a, b, a scalar real number. Then we define the scalar multiplication of a and T, which will denote by aT, this is a map from V to W we define this to be aT of v is equal to a times Tv.

Now, observe that aT, a priori did not have a meaning of its own. So, aT is just a symbol, aT is a function now, which acts on V and gives you some vector in W. And what is that vector? We already know that T is a linear transformation. So, Tv is a vector in W and you look at this scalar multiplication of a with the vector T. We define aT of v to be a of T rather a times Tv. So, an exercise just like in the previous case, check that aT is a linear transformation. So, straightforward check and I will leave it as an exercise.

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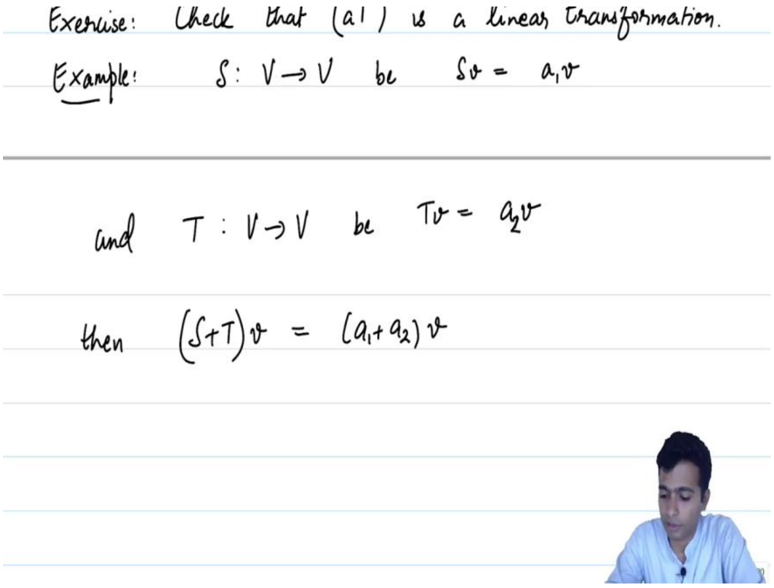

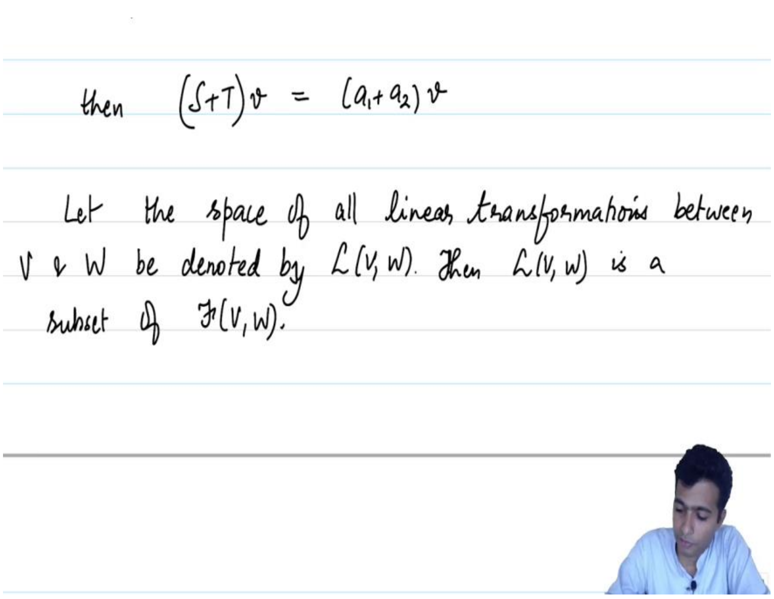

So, let us look at an example. So, let us take a vector, let us take a linear transformation S from V to itself, be S of V being equal to say, a 1 v, it is a dilation by a 1 and let T from V to itself be Tv is equal to another dilation, does not need to be a dilation could be any linear map. But for the sake of this example, let us look at say S from R 3 to itself, given by S of any vector V is a 3 times the vector and T is the vector, which is dilating it by say 6 times, so T of v 6 times v.

Then what is our S plus T of v by definition is Sv plus Tv which turns out to be a 1 plus a 2 times v which is a dilation again. We know it should be a linear map, but in this case it turns out to be a dilation itself. Similarly, we can talk about, will be a straightforward concept. So, let me not spend too much time on the examples, the thing however, which I would like to notice that.

So, let the space of all linear transformations between V and W, let us denote the space of all linear transformations between V and W be denoted by L of V, comma W. Then the first thing to observe is that every linear transformation in particular is a function from V to W. So, hence then L of V, W is a subset of the familiar vector space F of V, comma W, recall that F of V, comma W is the vector space of all functions from V to W.

Here, the addition is defined the most obvious manner, you take F and G here, F plus G of any vector V is Fv plus Gv and a constant times F will be exactly like how we defined a minute back, the scalar multiplication of a vector and sorry a scalar and a linear transformation, the way we defined it exactly similar manner, we can define the scalar multiplication Fv of W. And we have realized L of v, comma w as a subset of this vector space V w.

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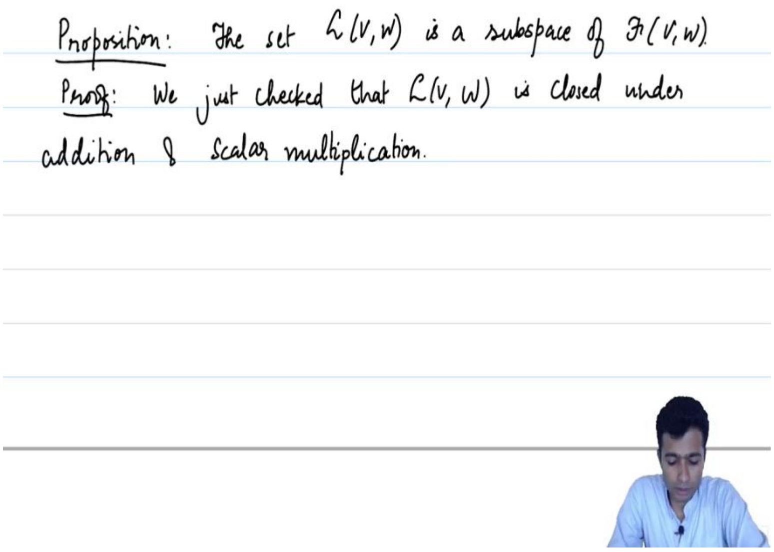

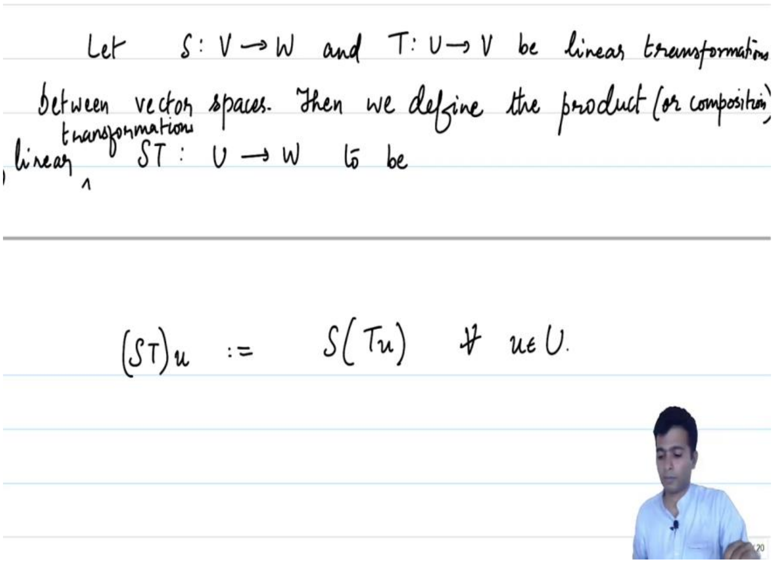

And further, what did we show? We showed that if you take 2 such linear transformations and added, you get back a linear transformation. If you look at the scalar multiplication of a scalar and a linear transformation, you get back a linear transformation. So, this prompts us to write down the proposition that the set L of V, comma W is a subspace of F of V, comma W.

So, the proof is quite checked already, we just checked that L of V, comma W is closed under addition and scalar multiplication and by a lemma or a proposition which we have proved earlier, L of V therefore turns out to be a vector subspace of F of V, Comma W in particular L of V, W is a vector space.

So, that is something so, we get from this proposition, the space of all linear transformations from V to W is a vector space. Alright, not only addition and scalar multiplication, we are also familiar with the notion of multiplication of product of matrices. When can we talk about the product of matters, suppose you start off with an L cross M matrix and then M cross N metrics, we can look at the product and obtain an L cross N metrics.

So, we would like to have an analog. So, I have not yet described or discussed the relationship between the addition that we have just defined up linear transformations and the addition of matrices but we will be coming to that in a minute. Before that, let me talk about product of linear transformations.

So, let S be a map from V to W. And T be a map from U to V be. Let linear transformation between vector spaces U, V and W between vector spaces. Then we defined the product, product of what? Product of ST which will be now from U to W to be the usual composition. So, this can also be written, we defined the product or composition, the more familiar notion of composition (())(13:39). This is the same, this is defined to be ST of u, not put brackets unnecessarily, u is defined as S of Tu for all u in capital U, so, this is the usual notion of composition of functions.

In our case, we would also like to call it product of linear transformations, the product of linear transformations.

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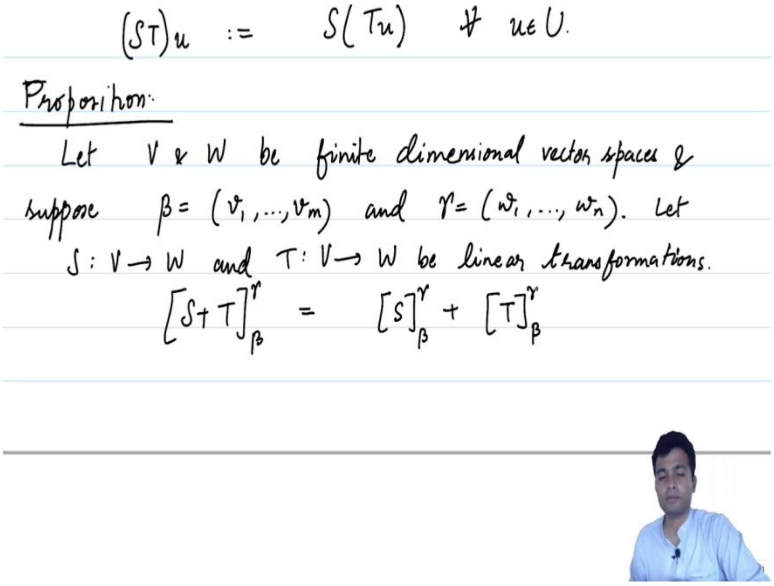

Alright, so it is time to now talk about how these notions we have just defined of additions, scalar multiplication and of product, how does it gel with the notions of the addition of matrices, product of scalar multiplication of matrix with the scalar and product of matrices? So, to do that, let us fix an ordered basis of so we will restrict our attention to finite dimensional vector spaces.

So, let V and W be finite dimensional vector spaces. You would like to talk about the matrix associated to it, so finite dimensional vector spaces are essential as a hypothesis. And not just that let us fix an ordered basis of V and W. And suppose beta equal to v 1 to v m and gamma equal to w 1 to w n, note that generally, we will keep track of the m and n very carefully. What we have effectively told is that our v is of dimension m and w is of dimension n, that is what we have just said.

So, suppose we start off with 2 finite dimensional vector spaces V and W and we fix ordered basis for V and W respectively. Further, let S from V to W and T from V to W be linear transformations. Now, the moment we have a linear transformation from a finite dimensional vector space to another finite dimensional vector space, we can talk about the matrix of the linear transformation once we have fixed the ordered basis.

So, here we have S and T, linear transformations, two linear transformations from V to W. We also know that S plus T is a linear transformation from V to W. Now, we would like to know what is the matrix of S plus T with respect to beta and gamma, then theorem, let me start off with a statement like this proposition, this is indeed a proposition and we will be proving that. Then the matrix of S plus T is actually equal to the matrix of S with respect to beat, gamma plus matrix of T with respect to beta, gamma. So, let us prove this statement.

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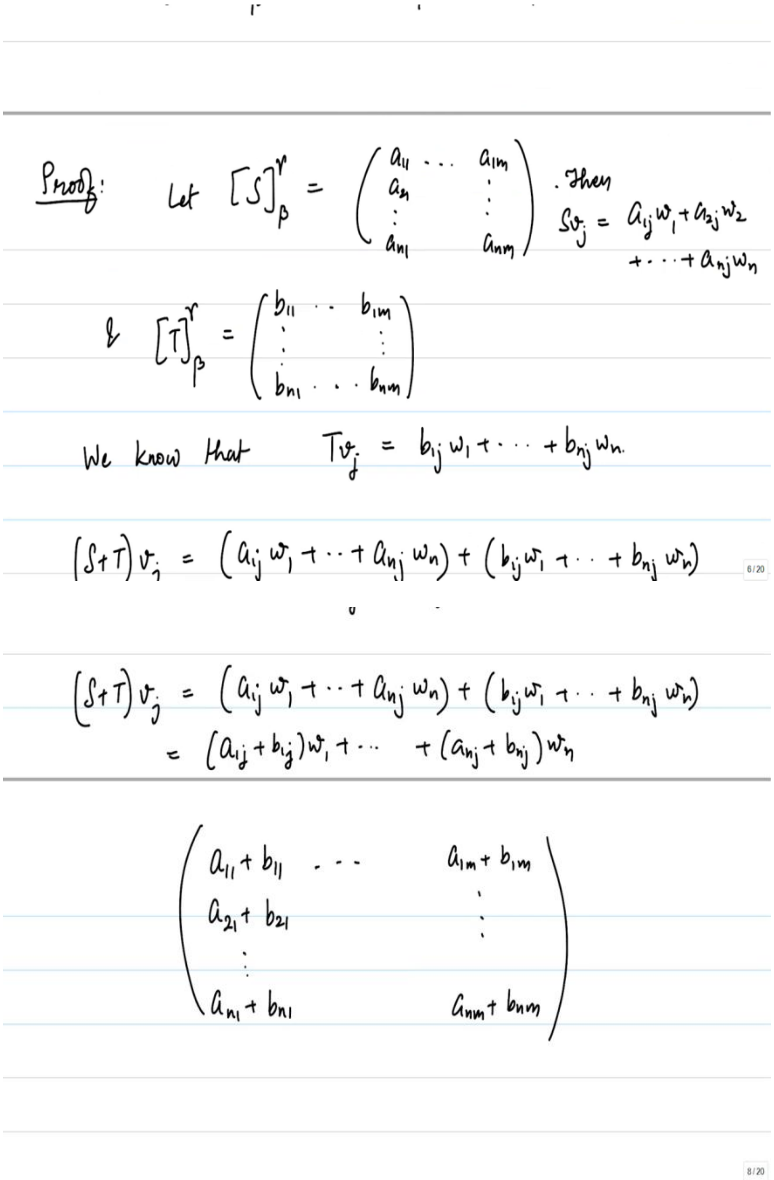

$$\begin{pmatrix} \vdots & & \\ a_{n1}+b_{n1} & & a_{nm}+b_{nm}\end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm}\end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1m}\\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm}\end{pmatrix} = [S]_\beta^\gamma + [T]_\beta^\gamma.$$

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So, what is our matrix of S with respect to beta, gamma and the matrix of T with respect to beta, gamma? Let us give it some names. First of all, let S be equal to say, a 11 remember, this is a linear transformation from an m dimensional vector space into an n dimensional vector space. So, this is going to be a 11, a 21, a n 1, a 1 m unto a n m. Alright and T beta, gamma be equal to b 11 to b 1 m, b n 1 n to b nm.

So, we know this is with respect to beta and gamma. So, we know that Tv j is equal to the jth column will come into the story. So, here we know that then Sv j is equal to the jth column will come in that is a 1j w 1 plus a 2j w 2 plus dot dot dot, a mj, nj w n, there are n basis vectors of W, so an n dimensional vector space. Similarly, we get T vj will be b 1j w 1 plus dot dot dot b nj w n.

So, now, let us look at what is S plus T of vj by very definition, it is going to be S vj plus T vj, which is going to be a 1j w 1 up to a nj w n plus Tv j, what is Tv j? b 1j w 1 plus b nj w n. We can do all the necessary rearrangements and finally obtain this is a 1j plus b 1j w 1 up to a nj plus b nj times w n. So, we have obtained the jth column of the matrix for S plus T. And therefore, jth column is going, the first column will look like a 11 plus b 11, a 21 plus b 21, a n 1 plus b n 1, jth column will be similarly written.

And finally, the mth column will be a 1 m plus b 1 m, a nm plus b nm which is exactly equal this matrix is exactly equal to a 11, a 21 up to a n 1, a 1 m, a nm plus b 11 to b 1 m, b n 1 to b nm which is equal to S beta gamma plus, did I write it wrong? Yeah, this is S beta gamma plus T beta gamma. So, we have proved the statement.

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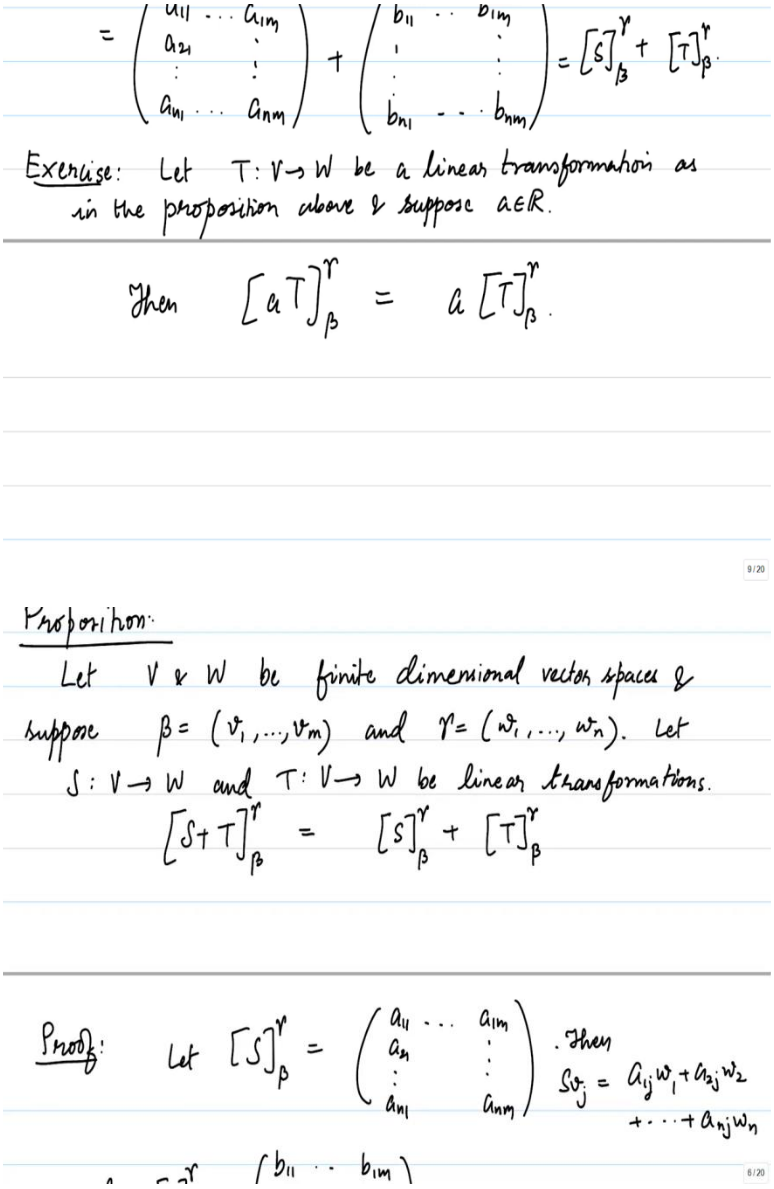

I will give it as an exercise to check that something similar can be concluded about the scalar multiplication as well. So, let T from V to W be as in the, be a linear transformations as in theorem above this, so as in the proposition above. And suppose, a is a real number then aT beta gamma, the matrix of aT with respect to beta and gamma, this will turn out to be a times the matrix of T with respect to beta gamma. So, I leave this as an exercise for you to check the proof runs in a similar flavor and I would advise you to strongly work of the details.

Alright, so we have now talked about what the matrix or how the matrix associated to a linear transformation behaves when we add to linear transformations. We have also given of course as an exercise about how if you look at the scalar product of a scalar vector a linear transformation, how does the matrix corresponding to it behaves.

Now, we would like to now, we would like to now explore how the linear transformation we would like to now explore how the matrix of a product of linear transformations behave. So, for that let me write down a proposition and directly tell you what the expected outcome would be.

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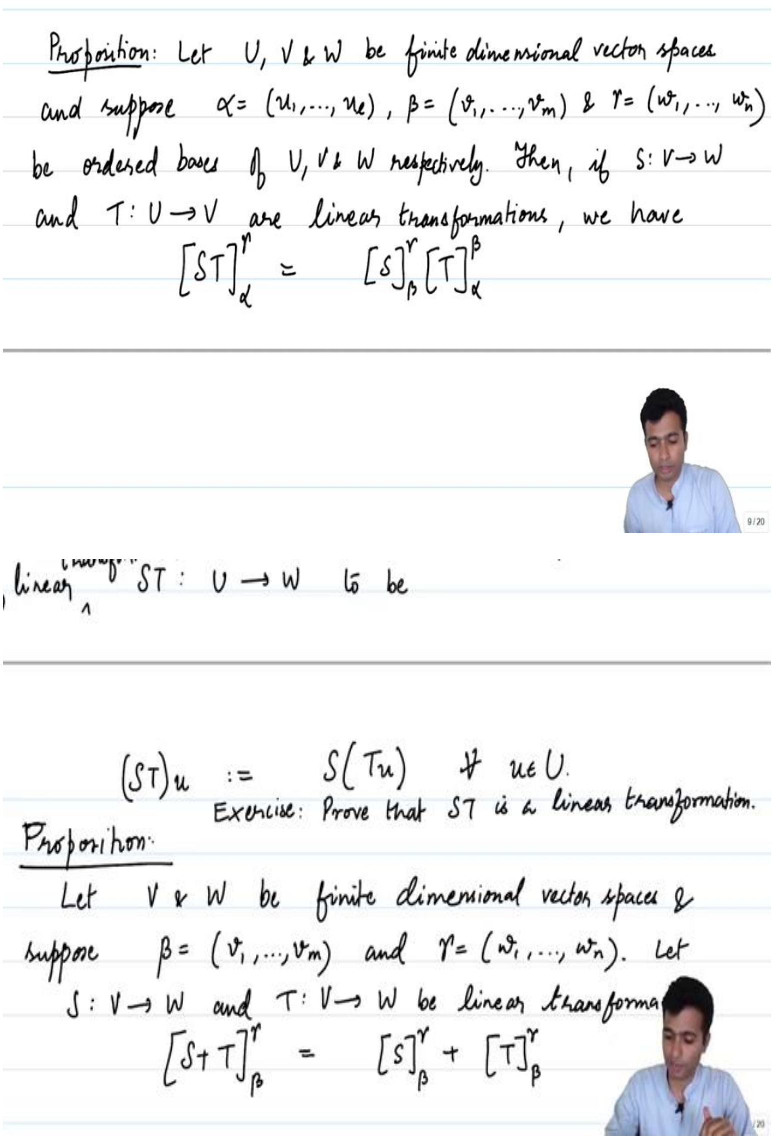

So, to talk about product of linear transformations, we need to start with 3 vector spaces, so let U, V and W be finite dimensional vector spaces again we are going to associate matrices. So let them be finite dimensional vector spaces. So, as you can see, we have not talked about U, V, W or U, V in the previous case being finite dimensional till we wanted to talk about matrices associated to this linear transformations. That is to tell you that these notions are not necessarily defined only in finite dimensional case. We can talk about the sum of 2 linear transformations in an infinite dimensional vector space as well.

We can talk about the composition in a finite dimensional vector spaces. Maybe after this proposition will give you an example in an infinite dimensional case, right now let us focus however, on finite dimensional cases, just so that we can see how the product and the matrix multiplication are related. So, here, we fix 3 basis, ordered basis for U, V and W respectively. So, let us fix that and suppose alpha which is equal to say u 1 to u L, beta which is equal v 1 to v m and gamma which is equal to w1 to w n be ordered basis of U, V and W respectively.

Then the proposition states that if S from V to W and T from U to V are linear transformations, then in particular there is a matrix associated to both S and, we have ST as you can see, this is a linear transformation, which we have already checked or have we checked? We have not checked that is actually a simple check.

So, I would just like to give an exercise here prove that ST is a linear transformation. If would talk about a composition of 2 linear transformations, you are not going to get some arbitrary function, you will again get back a linear transformation.

Maybe we can work it out here but before that, assuming that this is assuming that you have done the exercise, ST which is a linear transformation will have a matrix representation with respect to what, let us just try to analyze where our map ST is from. T is a map from U to V and S is a map from V to W and therefore, ST is a map from U to W, what is the, what are the basis in U and W? Alpha and gamma.

So, we would like to talk about the matrix of ST with respect to alpha and gamma, this turns out to be the matrix of S with respect to what is S, where is S from, S is a map from V to W. And therefore, this is going to be from beta to gamma and times T from alpha to beta. So, I would like to note at this point that the operations of product this is not necessarily commutative operation.

So, in other words ST is not equal to T S, in fact, TS need not even makes sense if the vector spaces U, V and W are not suitably picked. It is almost the same case with the product of matrices. If you look at matrix, which say 1 cross 2 and matrix, which is 2 cross 3, if you multiply you will get 1 across 3 matrix. However, you cannot look at the commutative version of it. So, if a was the 1 cross 2 and b was the 2 plus 3 matrix, then ab makes sense, but ba does not even make sense. So, that goes here. So, the product of s and t is in this order.

So, before we go further ahead, I would like to give you an example of the fact that the product ST need not be equal to TS, even in the finite dimensional case. Of course, if you already prove this proposition, then we are through, we know that there are matrices which do not commute. And for each matrix we can talk about a linear transformation and therefore, we will get hold of linear transformations which do not (())(30:21).

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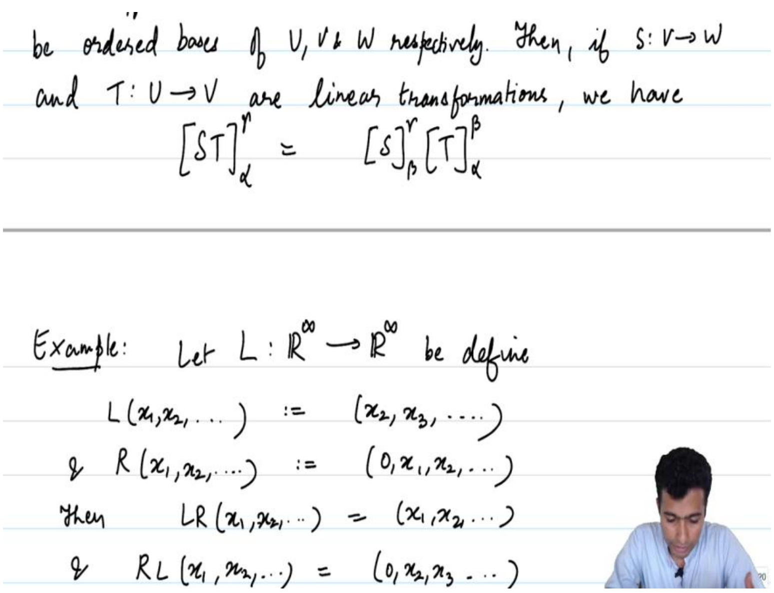

But, before going into all these propositions, let us just give an example to indicate that in the infinite dimensional case, also, this is not true. And even if the domain is from sorry, the linear transformation is from the vector space to itself even then does not make sense to talk about commutativity, why? Let us look at an example.

Let U be a map from say R infinity to itself. Remember R infinity is the vector space of all sequences x 1, x 2, x 3, infinite sequences. So, let us define U of, in fact, let me call it something else, let me call it the left shift. So, let me call it L. I hope, I am not going to confuse you, L of say x 1, x 2, dot, dot dot. This is the left shift let this be defined to be we have seen this example.

This is the inner transformation L that we would like to define, check that this is a transformation and R be the right shift operator. This time R is not the range, R is the right shift operator which is defined as 0, x 1, x 2 and so on. Then check that LR is just the identity map. This is just x 1, x 2, dot dot dot.

And what is RL? Check that RL of x 1, x 2 and so on. First, it will do a left shift by throwing out x 1 and then it will do a right shift and therefore, you will be ending up with something like 0, x 2, x 3 and so on, which is not the same as LR of x 1, x 2, so on, if x 1 is not equal to 0. Alright, so we have a straightforward example of when 2 linear transformations do not compete with each other.

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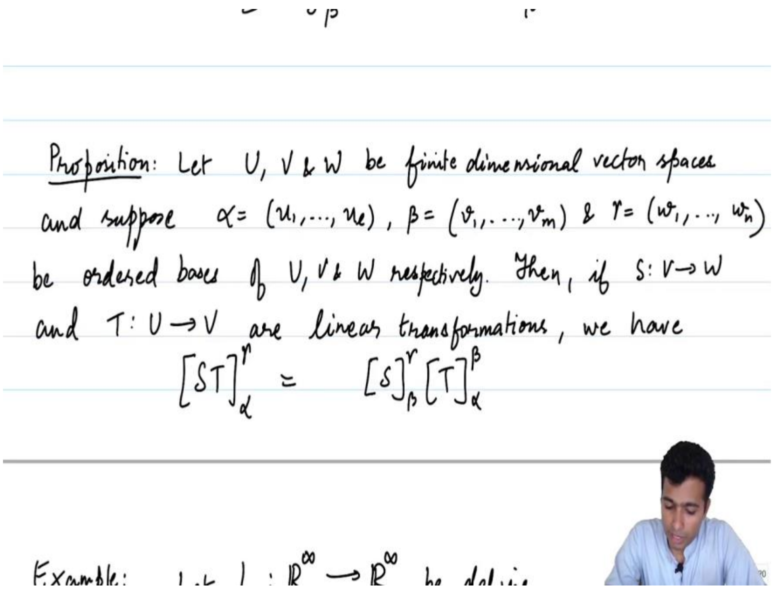

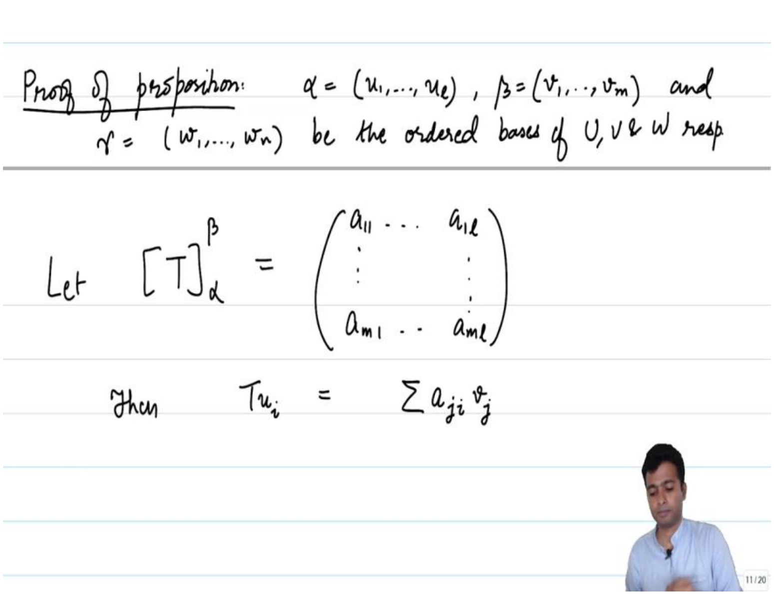

Let us further established by this proposition which tells that if you have the product of 2 linear transformations and if you look at the matrix corresponding to it, it is going to be the corresponding product of the matrices. So, let us go ahead and prove this proposition. So, proof of the proposition. It is more or less keeping track of the indices that will give us the proof. So, let us quickly go through the proof.

So, recall that alpha is equal to u 1 up to u L beta is equal to v 1 to v m and gamma equal to w 1 to w n be ordered basis of U, V and W respectively and remember that S is a map from V to

W, T is a map from U to V. So, let T is a map which is from U to V. So, this will be alpha to beta or rather alpha beta. Let this be given by, let us put it in the matrix notation a 11. So, our U is of dimension L and V is of dimension m, so this is going to be a m 1 to a m.

So, this final one would be for the Lth vector. So, this is going to be a 1 L to a ml be matrix of T. So, then we can write in compact notation, what can we write in compact notation T u I, this is equal to will be given by the jth column there that will be summation. So, will be the ith column there will be a ji v j.

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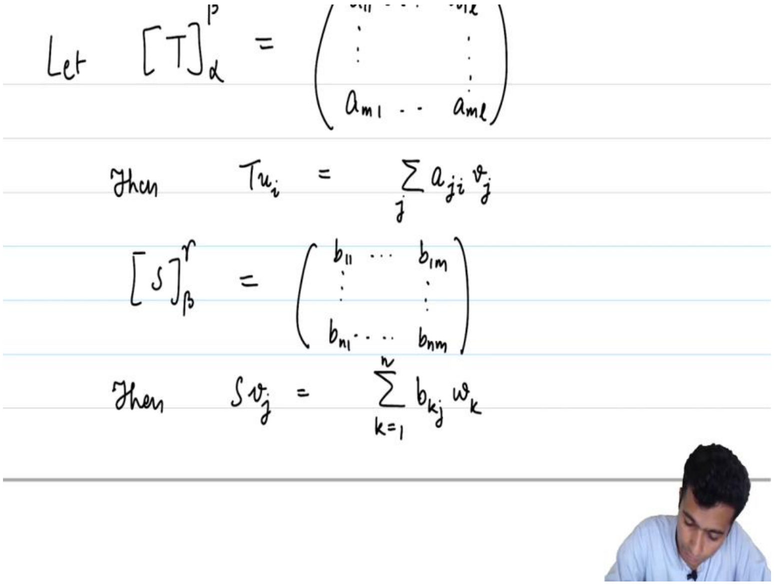

Similarly, let us give some give the matrix representation of S. Now, S is a map from V to W, V has an ordered basis beta. W has an ordered basis gamma. So, this let it be b 11 to b this is I am think from V to W, W has dimension n. So, this will be n 1. And there are L such vectors sorry m sub vectors this will be b n m.

So, we have to check that the indices are making sense after all the dimensions of V and W are clear. So, this is right and this particular matrix representation tells us that S vj is equal to summation, L be the jth column called, jth column would be b kj w k where k is going from 1 to n and here j is going from 1 to m.

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and suppose $\alpha = (u_1, \ldots, u_\ell)$, $\beta = (v_1, \ldots, v_m)$ & $\gamma = (w_1, \ldots, w_n)$ be ordered bases of U, V & W respectively. Then, if $S: V \to W$ and $T: U \to V$ are linear transformations, we have

$$[ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta$$

Example: Let $L : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be define

$$L(x_1, x_2, \ldots) := (x_2, x_3, \ldots)$$

$$\& \quad R(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$$

Then $LR(x_1, x_2, \ldots) = (x_1, x_2, \ldots)$

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Then $$S v_j = \sum_{k=1}^{n} b_{kj} w_k$$

$$(ST) u_i = S(T u_i) = S\left(\sum_{j=1}^{m} a_{ji} v_j\right)$$

$$= \sum_{j=1}^{m} a_{ji} (S v_j) = \sum_{j=1}^{m} a_{ji} \left(\sum_{k=1}^{n} b_{kj} w_k\right)$$

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$$= \sum_{j=1}^{m} a_{ji} (S v_j) = \sum_{j=1}^{m} a_{ji} \left(\sum_{k=1}^{n} b_{kj} w_k\right)$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{m} b_{kj} a_{ji}\right) w_k$$

Let $$c_{ki} = \sum_{j=1}^{n} b_{kj} a_{ji}$$

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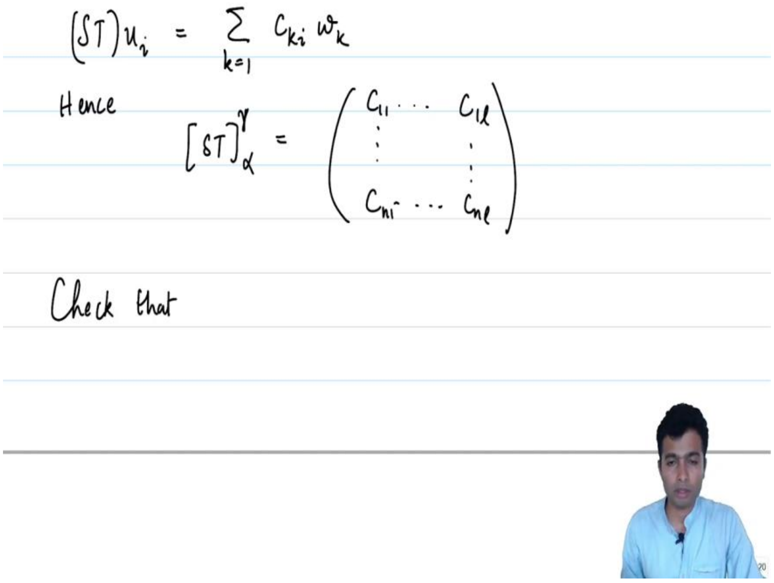

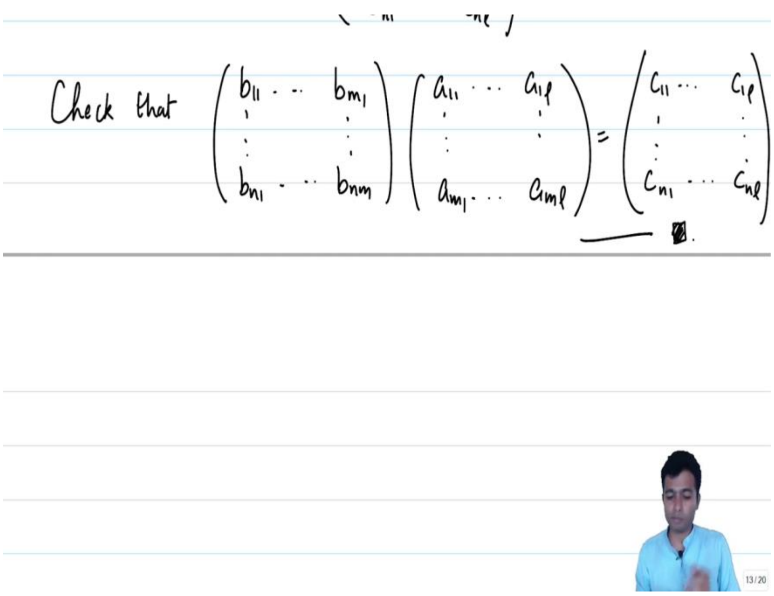

All right, so now let us try to see what is what are we trying to do? We are trying to show that ST, you are trying to show this, you are trying to show that ST alpha gamma is equal to S beta gamma times T alpha beta. So, let us look at what is ST of u j. To talk about ST of alpha gamma, let us look at ST of u j, u j will be S of by definition T u j.

But we know exactly, let us call it u i, so that our indices do not get confused. So, we know exactly what T u i is this is equal to S of summation j is equal to let us bring it up so that you can see it this is equal to 1 two m a ji v j.

But we know that S is a linear transformation and therefore this is equal to summation j is equal to 1 to m S a ji times S vj but we also know what S vj is from our matrix representation. This is nothing but summation j is equal to 1 to m, a ji summation k is equal to 1 to n, S vj is b kj w k. And there we are, we are in good shape now, why?

Because these are, is a finite sum you can write it as summation a k, summation k is equal to 1 to n, summation j is equal to 1 to m, b kj a ji times w k. So, what we will do is, we will define c k i to be let c k i be equal to summation b kj a ji where j is equal to 1 to n, it is already start seeing the similarities with the matrix multiplication, so this is nothing but summation.

So, what is nothing but summations ST of u i, let just write that here, ST of u i is nothing but summation k is equal to 1 to n, c ki times w k. So, what is the matrix of ST? Hence the matrix of ST with respect to alpha gamma, this is nothing but c 11 to c n 1, c 1 L to c nL. I would like to leave it as an exercise right now to check that by familiar of notion of the product of matrices.

So, check that b 11 to b m 1, b n 1 to b nm this times a 11 to a m 1 here, a 1 L to a m L. This is equal to c 11 to c 1L, c n 1 to c nL. And therefore we are done. In next video, we will take the story of how matrices and linear transformations play along with each other further.