Linear Algebra
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Lecture - 4.1
Problem session

So, this video is problem session, which is based on the material that was covered in the first two weeks of this course. The main intention of the problems session is to supplement the problems that have already been given in your assignments. So, I hope that you have given a considerable amount of thought to the problems that were given in your assignments. So, let us now look at a few more problems.

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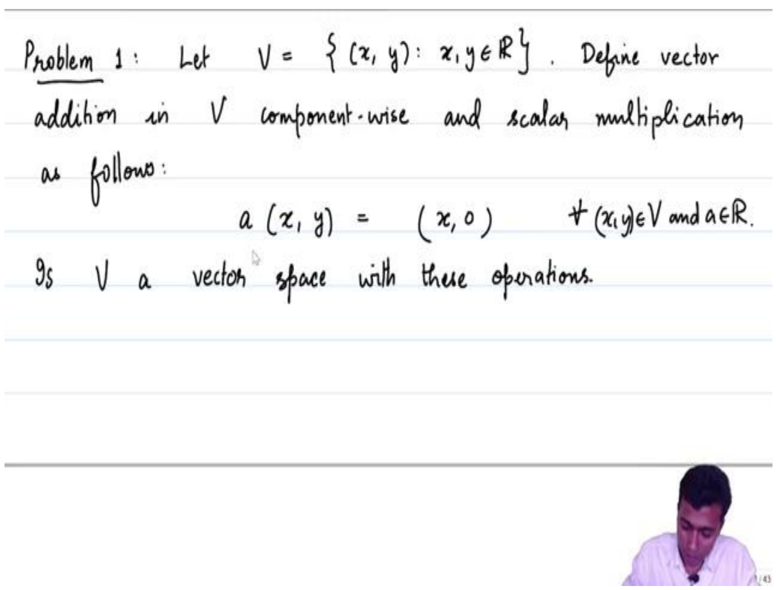

So, Problem 1. So, the problem 1 is as follows. Let V be the set of all elements x, y such that x, comma y are in R. Basically it is the Cartesian product of R with itself. Define vector addition in V component wise and scalar multiplication as follows. What is the scalar multiplication? The scalar multiplication is a times x, comma y is equal to x, comma 0 for all x, comma y in capital V and a in the field of scalars. So, the problem is to check is V a vector space with these operations.

So, notice that we are looking at the same set V which is R2, the only thing is we are tweaking the vector addition actually vector addition is the same. The scalar multiplication has been tweaked to the new one, which is underlined in green. And our task here in this problem is to check whether V is a vector space in these operations, alright. So, what do we need to do in order to establish this or solve this problem?

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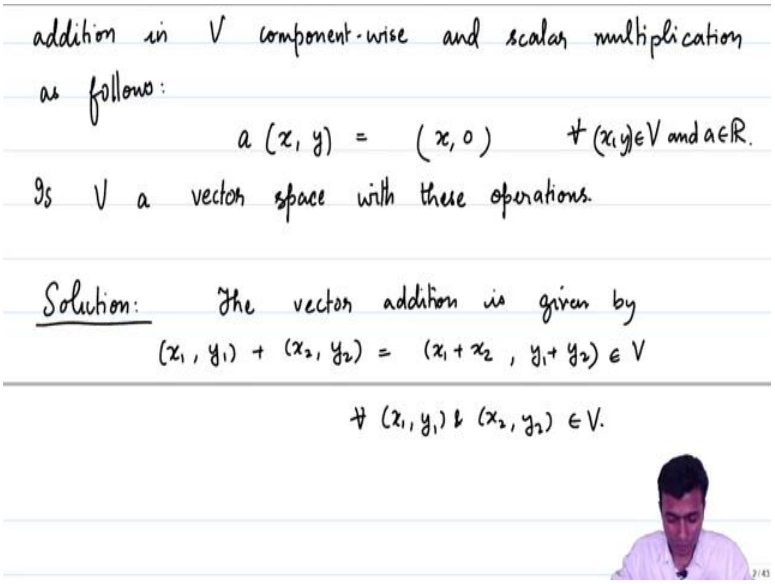

So, let us look at the solution. The first thing to notice is first thing to check is whether V is closed under vector addition and scalar multiplication. So, the vector addition, so recall the vector addition, which is component wise is given by x1, y1 plus x2, y2 is equal to x1 plus x2 component wise, y1 plus y2 which is an element in capital V for all x1, y1 and x2, y2 in capital V.

So, this is something which we have already seen in the case in the example where we check that R2 is a vector space because vector addition in this problem is the same as the vector addition that, so yes, V is closed under vector addition.

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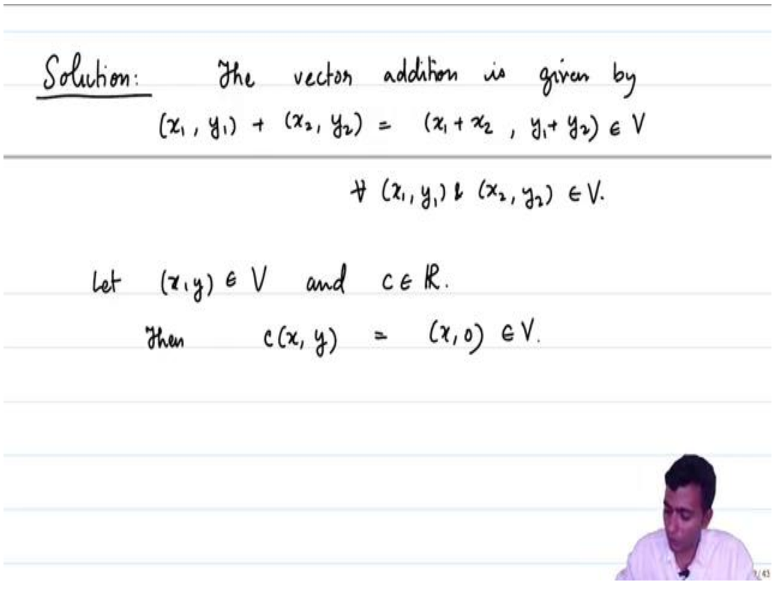

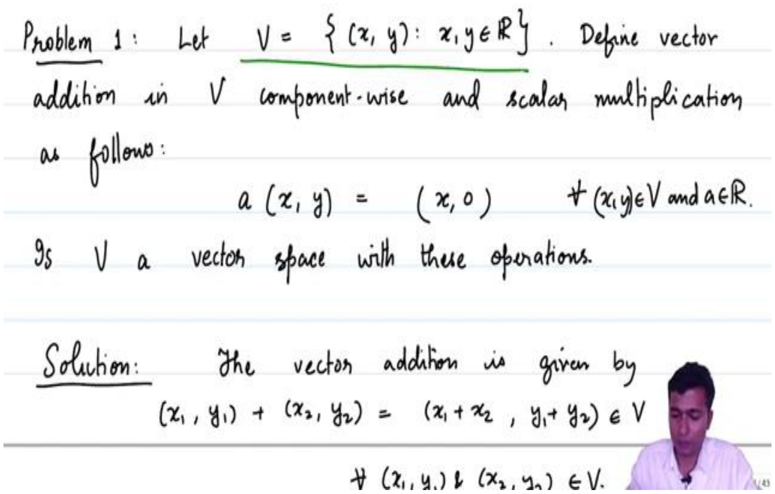

How about scalar multiplication? So scalar multiplication. So, let x, comma y be in R2 in V. V is the same as R2 but let me just call it V because that is what the vector space is being called as. So, let x, y be in V and C be an element from the field of scalars.

Then C times the scalar multiplication how is it defined c times x, y is x, comma 0 which is an element in capital V, right it is after all, an element in capital V, which has the second coordinate 0. Recall that V is nothing but the set of all tuples x, comma y with x, y in R. So, yes, this is also in V.

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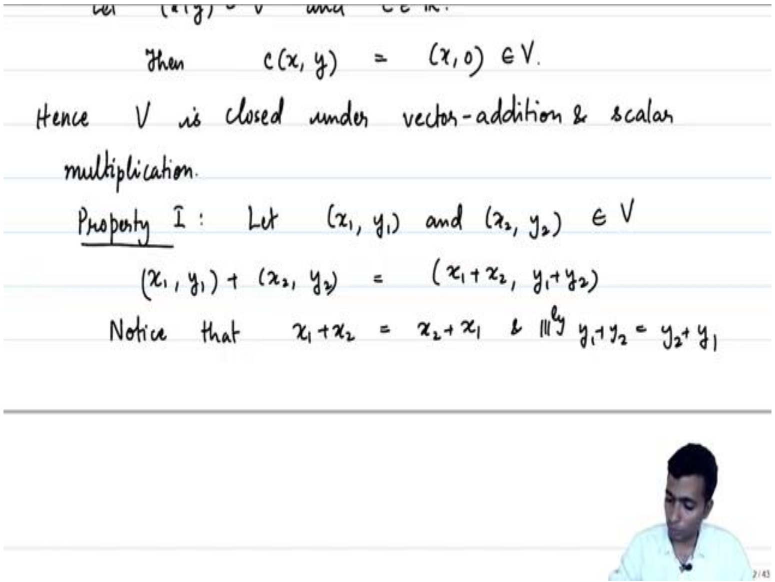

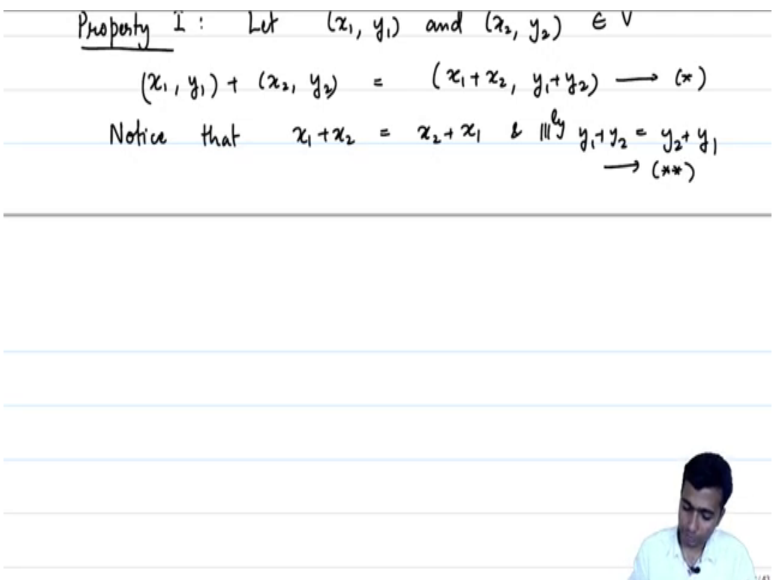

So, hence V is closed under both vector addition and scalar multiplication, we take two vectors and look at the vector addition of that that gives you back a vector in ok, so if you take two elements in V and if you look at the addition component wise, it is giving back an element in V. And therefore, it is closed under vector addition.

And similarly, if you take any element x, comma y in any scalar, look at the scalar multiplication as defined here it is giving us back an element in capital V. Therefore, V is closed under vector addition and scalar multiplication. So, what is now needed to be checked for these two operations? Properties 1 to 8 need to be checked. All the properties 1 to 8 listed in the definition of the vector space should be satisfied for V to be a vector space with these operations.

So, let us now check the properties involved in the definition. So, let us now check for the properties 1 to 8, which is listed in the definition of the vector space. So, property 1, so what was the first property? Property 1 dealt with whether this vector addition is commutative. So, let us take two vector, two elements in V x1, y1 and x2, y2 be in capital V. What do we need to do? We need to check that if v1 and v2 are two vectors in capital V, two elements in capital V, v1 plus v2 should be the same as v2 plus v1.

So, let us see what is x1, y1 plus x2, y2. x1, y1 plus x2, y2 is just component wise addition which is x1 plus x2, y1 plus y2. And what is, so notice that x1 plus x2 is just addition of two scalars. So, notice that x1 plus x2 is equal to x2 plus x1. And similarly, y1 plus y2 is the same as y2 plus y1, why is this the case? Because addition of scalars is commutative, real numbers if you add in whatever order you wish the answer is going to be the same that is the reason. So let us call this star and call this observation star star.

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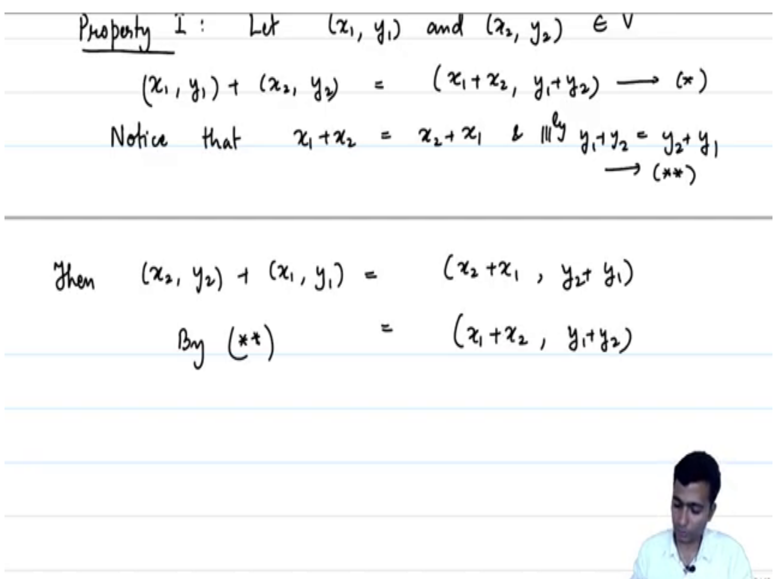

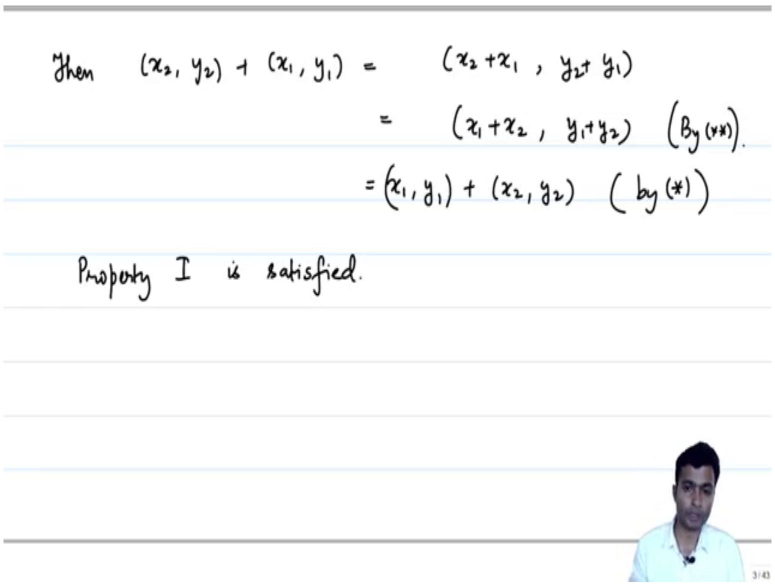

Then what is x2, y2 plus x1, y1 this again by component wise addition is going to be x2 plus x1, comma y2 plus y1, just component wise addition and by stars star this is equal to x1 plus x2, y1 plus y2. So, let me write this by star star here, by star star. And what is this? This is equal to x1, y1 plus x2, y2 by star above. So, basically what we have established is x2, y2 plus x1, y1 is the same as x1, y1 plus x2, y2 thereby establishing commutatively. So, yes, property 1 is satisfied. Property 1 is satisfied.

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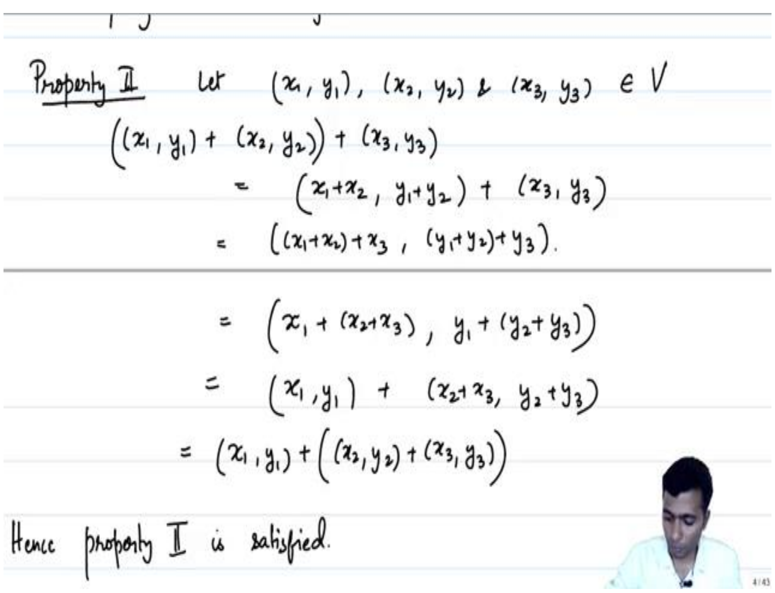

Now, let us look at property 2. Property 2 dealt with associativity. So, if you take three vectors, v1, v2, v3 the question of whether, if you look at v1 plus v2 plus v3 the question of whether v1 plus v2 is added first then added to v3 should not matter as compared to whether v1 is added to the vector addition of v2 and v3. So, let for that we need to take three vectors v1, v2, v3 and three elements v1, v2, v3, elements here typically look like x1, y1, x2, y2 and x3, y3, v1, v2, v3 be element in capital V.

So we are interested in what is x1, y1 plus x2, y2 plus x3, y3 whether this the same as yeah, we will come to that, so this, if you notice this is just equal to x1 plus x2, y1 plus y2 plus x3, y3 we added the first two vectors first and now this is going to be equal to x1 plus x2 plus x3, y1 plus y2 plus y3.

But what do we know about the sum of scalars, sum of real numbers? We know that that is a associative addition. So, this is equal to x1 plus x2 plus x3. The order here does not matter. So, we will make use of that to write it like this.

But notice that this is nothing but x1, y1 plus x2 plus x3, comma y2 plus y3 and what is this? This is nothing but x1, y1 plus x2, y2 plus x3, y3. And that establishes that property 2 be satisfied. So, hence property 2 is satisfied. So, what was property 3?

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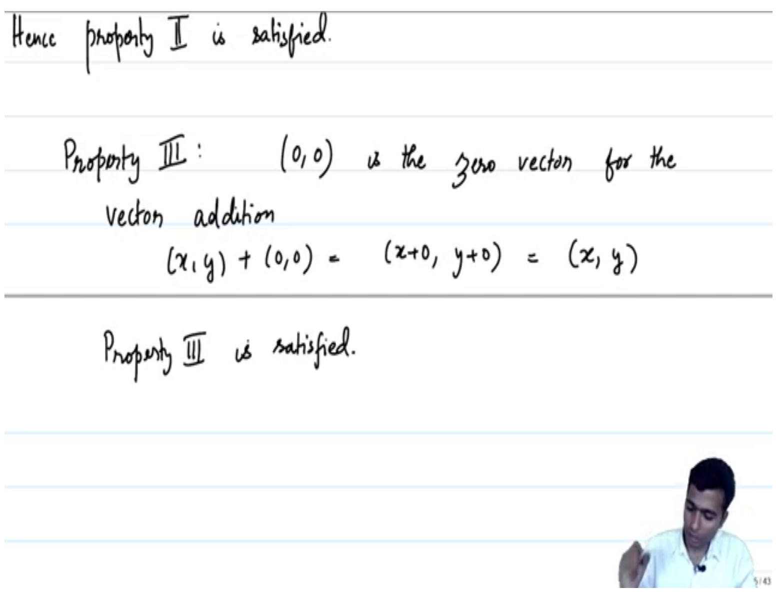

Property 3 talked about the additive identity. The existence of a 0 vector. So, my claim is 0, 0 is the 0 vector for the vector addition. So, in particular, 0, comma 0 is an element of V. So, if you look at x, comma y plus 0, comma 0, what do we have? This is equal to x plus 0, y plus 0.

But any number added to 0 should give back the same number. This is equal to x, y. So, any vector v added to the 0 vector is giving us back v. So, yes, property 3 additive identity does exist, property 3 is satisfied. How about property 4?

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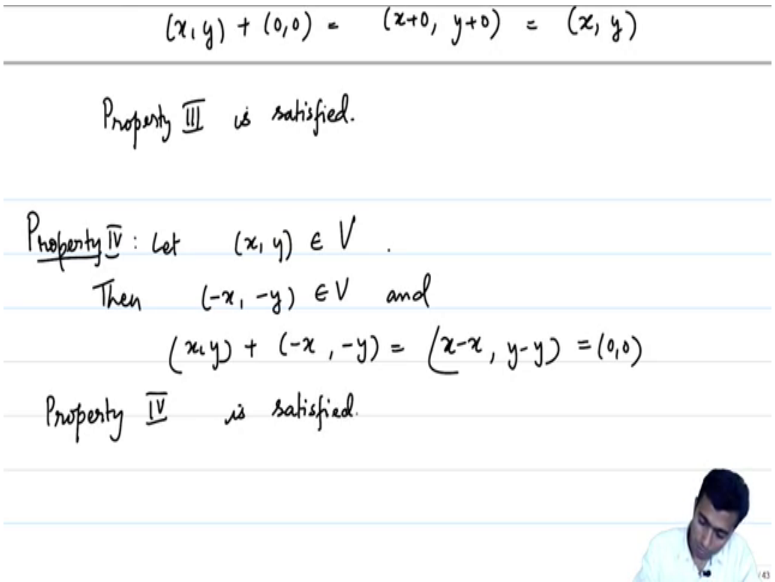

Property 4 talked about additive inverse given any vector v we would like to look for another vector w such that v plus w is the 0 vector, 0 element. So, let, so this is Property 4. Let us see if this is getting satisfied. Property 4 demands that let x, y be a vector, be an element in capital V. The addition is the same as the addition in the vector space R2.

So, we know what to expect and hence minus x, minus y is a candidate. Then minus x, comma minus y is an element of capital V and x, comma y plus minus x, comma minus y is just x minus x, x plus minus x which is x minus x, y plus minus x which is y minus y which is nothing but the 0 element. So, yes property 4 also is satisfied. Every vector v has an additive inverse. What was the 5th property? 5th property is the existence of multiplicative identity.

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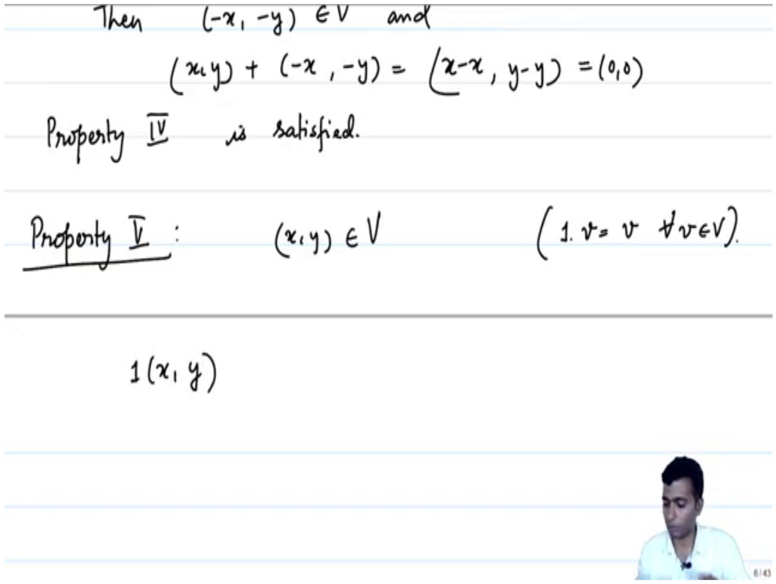

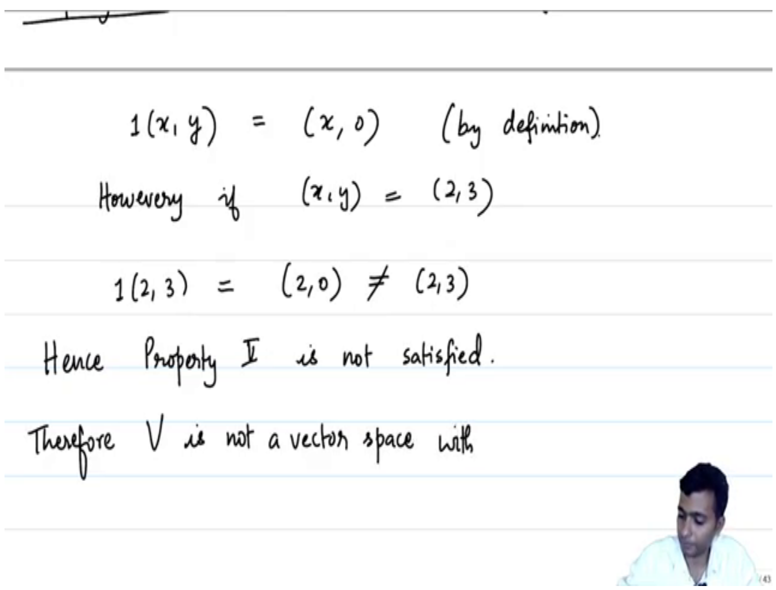

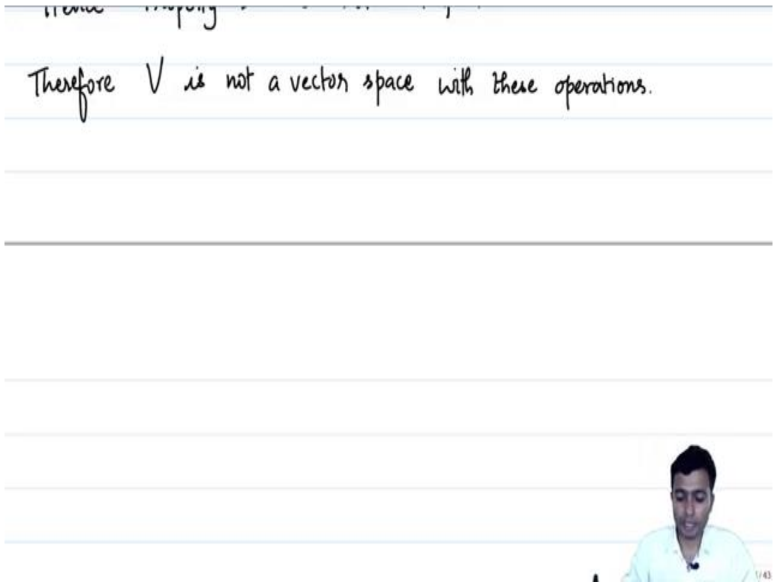

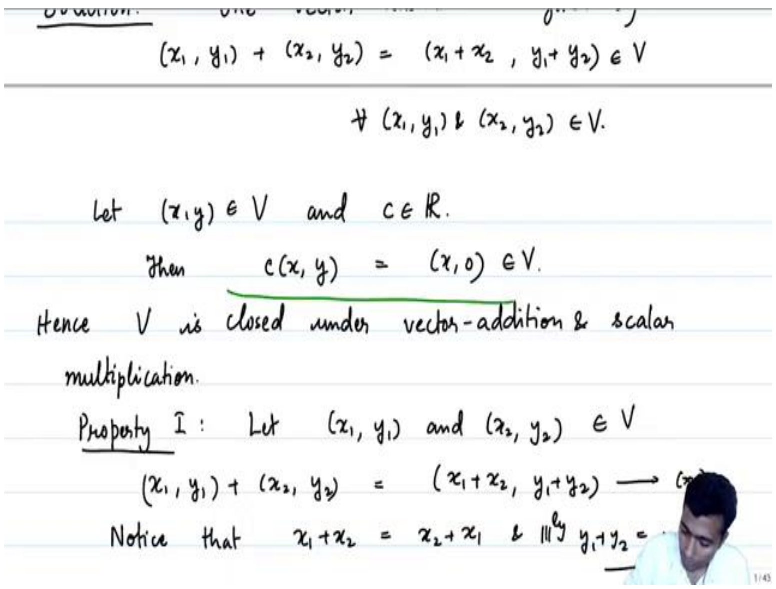

So, property 5, so let us look at a vector v which is say x, y in capital V. So, an element in capital V is being taken we would like to check. So, if, so v be equal to this so what was the multiplicative identity demanding? It was demanding that one times v is equal to v for all v in capital V, this is what we should, we would like to check.

But what is 1 times x, comma y so to do that let us go and recollect what, was the definition of the scalar multiplication which I am now underlining in the green here. Any scalar c times x, comma y is giving us back x, comma 0. As you can see, so 1 times x, comma y will give you back x, comma 0, so it does not matter what c is at. Every scalar should give you back the vector which is the first coordinate and 0 is I am putting this in the second coordinate. So, this is the definition, by definition, this is what it is. But if y is not equal to 0, then x, y is not equal to x, 0.

So, this however, if say x, comma y is the vector 2, comma 3 let us say 3 not equal to 0. So, 1 times 2, comma 3 here by definition is equal to 2, comma 0 which is not equal to 2, comma 3, we should have got 2, comma 3. If the property 5 is to be satisfied. So, hence property 5 is not satisfied. So, therefore, V is not a vector space with these operations. We have already solved the problem establishing that with these operations, V cannot be a vector space because the multiplicative identity, the property involved in the multiplicative identity is not getting satisfied.

Well, out of curiosity, we could ask what about the remaining properties? It does not matter because we have already established that V is not a vector space.

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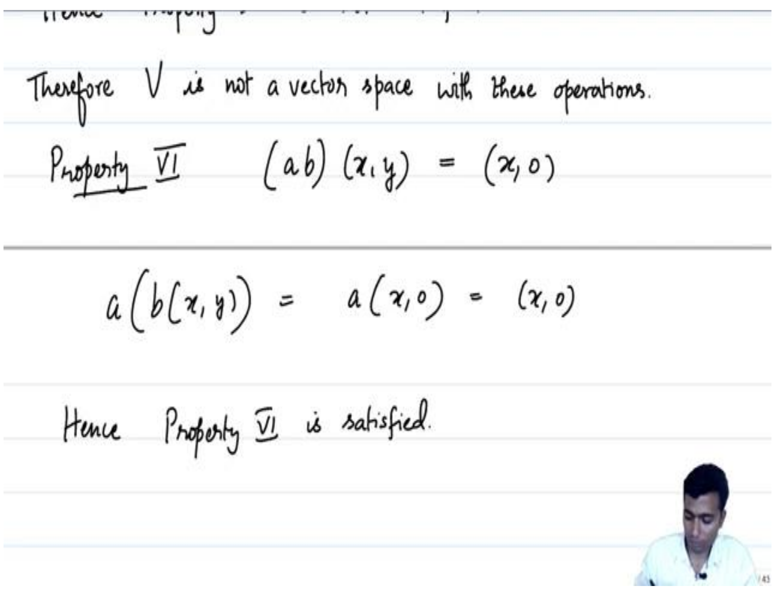

However, just to satisfy our curiosity, let us look at the remaining properties as well, property 6. Property 6 so was about the multiplicative associativity, so if you look at ab times say x, comma y let me now do a quick observation. This is just going to be any vector, sorry any scalar times x, comma y by scalar multiplication is just going to be x, comma 0 the first coordinate and the 0 in the second coordinate.

But we demand that this be equal to a times b of x, comma y right. And what is this? This is just a times x, comma 0. B of b times x, comma y is x, comma 0 and a times x, comma 0 will again be equal to x, comma 0. So, yes, this is equal if you can, if you have notice this is equal. Therefore, property 6 is actually getting satisfied. So, that is interesting. So, even though property 5 is not satisfied, Property 6 is still getting satisfied. How about property 7?

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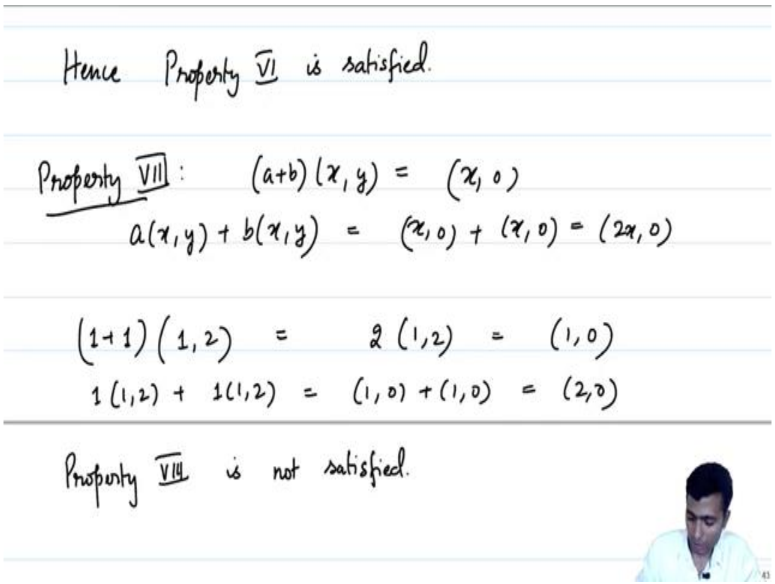

Property 7 demands that a plus b times x, comma y let us see what this is. This is equal to, it does not matter what a plus b say c. C times x, y this is going to be x, comma 0 and what is a times x, comma y plus b times x, comma y. Oh, this was Property 8, I guess. So, let me just put it here, property 8 satisfies or rather it is satisfied or not let us check. So, this is going to be equal to x, comma 0 plus x, comma 0 which is actually equal to x plus x, comma 0 which is 2x, comma 0, so this is not necessarily if x is say non-0 then this is not going to get satisfied.

So, for example, look at 1 plus 1 on 1, comma 2 this by the first part will or rather direct. So, this is just two times, let me not use the green. This is just two times 1, comma 2 which is equal to 1, comma 0 by the scalar multiplication. Any scalar times a vector gives you the same coordinate in the first, it is the same first coordinate. But what about 1 times 1, comma 2 plus 1 times 1, comma 2? This is just going to be equal to 1, 0 plus 1, 0, which is equal to 2, comma 0. This is not equal as you can notice, and therefore property 8 is not getting satisfied. So, we have one more property, which is not getting satisfied. Not satisfied.

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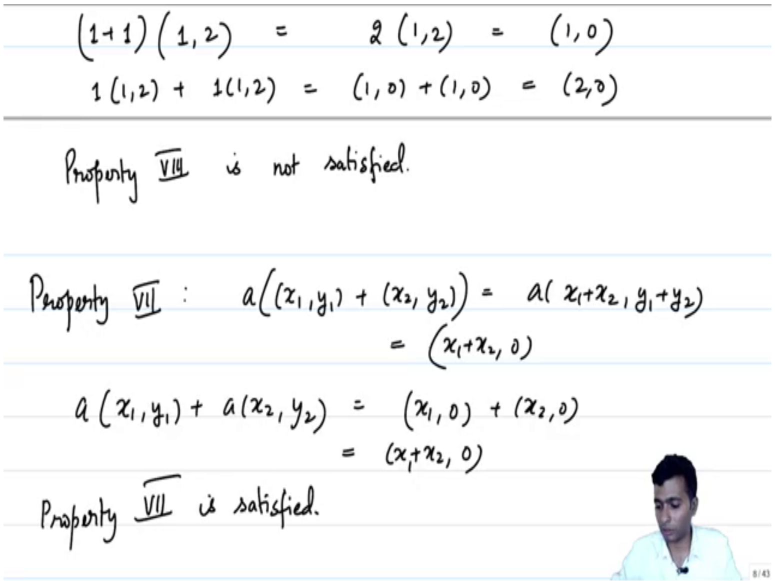

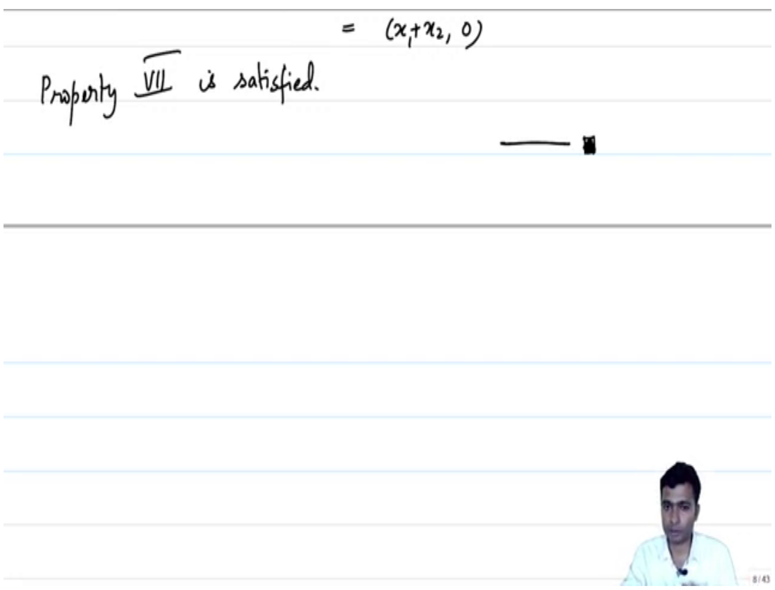

Actually, let me not now bother about property 7, but let me just tell you that or maybe I will just write it. We need to check that a times x1, y1 plus x2, y2 this is equal to a times x1 plus x2 y1 plus y2 which is equal to x1 plus x2, comma 0. And what should this be equal to a times x1, y1 plus a times x2, y2. But what is that? That is equal to x1, comma 0 plus x2, comma 0 which is equal to x1 plus x2 0 which actually are equal.

And therefore, property 7 is satisfied. So, if you start worrying about all the properties, we will notice that the 5th property and the 8th property are not satisfied. Even if one of the properties are not satisfied, it cannot be a vector space. We just checked the remaining three properties out of curiosity, I would say. All right, so we have completed the first problem and concluded that the set V with the vector space operations, as defined in the problem, cannot be a vector space.

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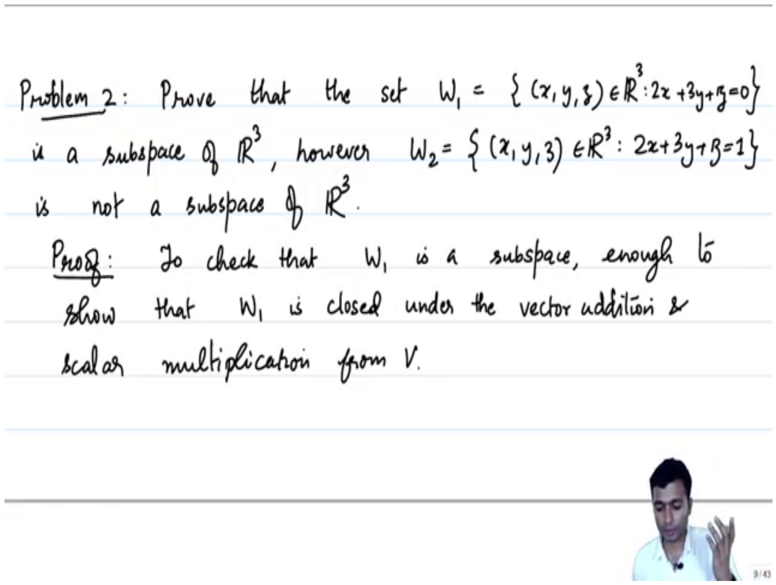

Now, let us look at the next problem, problem 2. Prove that the set w1, which is say x, y, z in R3 such that 2x plus 3y plus z is equal to 0 is a sub space of R3. However, w2 which is the set of all x, y, z in R3 such that 2x plus 3y plus z is equal to 1 is not a subspace of R3.

So, after looking through the solution, you will notice that 2x plus 3y plus z is equal to any non-0 number not necessarily 1, you look at wk to be equal to x, y, z in R3 such that 2x plus 3y plus z is equal to K that will not be a subspace, it has to be equal to 0 otherwise, it will not be a sub space. This is the same proof we will go through.

So, let us look at a solution. So, this is more like a, prove that, so let us call it a proof. So, w1 is subspace of R3 that is what we would like to prove here, but what, when does some subset become a subspace. You recall the definition, it is going to be a subspace if in the borrowed vector space operations, it is closed under both the operations.

So, enough to show to check that w1 is a subspace enough to show that w1 is closed under the vector addition and scalar multiplication which is borrowed from capital V. So, let us take two elements and look for whether vector addition of those two elements in w1 gives us back an element in w1.

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Let (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in W_1$

$\Rightarrow 2x_1 + 3y_1 + z_1 = 0$ & $2x_2 + 3y_2 + z_2 = 0. \longrightarrow (*)$

Then

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Then $2(x_1 + x_2) + 3(y_1 + y_2) + (z_1 + z_2)$

$= 2x_1 + 2x_2 + 3y_1 + 3y_2 + z_1 + z_2$

$= (2x_1 + 3y_1 + z_1) + (2x_2 + 3y_2 + z_2)$

$= 0 + 0$ (by $(*)$)

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$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Then $2(x_1 + x_2) + 3(y_1 + y_2) + (z_1 + z_2)$

$= 2x_1 + 2x_2 + 3y_1 + 3y_2 + z_1 + z_2$

$= (2x_1 + 3y_1 + z_1) + (2x_2 + 3y_2 + z_2)$

$= 0 + 0$ (by $(*)$)

$= 0$

$\therefore (x_1, y_1, z_1) + (x_2, y_2, z_2) \in W.$

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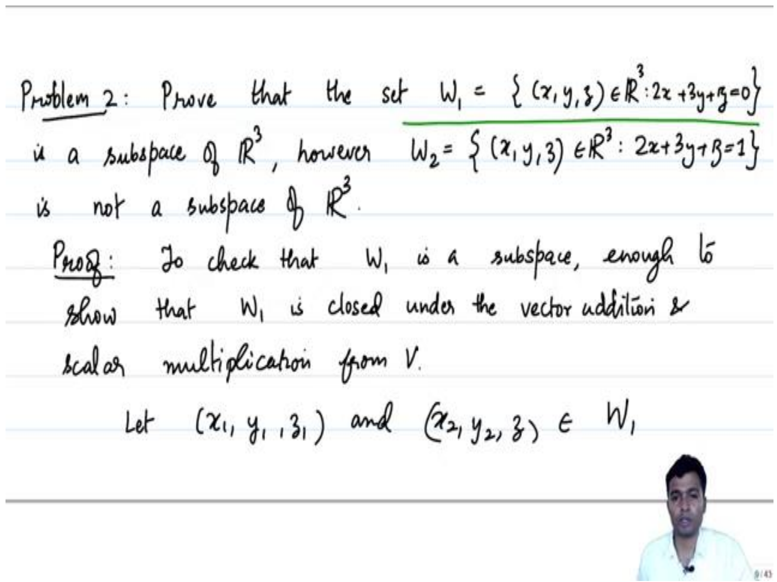

So, let x1, y1, z1 and x2, y2, z2 be two vectors in w1. What does it mean to say that something is in w1? Let me go back and remind you what w1 is. W1 is the set of all x, y, z such that 2x plus 3y plus z is equal to 0. So, this implies 2x1 plus 3y1 plus z1 is equal to 0 and this implies the fact that it x2, y2, z2 is in w1 implies that 2x2 plus 3y2 plus z2 is equal to the scalar 0.

We would like to see what happens to x1, y1, z1 plus x2, y2, z2 whether this particular vector is in w but what is the vector addition of this? This is component wise addition if you recall, this is going to give you x1 plus x2, y1 plus y2, z1 plus z2. And let us see if this belongs to the set w1 but what is the requirement for this particular element to be in W1. It should satisfy the condition that 2x1 plus x2 plus 3y1 plus y2 plus z1 plus z2 this should be equal to 0. If at all this sum should be in w1, then this is what will happen.

Then this is equal to let us write it out. This is 2x1 plus 2x2 plus 3y1 plus 3y2 plus z1 plus z2 notice that all these are scalars. 2x1 is a real number 2x2 is a real number 3y1 is a real number 3y2 is a real number and so on. And the vector addition is commutative and using that we can write this as 2x1 plus 3y1 plus z1 plus 2x2 plus 3y2 plus z2. Let us just go up notice that whatever I am putting now as star, this is equal to 0 plus 0 by star which is equal to 0 and therefore x1 y1 z1 plus x2 y2 z2 belongs to capital W by what we have just observed. So yes, w is closed under the vector addition, which is borrowed from R3.

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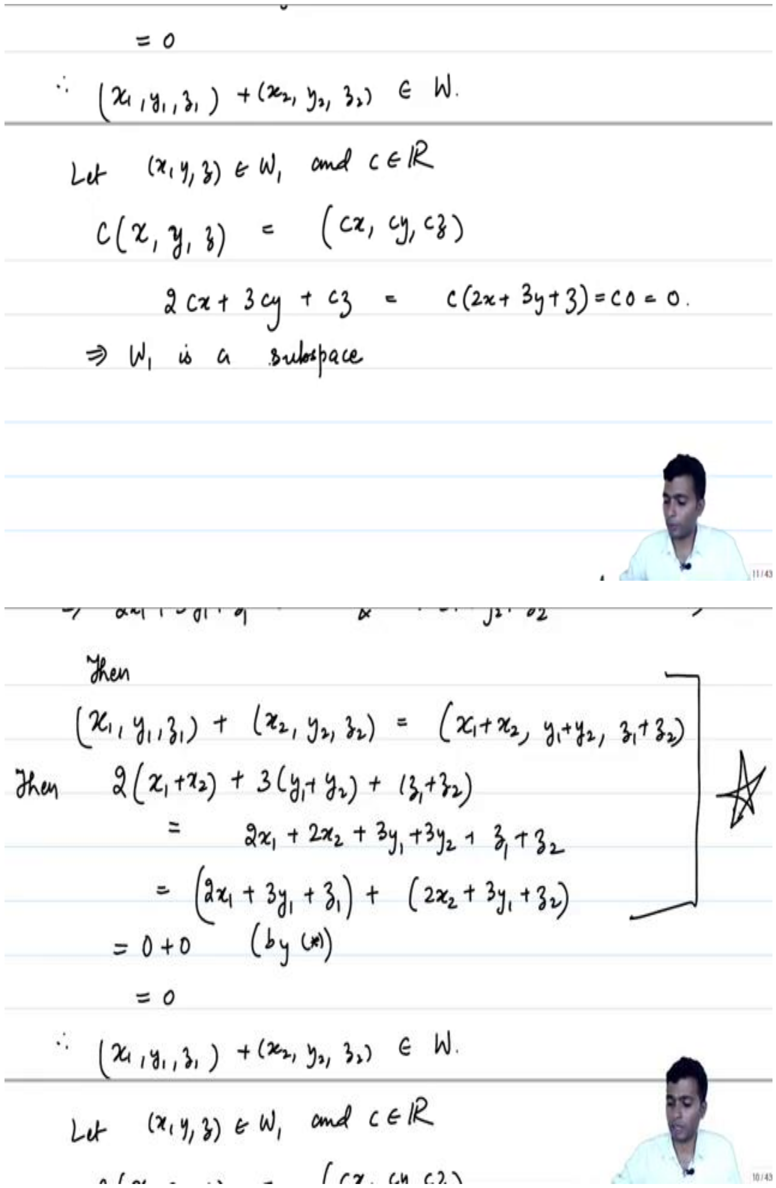

How about scalar multiplication we have already seen similar ideas. So, let x, y, z be in capital W1 and C be in R then what is this? This is equal to cx, cy, cz and let us look at whether this vector in the right cx, cy, cz does it belong to the vector space W. So, you need to look for two times cx plus 3 times cy plus cz is equal to 0, which is a cz is this equal to 0 that is the question. I will just write it as C times 2x plus 3y plus z. But we know that 2x plus 3y plus z is 0 because x, y, z belongs to W1 and this is equal to C times 0 which is equal to 0.

So, yes this implies W1 is a subspace. Now, what happens to W2? So, I would say that W2 let me do one thing. Let me note this part has maybe a star. I am going to use the star to conclude.

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Is W_2 a subspace?

$(x_1, y_1, z_1) \in W_2$ and $(x_2, y_2, z_2) \in W_2$

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

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$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$2(x_1 + x_2) + 3(y_1 + y_2) + z_1 + z_2) =$

$(2x_1 + 3y_1 + z_1) + (2x_2 + 3y_2 + z_2)$

$1 + 1 = 2$

Hence W_2 is not closed under vector addition & therefore not a subspace.

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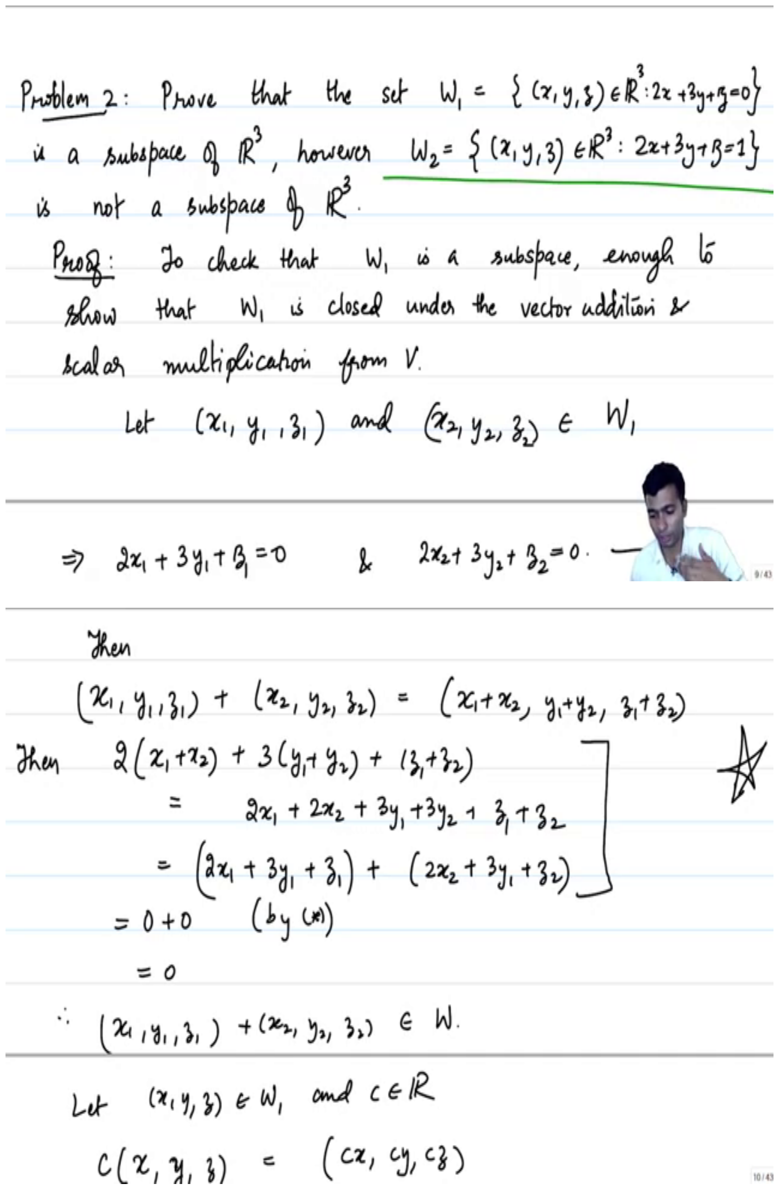

So, now is W2 is a subspace? That is what we would like to answer or that we would like to prove that w2 is not a subspace. So, in order to do that, we should show that it is not closed under either vector addition or the scalar multiplications.

So let us take x1, y1 and z1 in 2 in W2 and x2, y2, z2 which is in W2. We would like to see if the vector addition is in W2, but we already checked this out. What is x1 y1 plus sorry x1, y1, z1 plus x2, y2, z2, this is just equal to x1 plus y1. Let me not, let me quickly write it and we would like to see whether this belongs to W2. What is W2? Recall that W2 is defined in this manner, so I am underlining it in green. 2x plus 3y plus z the sum of the coordinates with these linear combinations should be equal to 1, that is the requirement.

So, let us look at 2 times x1 plus x2 plus 3 times y1 plus of y2 plus z1 plus z2. Now let me show you what was put in star. Yeah. So, maybe this is what I would like to put in Star. This tells us that this is equal to 2x1, same argument tells us that this is equal to 2x1 plus 3y1 plus z1 plus 2x2 plus 3y2 plus z2. But what is 2x1 plus 3y1 plus z1 that is equal to one because x1, y1, z1 belongs w1. Similarly, this is also equal to 1 and sum of these two will give you 2 but what was our requirement?

Our requirement was that this vector, if you look at 2 times the first coordinate plus 3 times the second coordinate plus the third coordinate, we should have got 1 for it to be in the vector subspace. Here we are getting 2 and therefore, hence W2 is now this is not, so this vector, the vector sum is not in W2. W2 is not closed under vector addition. So, w2 is not a subspace. And hence, and therefore not a subspace. If you had taken scalar multiplication and checked even that would not have landed up here, it would give you something like C times 1 right and therefore not a subspace.

So, even if one of the two conditions are not satisfied, it will not be a subspace. All right so we have proved or rather completed problem 2.

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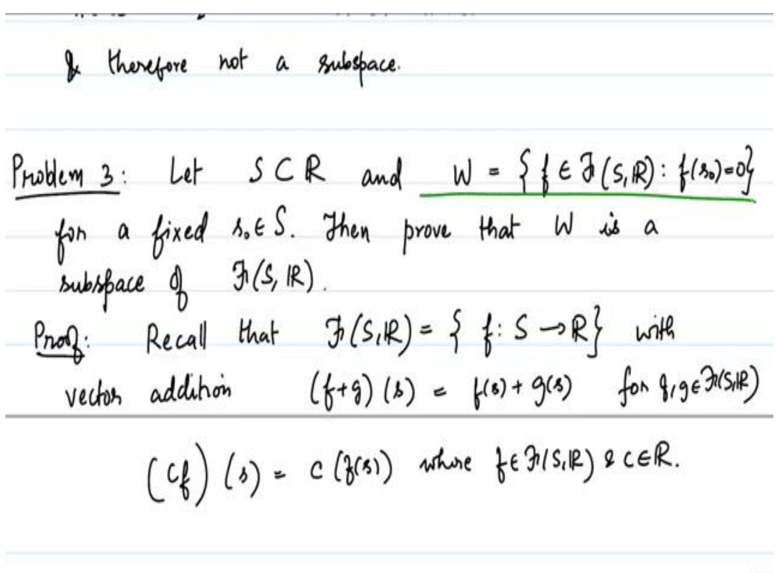

Ok, so the next problem is in the vector space of all functions from set 2 are, so let S be a subset of the real numbers and w be the set of all F in F of S, comma R such that F of S naught is equal to 0 for fixed x naught in capital S. So, what is F of S, comma recall. I will come to that. So, then prove that W is a subspace of F of S, comma R. Okay, let us look at solution to this problem rather proof because that is a proof that problem.

So let us call it a proof. So, what was F of S, comma R? Recall that F of S, comma R is the set all functions from S to R. And what was the vector addition and scalar multiplication there it was point wise. So, with vector addition if you take two functions and if you look at F plus G, we would like to say that it is a function from S to R. So, we will define what this is set of point s in capital S. This is just F of S plus G of S which makes it into an honest function from S into R and what about cf of S.

So, for, this is for F, comma G in F of S, comma R and this is equal to c times F of S where f is in capital F of S, comma R and c is in so, so this is vector addition and scalar multiplication, which we had defined during the lectures and it was left as an exercise for you to check that this is indeed a vector space. So, like in the first problem, you should have checked for whether it is closed under vector addition and scalar multiplication and see in what the properties 1 to 8, whether properties 1 for 8 are indeed getting satisfied.

Yeah, in the process, you would had to guess what the additive identity is what the inverse of a given function is and so on. But they are all quite straightforward. Ok so our goal here in this problem is to show that we are given a very particular subset.

This is the set of all those functions F from S to R where F satisfies the condition that F of s naught is equal to 0 where s naught is some fixed point. So, if say S is the open interval is 0 1 and s is S naught is half, this will just turn out to be all functions from 0 1 to R such that F of R is equal to 0 one such example.

Ok. So, we will show that W is always a subspace. Again like in the previous problem, we just have to show that W is closed under vector addition and scalar multiplication, but that is quite straightforward.

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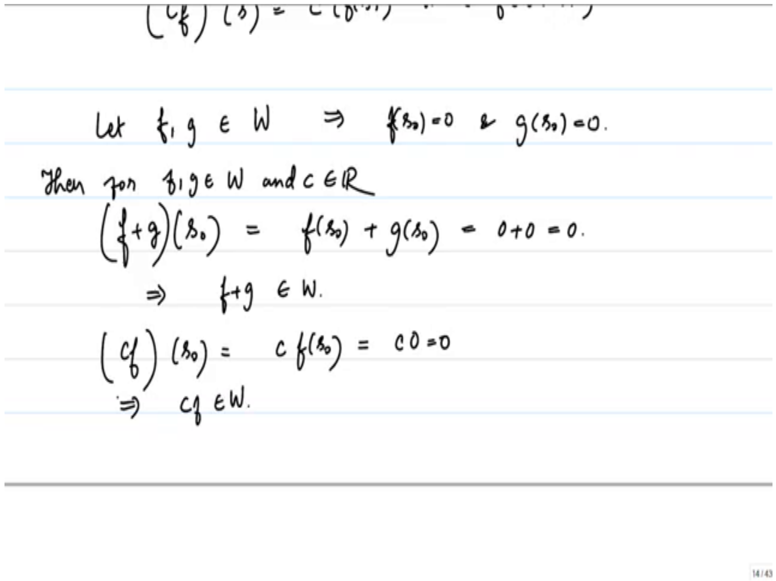

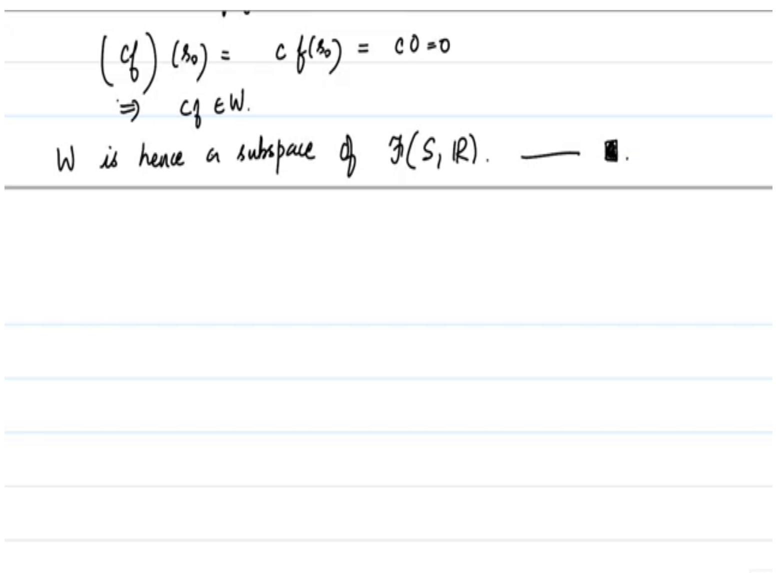

So, let F, comma G be in W. This implies that F of S naught is 0 and G of S naught is 0. Ok, so now let us look at the vector addition of F plus G. And look at what is the value at S naught but by the definition. F plus G of S naught is equal to F of S naught plus G of S naught which is equal to 0 plus 0, which is equal to 0. So, therefore, it implies that F plus G belongs to capital W.

How about scalar multiplication cf times S naught? So this is for, then for F G in W and c in R, this is what is getting satisfied. cf at S naught is equal to c times F of S naught but F is in capital W means that F of S naught is 0. This implies that C times 0, which is equal to 0 hence cf is in W. So, W is closed under both vector addition and scalar multiplication, W is hence a subspace of F of S, comma R. Next, let us do a problem which is similar to one of the assignment problems.

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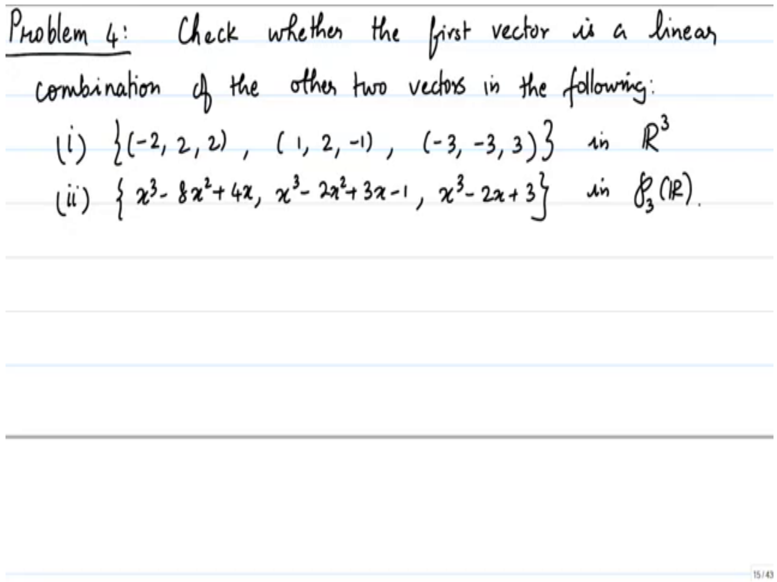

Problem 4 demands that we check whether the first vector can be written down as a linear combination of the other two vectors. So, check whether the first vector is a linear combination of the other two vectors in the following. The first one is minus 2 2 2, 1 2 minus 1, minus 3 minus 3 3 in R3 and how about the second one? Let us just to two of them.

This is just going to be x cube minus 8x square plus 4x, x cube minus 2x square plus 3x minus 1, x cube minus 2x plus 3. This is in P3 of R. So, we need to check whether the first vector can be written as a linear combination of the other two vectors. Let us look for whether it can be done.

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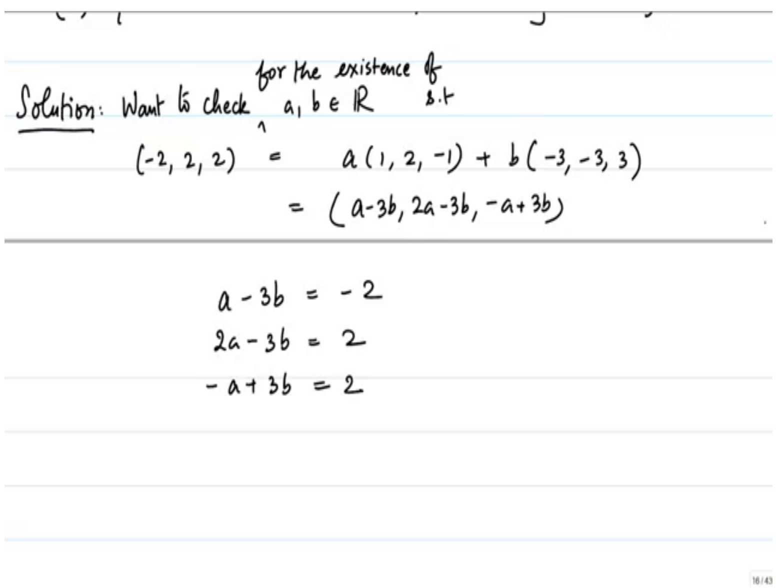

So, solution. So, we would like to see if so want a, comma b in the field of scalars such that minus 2, 2, 2 is equal to a times 1, 2, minus 1 maybe I should just change this, this makes it, this is not going to make it easy, plus B times minus 3, minus 3, 3.

Yeah. Let us see if we can do that or want to check. Let me just reword what I want, want to check for the existence of a, comma b in R such that this happens. Ok, so let us write it down. Let us write down what this means. What is the right hand side here, the right hand side just tells us that this is equal to a plus minus 3b which is a minus 3b. This is 2a minus 3b, and minus of a plus 3b. So, what we want is a solution for the system of equations which is written down here 2 and minus of a plus 3b is equal 2.

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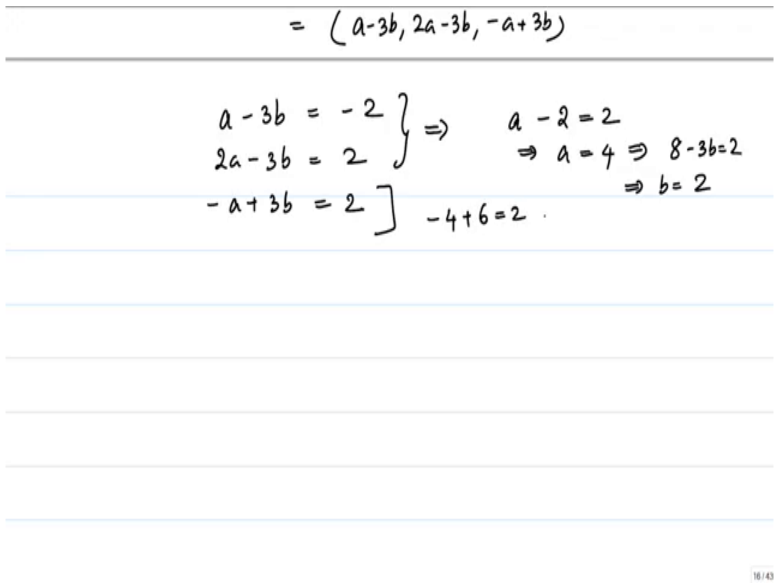

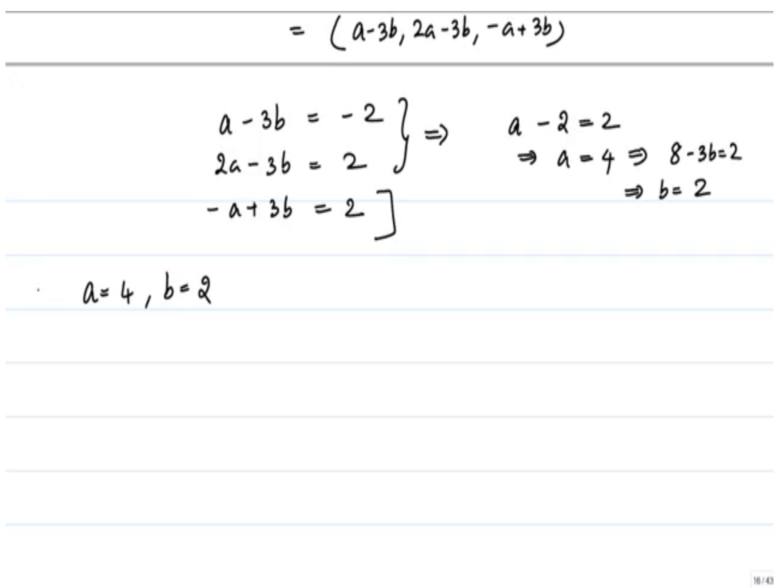

Ok so what can we say from here, a minus 3b is equal to minus 2, these two directly implies a minus 2 is equal to 2, which implies a is equal to 4 and this implies two times 4 minus 3b is equal 2. Oh, sorry. 8 minus 3b is equal to 2 which implies b is equal to minus of 6 by minus 3, which is 2. And is it consistent here? We have minus of 4 plus 3 into 2 is 6 which is equal to 2. Yes. So, my claim is, so a equal to 4, b equal to 2 will satisfy the relevant equations that has been written down. So let us see.

(Refer Slide Time: 41:04)

$$(-2, 2, 2) = a(1, 2, -1) + b(-3, -3, 3)$$
$$= (a-3b, 2a-3b, -a+3b)$$

$$\left.\begin{aligned} a - 3b &= -2 \\ 2a - 3b &= 2 \end{aligned}\right\} \Rightarrow \quad a - 2 = 2 \Rightarrow a = 4 \Rightarrow 8 - 3b = 2 \Rightarrow b = 2$$
$$-a + 3b = 2$$

$\Rightarrow a = 4, \ b = 2$

Hence $(-2, 2, 2) = 4(1, 2, -1) + 2(-3, -3, 3)$

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So, hence easy to check that minus 2, 2, 2 is equal to 4 times 1, 2 minus 1 plus 2 times minus 3, minus 3, 3. So, yes, therefore, we can write the first vector as a linear combination of the other two in R3. Ok how about the next one? Okay. So, let me write this down here.

(Refer Slide Time: 41:34)

$\Rightarrow a = 4, \ b = 2$

Hence $(-2, 2, 2) = 4(1, 2, -1) + 2(-3, -3, 3)$

(ii) $\{x^3 - 8x^2 + 4x, \ x^3 - 2x^2 + 3x - 1, \ x^3 - 2x + 3\}$ in $P_3(\mathbb{R})$

Want to check if $\exists \ a, b \in \mathbb{R}$ s.t.

$$x^3 - 8x^2 + 4x = a(x^3 - 2x^2 + 3x - 1) + b(x^3 - 2x + 3)$$
$$= (a+b)x^3 + (-2a)x$$

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Want to check if $\exists\ a, b \in \mathbb{R}$ s.t

$$x^3 - 8x^2 + 4x = a(x^3 - 2x^2 + 3x - 1) + b(x^3 - 2x + 3)$$
$$= (a+b)x^3 + (-2a)x^2 + (3a - 2b)x + (3b - a)$$

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combination of the other two vectors in the following:

(i) $\{(-2, 2, 2),\ (1, 2, -1),\ (-3, -3, 3)\}$ in $\mathbb{R}^3$

(ii) $\{x^3 - 8x^2 + 4x,\ x^3 - 2x^2 + 3x - 1,\ x^3 - 2x + 3\}$ in $P_3(\mathbb{R})$.

<u>Solution:</u> Want to check for the existence of $a, b \in \mathbb{R}$ s.t

$$(-2, 2, 2) = a(1, 2, -1) + b(-3, -3, 3)$$
$$= (a - 3b,\ 2a - 3b,\ -a + 3b)$$

$$\left.\begin{array}{l} a - 3b = -2 \\ 2a - 3b = 2 \end{array}\right\} \Rightarrow \quad a - 2 = 2$$
$$\Rightarrow a = 4 \Rightarrow 8 - 3b = 2$$
$$\Rightarrow b = 2$$
$$-a + 3b = 2$$

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So, this is 2. So, let us now see what 2 is. 2 was x cube minus 8x square plus 4x, x cube minus 2x square plus 3x minus 1 and x cube minus 2x plus 3, this in P3 of R, all polynomials of degree less than or equal to 3.

Ok, so if we can indeed do that, so want to check if there exist a, comma b in R such that x cube minus 8x square plus 4x is equal to a times x cube minus x square I am sorry x square plus 3x minus 1 plus b times x cube minus 2x plus 3. But what does this mean? This means what is the right hand side. This is equal to a plus b times x cube plus minus of 2a times x square plus 3a minus 2b times x plus 3b minus a. So, this is exactly what the right hand side is going to be. We would like to see if there exist a and b such that the right hand side is equal to the left hand side here.

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$$= (a+b)x^3 + (-2a)x^2 + (3a-2b)x + (3b-a)$$

$$\Rightarrow \left.\begin{array}{l} a+b=1 \\ -2a=-8 \end{array}\right\} \Rightarrow a = 4 \Rightarrow b = 1-4=-3.$$

$$\left.\begin{array}{l} 3a-2b=4 \longrightarrow 12+6 = 18 \neq 4 \\ 3b-a=0 \end{array}\right]$$

Ok. So, what does this imply by equating the coefficients of the monomials involved we have a plus b is equal to 1 minus of 2a is equal to minus of 8, 3a minus 2b is equal to 4, 3b minus a is equal to 0. Ok, so let us see. This implies so let us see the first, second equation minus 2a is equal to minus 8 implies a is equal to minus of 8 by minus of 2 is equal to 4 and a plus b is equal to 1 implies b is equal to 1 minus 4, which is equal to minus 3.

Now, let us see if the final does, so these two implies this. Let us see if the next two equations are consistent. If it is not, then we do not have a linear combination. So, basically with a equal to 4 and b equal to minus 3, let us look for whether the third and the fourth equation are satisfied. Ok so what is this going to be? This is just going to be 4 into 3, 12, minus 2 into minus 3 which is 6 which is equal to 18 which is not equal to 4, so that that cannot be a consistent choice of a and b, which satisfies all these equations.

So, we do not even need to bother about whether the fourth equation can be satisfied. So, the first two equation forces a to be 4 and b to be minus 3, but with 4 and minus 3, the third equation cannot be satisfied.

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$$3b - a = 0$$

Hence $\nexists\ a, b \in \mathbb{R}$ s.t $(*)$ is satisfied.

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$\Rightarrow a = 4,\ b = 2$

Hence $(-2,2,2) = 4(1,2,-1) + 2(-3,-3,3)$

(ii) $\{x^3 - 8x^2 + 4x,\ x^3 - 2x^2 + 3x - 1,\ x^3 - 2x + 3\}$ in $P_3(\mathbb{R})$

Want to check if $\exists\ a, b \in \mathbb{R}$ s.t

$$x^3 - 8x^2 + 4x = a(x^3 - 2x^2 + 3x - 1) + b(x^3 - 2x + 3) \longrightarrow (*)$$

$$= (a+b)x^3 + (-2a)x^2 + (3a - 2b)x + (3b - a)$$

$$\Rightarrow \left.\begin{array}{l} a + b = 1 \\ -2a = -8 \end{array}\right\} \Rightarrow a = 4 \Rightarrow b = 1 - 4 = -3$$

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Hence, there does not exist a, comma b in R such that, well the first let me not, so let me put it this way such that star is satisfied. Ok, so we have checked for one case where the first vector could be written down as a linear combination of the other two and another in which infrastructure cannot be written as a linear combination of the other.

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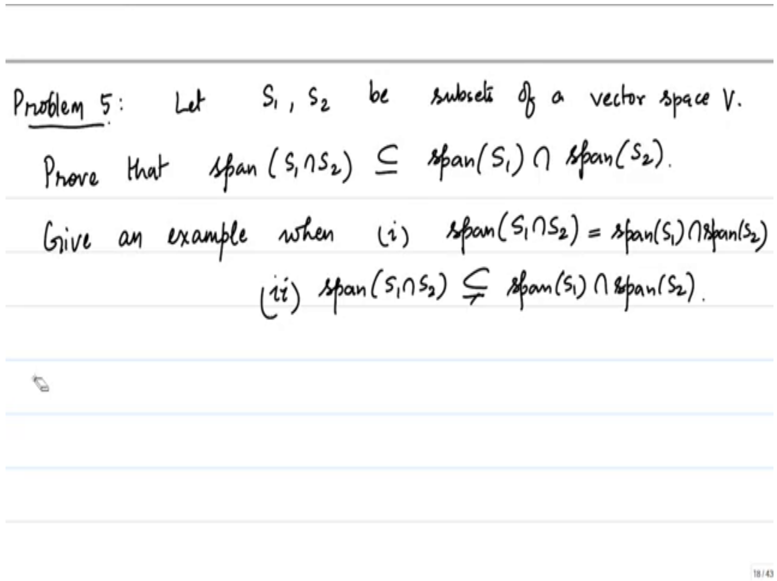

Okay, so the next problem involves the span. So, let me call it problem 5. So, let s1 s2 be subsets of a vector space V prove that span of s1 intersected with s2 is contained in the span of s1 intersected with span of s2. Moreover, give an example when one span of S1 intersected with S2 is equal to span of S1 intersected with span of S2. And 2, when the span of S1 intersected with S2 is a strict subset of span of s1 intersected with span of s2.

So, the problem not only demands us to prove some statement, it also asks us to come up with an example. This is one of the cases where you will have to sit down, look at various examples you have already seen, look for examples of the span and see that there you know, the conditions are being satisfied or not. This is a good problem for you to think about the various vector spaces that you have already seen. So, this is an interesting problem in that sense.

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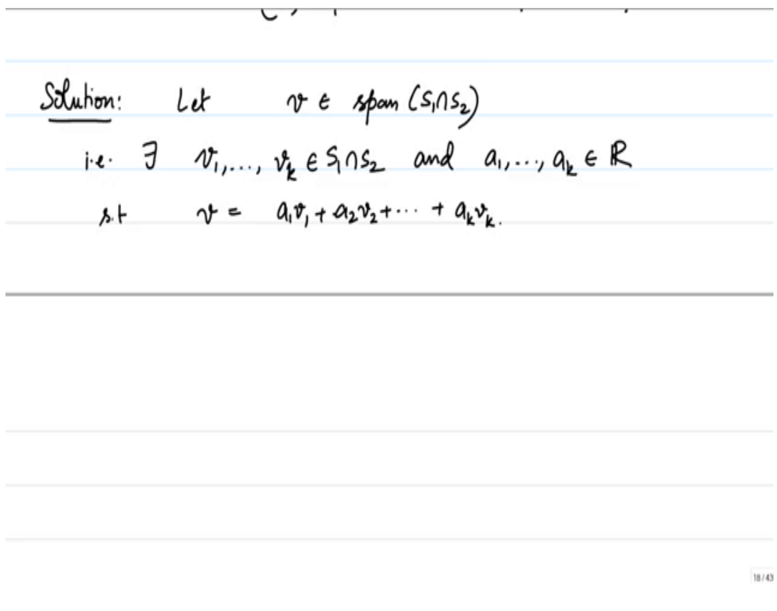

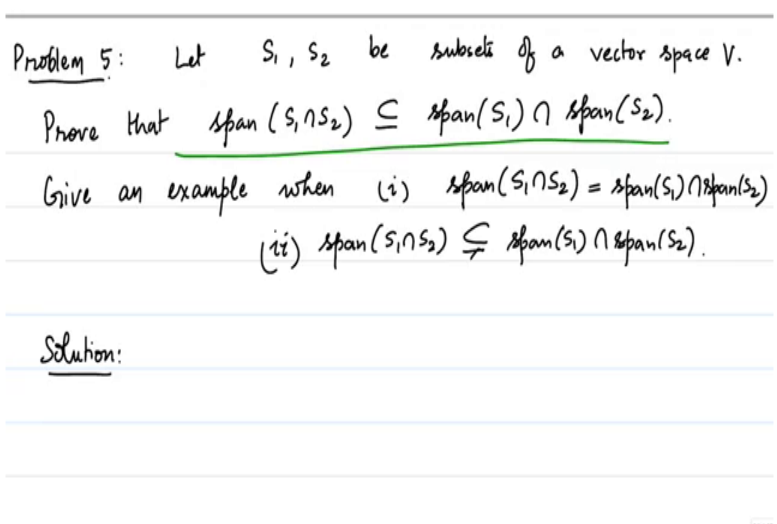

Okay, so let us look at a solution. So, we need to check for the span of S1 intersected with S2 being a subset of span of S1 intersected with span of S2. So, let us take how do we go about proving such a statement? We take some arbitrary element in the left and prove that it is also necessarily an element in the right. But here it is quite straightforward.

So, let V be in span of s1 intersected with s2. What does it mean for a vector v to be in the span of some s1 intersected with s2 that means that there exist some v1 v2 up to vk in s1 intersected with S2, such that v is a linear combination of v1 v2 up to vk i.e. there exist v1 to vk in S1 intersected with S2 and a1 to ak scalars such that v is equal to a1v1 plus a2v2 plus up to akvk.

Ok and what is our goal? Our goal is to show that v is in the span of s1 intersected with span of s2.

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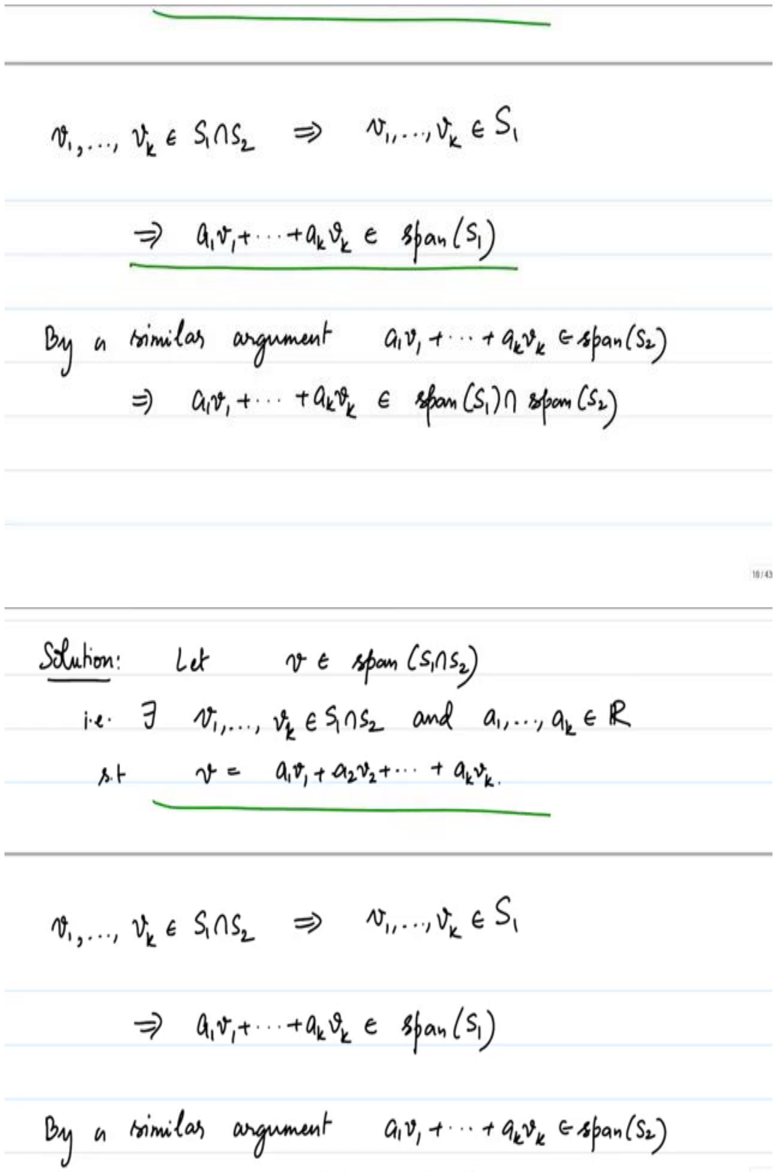

Okay, so what does it mean to say that v1 v2 up to vk belongs to S1 intersected with S2 this implies in particular that v1 to vk belongs to s1 as well belongs to both s1 and s2 that is what it means for it to be in the intersection. So, in particular it belongs to s1. And since span of s1 is going to consist of all linear combinations of the vectors in s1 this implies, so this implies a1v1 plus up to akvk belongs to the span of s1 by a very similar argument though this is by a similar argument instead of S1 if you had just picked s2. You would have got that a1vi plus a2v2 up to akvk belongs to span of s2 as well.

So, in particular, it belongs to both span of S1 and span of S2 and therefore, it belongs to the intersection. So, we used two aspects here. The one I am underlining in green was because v is in the span of s, span of s1 intersected with s2 and therefore there exist some a1 a2 up to ak such that we get a linear combination of v1 to vk. And the one I am underlining in green right now follows because every linear combination of v1 v2 up to vk which are in s1 should also be in the span of s1 so both the aspects are being used to establish this proof, but that is precisely what we wanted. In fact, we have established the proof as the point.

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$$\Rightarrow a_1v_1 + \cdots + a_kv_k \in span(S_1)$$

By a similar argument $a_1v_1 + \cdots + a_kv_k \in span(S_2)$

$$\Rightarrow \underline{a_1v_1 + \cdots + a_kv_k} \in span(S_1) \cap span(S_2)$$

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$$\Rightarrow \quad a_1v_1+\cdots+a_kv_k \in span(S_1)$$

By a similar argument $\quad a_1v_1+\cdots+a_kv_k \in span(S_2)$

$$\Rightarrow \quad a_1v_1+\cdots+a_kv_k \in span(S_1)\cap span(S_2)$$

$$\Rightarrow \quad v \in span(S_1)\cap span(S_2)$$

Hence $\quad span(S_1\cap S_2) \subseteq span(S_1)\cap span(S_2)$.

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Problem 5: Let S_1, S_2 be subsets of a vector space V.

Prove that $span(S_1\cap S_2) \subseteq span(S_1)\cap span(S_2)$.

Give an example when (i) $span(S_1\cap S_2) = span(S_1)\cap span(S_2)$

(ii) $span(S_1\cap S_2) \subsetneq span(S_1)\cap span(S_2)$.

Solution: Let $v \in span(S_1\cap S_2)$

i.e. $\exists\ v_1,\ldots,v_k \in S_1\cap S_2$ and $a_1,\ldots,a_k \in \mathbb{R}$

s.t. $v = a_1v_1+a_2v_2+\cdots+a_kv_k$.

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So, this vector which I just underlined is the vector V. So, this implies v is in span of s1 intersected with span of s2. So, we take any arbitrary vector in span of s1 intersected with s2 that will be in span of s1 intersected with span of s2. Therefore, or rather hence span of s1 intersected with s2 is contained in span of s1 intersected with span of s2. Ok, so now we have proved one part.

So, we have established this part. So, what is left is to give examples of when span of s1 intersected with S2 is equal to the span of s1 intersected with span of s2 and when it is not equal. So, I will not go too much into it.

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Example: (i) In $\mathbb{R}^3$ consider

$$S_1 = \{(1,0,0), (0,1,0), (1,1,0)\}$$

$$S_2 = \{(0,1,0), (1,1,0), (1,2,0)\}.$$

$$S_1 \cap S_2 = \{(0,1,0), (1,1,0)\}$$

$$\text{span}(S_1 \cap S_2) = \{(x, y, z) \in \mathbb{R}^3 : z = 0\} = W$$

$$\text{span}(S_1) = W = \text{span}(S_2).$$

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Let me just give you the examples straight away and let me leave it for you to check that if so, the first case where it is equal. Let me just take the simple vector space that can come to that we can consider which is say R2 or maybe R3.

In R3 consider s1 to be say the vector 1 0 0, 0 1 0 and 1 1 0. This is our S1 and what about s2, s2 is 0 1 0, 1 1 0, well 1 2 0. So, you should check that the span of, what is s1 intersection with s2? s1 intersected with s2 is equal to 0 1 0, 1 1 0. And what is the span of s1 intersected with s2? This will be all linear combinations of the vectors returned to the right and you should take that this is the set of all x, y, z in R3 such that z is equal to 0.

In fact you should check that span of s1 is also let me call this w, it is also equal to W, which is the same S span of S2. So, yes, this is a case where all three are equal and therefore span of s1 intersected with s2 is equal to span of s1 intersected with span of s2. How about the second case? Yeah, actually, we could have also arranged for a case where all the three are not equal.

And still we have this, but I will leave that as a thing to think about for you, this is just, after all picking the right vector. So, you should pick various choices and see how it works out. How about the second?

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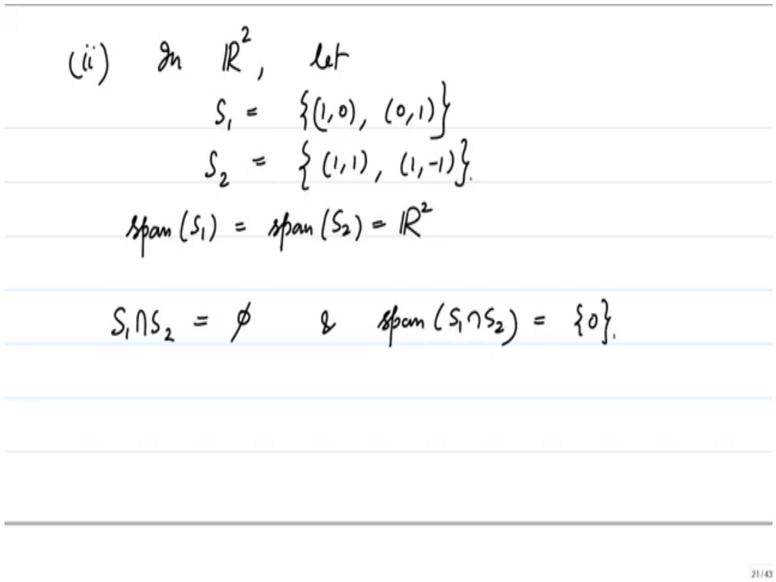

So, let me just give you an example and leave the rest to you. We would like to get hold s1 and s2 such that span of s1 intersected with s2 is strict subset of span of s1 and s2. That is not difficult. Well, we just take, let us do this, let us look at in R2, let S1 be equal to the vectors e1 e2 which is 1 0, 0 1. And s2 be the set 1, comma 1; 1, comma minus 1 and let me leave it for you to check that span of s1 is equal to span of S2 is equal to R2.

By now you know that this is a basis and therefore the intersection is also R2. But what is s1 intersected with s2? This is empty and the span of s1 intersected with S2 is just the span of the empty set which is the 0 subspace. Yes, this is a clear case where it is a proper subspace or rather it is a subspace which is not equal to span of s1 intersected with span of s2. Ok, the next problem demands that we check whether a particular set is linearly independent or not.

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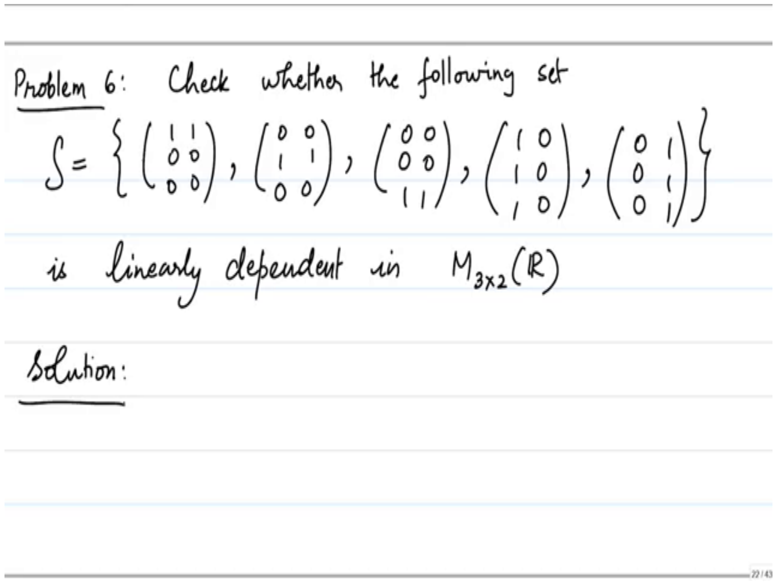

So, problem 6. Check whether the following set what is the set here? The set here is 1 1 0 0 0 0, 0 0 1 1 0 0, 0 0 0 0 1 1, 1 1 1 0 0 0, 0 0 0 1 1 1. So, check whether this following set s is linearly dependent in which is the vector space this is in the vector space of all 3 cross 2 matrices over R. Okay, so solution. So, what do we need to do to check whether a given set is linearly independent or not? We should check for whether there exist a linear combination of it, which is equal to 0. Okay, so suppose there is a linear combination.

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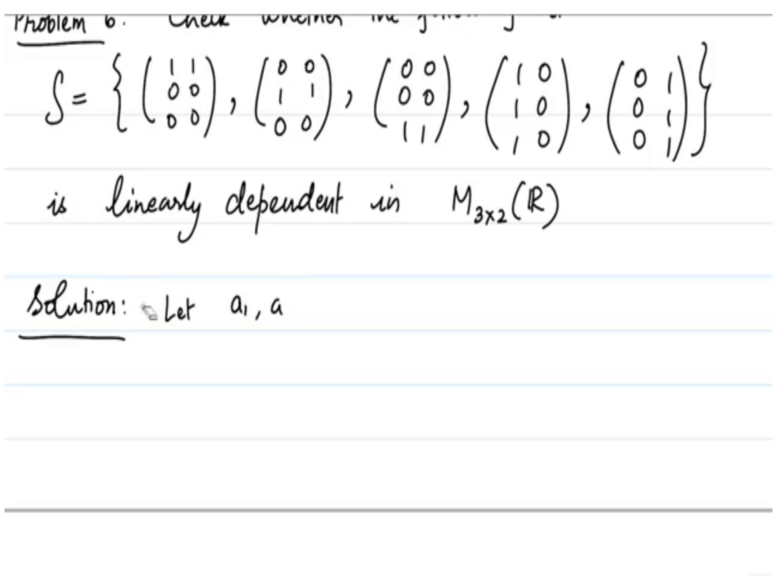

Solution: To check whether S is linearly dependent, we want to check for the existence of $a_1, a_2, \ldots, a_5 \in \mathbb{R}$

s.t. $a_1 \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

So, let a1 a2 ok we would like to, to check whether S is linearly dependent or independent either way, we want to check for the existence. The existence of what? The existence of scalars a1 a2, how many are there? 1, 2, 3, 4, 5 a1, a2, a3 up to a5 in field of scalars such that the linear combination a1 times 1 1 0 0 0 0 plus a2 times 0 0 1 1 0 0 plus a3 times 0 0 0 0 1 1 plus a4 times (0 0 0 0) 1 0 1 0 1 0 a5 times 0 0 0 1 1 0 is ok right.

What is the 0 vector here? Remember that the vector space that we are talking about is the vector space of all 3 cross 2 vectors, sorry 3 cross 2 matrices over R and the 0 vector there is the 0 matrix which consist of the 0 as it entries. This is 0 vector which is 0 0 0 0 0 0 this is precisely what we would like to check, whether there exist a1 a2 a3 a4 a5 such that this linear combination is the 0 vector.

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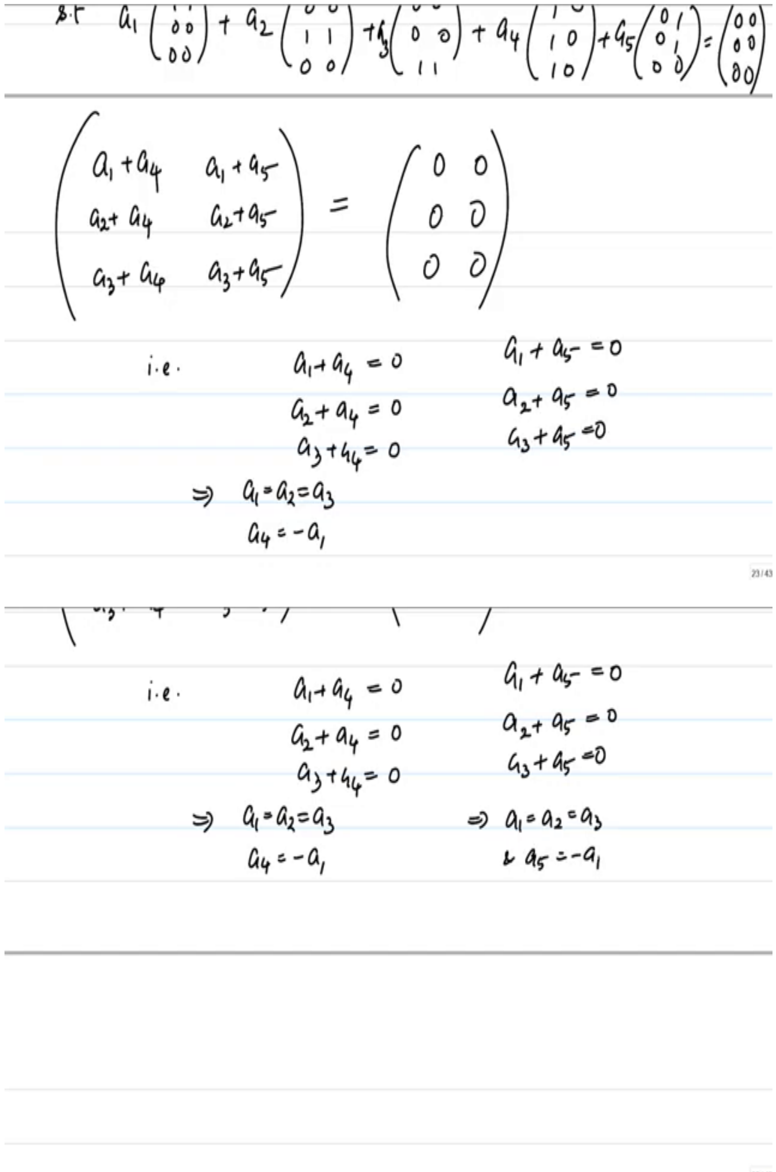

Ok, so if we have to translate it using the vector addition and the scalar multiplication that is involved in M3 cross 2 of R, which is basically component place in both the cases. This is just going to be let me write down the answer directly. This is going to be a1 plus a4, a1 plus a5, this is a2 plus a4, a2 plus a5, a3 plus a4, a3 plus a5. This is the matrix to the left after doing the calculations and this we are demanding to be equal to the 0 vector in the vector space of 3 cross 2 matrices over R.

So, this is what the demand is. But what does that mean? This means that component wise they are equal. That means a1 plus a4 is 0, a2 plus a4 is 0, a3 plus a4 is 0 and the three implies that a1 is equal to a2 is equal to a3 and how about the other one? a1 plus a5 is equal to 0. And this also tells us that a4 is equal to minus of a1 whatever the value of that is. This is also a2 plus a5 equal to 0, a3 plus a5 equal to 0 which implies that a1 is equal to a2 is equal to a3 again, which is consistent and a5 is equal to minus of a1.

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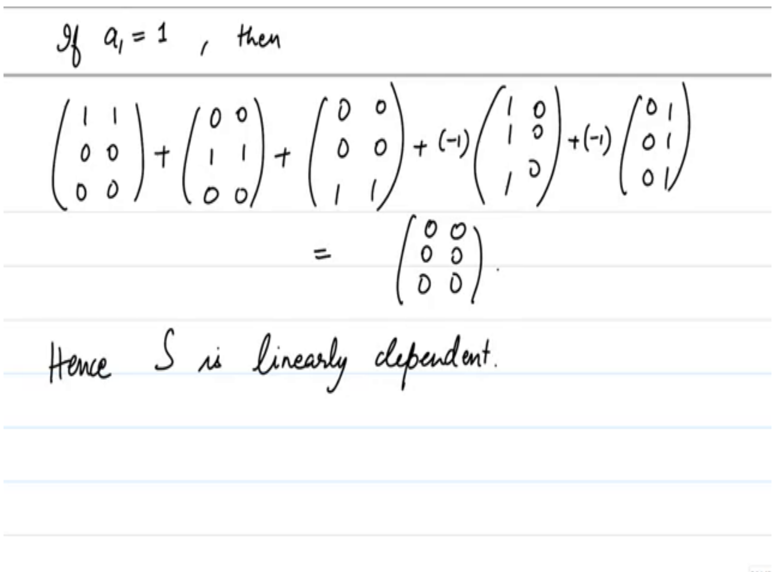

So in particular if a1 is equal to 1, then a1, a2, a3 are all 1, a4 is minus 1 and a5 is also minus 1. So, then what do we have? Then let us check for 1 1 0 0 0 0 plus 0 0 1 1 0 0 plus 0 0 0 0 1 1 plus minus 1 times 1 1 1 0 0 0 or otherwise 1 0 1 0 1 0 plus minus 1 times 0 1 0 1 0 1. This will just turn out to be the 0 vector which is true and therefore the set is linearly dependent. Hence, S is linearly dependent. The next problem demands that we check for a subset to be linearly independent.

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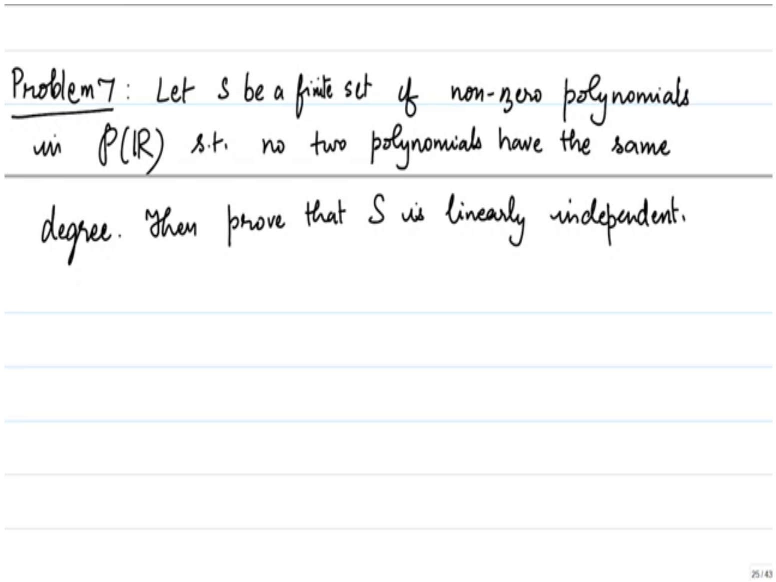

So, this is problem 7. So, let S be a set of non-0 polynomials in P of R. Let us do one thing, let us put it as a finite set. Let S be a finite set. I do not need to impose this you should think about why this is true when it is not impose; the finiteness condition is not being imposed. They should still be true but nevertheless, let S be a finite set of non-0 polynomials in P of R such that no two polynomials have the same degree.

So, if one polynomial is a X square plus 1, then that other polynomials, none of them can have degree 2 because x square plus 1 is already having degree 2, so all the polynomials have distinct degrees. If at all there is a polynomial of say degree 2 there would be only one polynomial which has degree 2.

So, they have distinct degrees ok no two of them have the same degree. Then prove that S is linearly independent. So, as much as the problem might sound sophisticated. It is actually quite simple to prove if you make the right observations.

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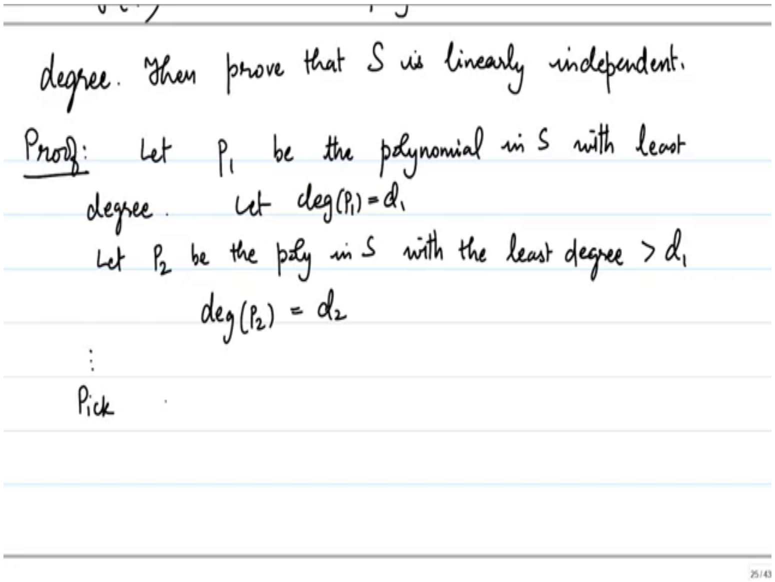

So, let us give a proof a solution to this problem. So, we know that S has, S is a finite set which has distinct degrees. So, what we will do is we will pick the polynomial in this, which has the least degree. So, let P1 be the polynomial in S with least degree. Let us say it is d1. P2, similarly, be the polynomial which has least degree, however greater than d1.

We know that least degree is d1. So, let us, let P2 be the polynomial in S with degree with degree d2 with the least degree. Let us call this d, degree let us call it. Let us call the degree let of P1 be equal to d1. So, d2 degree, least degree greater than d1 so there will be only one such polynomial and let degree of P2 be equal to d2. So, notice is that d1 is less than d2 and so on. So, pick pk.

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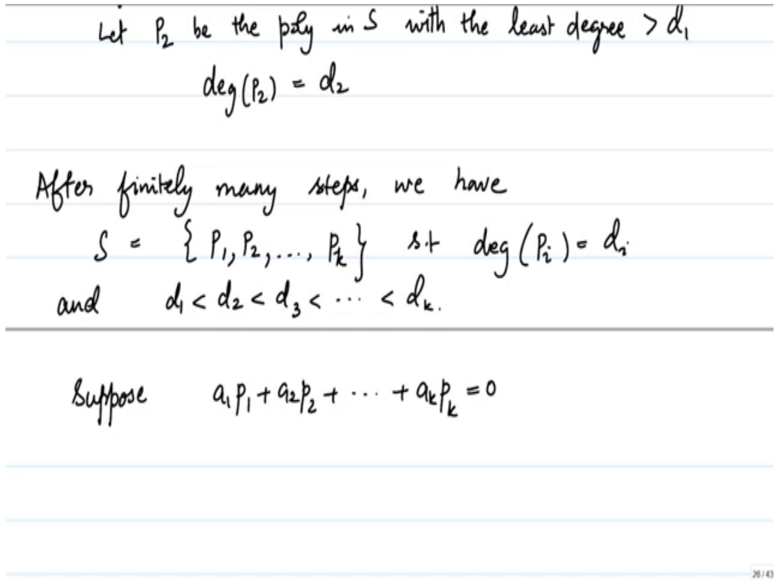

So, therefore, after finitely many steps such in the algorithmic process. We have S is equal to P1 P2 up to say Pk, there are finitely many of them such that degree of Pi is equal to di and d1 is less than d2 is less than d3 less than dot dot dot dk. So, pk has the highest degree and Pk minus 1 has degree less than the degree of Pk and so on. Okay, now we are almost done. Suppose, a1p1 plus a2p2 plus up to akpk is equal to 0 then what do we know?

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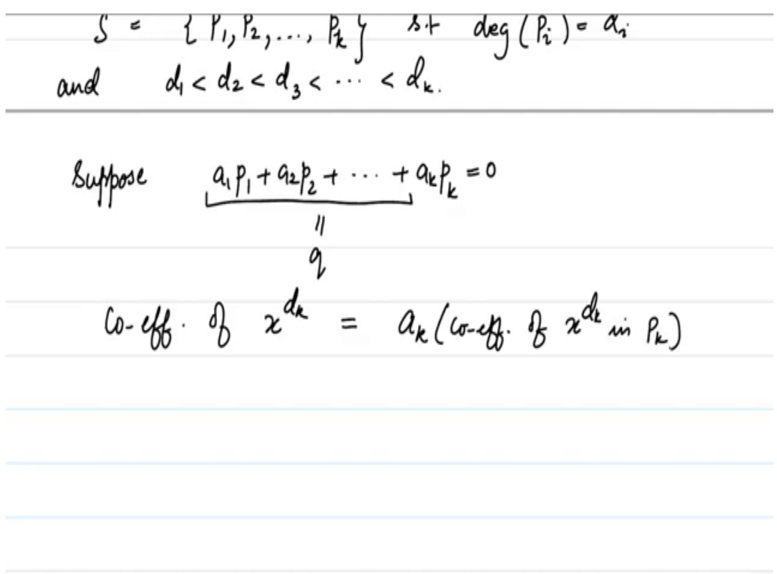

$S = \{p_1, p_2, \ldots, p_k\}$ s.t. $\deg(p_i) = d_i$

and $d_1 < d_2 < d_3 < \cdots < d_k$.

Suppose $\underbrace{a_1 p_1 + a_2 p_2 + \cdots + a_k p_k}_{= q} = 0$

Co-eff. of $x^{d_k} = a_k b_k$

where b_k = co-eff. of x^{d_k} in $p_k(x)$.

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Let me ask you what the degree, so let this be equal to q, check what will be the coefficient of x to the power dk will be equal. Check that this is equal to ak because a1 a2 up to ak minus 1 do not have x to the power dk in its polynomial expression because all of them have a degree less than dk, only pk has the monomial x to the power dk.

And therefore, the coefficient of pk would be ak times the so ok. So, I have to be a bit more careful times the coefficient of x to the power dk in pk, which is an non-0 number, so let us do one thing, let us call it bk where bk is equal to the coefficient of x to the power dk in pk of x. But we know that we are looking at this being equal to the 0 polynomial.

(Refer Slide Time: 68:19)

Co-eff. of $x^{d_k} = a_k b_k$

where b_k = co-eff. of x^{d_k} in $P_k(x)$.

$\Rightarrow a_k b_k = 0 \Rightarrow a_k = 0$ (since $b_k = 0$ implies that $\deg(P_k) \neq d_k$).

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After finitely many steps, we have

$S = \{P_1, P_2, \ldots, P_k\}$ s.t. $\deg(P_i) = d_i$

and $d_1 < d_2 < d_3 < \cdots < d_k$.

Suppose $\underbrace{a_1 P_1 + a_2 P_2 + \cdots +}_{= q} a_k P_k = 0$ — (*)

Co-eff. of $x^{d_k} = a_k b_k$

where b_k = co-eff. of x^{d_k} in $P_k(x)$.

$\Rightarrow a_k b_k = 0 \Rightarrow a_k = 0$ (since $b_k = 0$ implies that $\deg(P_k) \neq d_k$).

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Hence, equating the coefficients this implies that ak bk is equal to 0. But we know that bk is not equal to 0, why? Because pk has degree x to the power, pk has degree dk and therefore, if its coefficient is 0. Then we cannot have its degree to be equal to dk. This implies ak is equal to 0. Why? Let me just note down the reason since bk is not equal to 0 or rather, since bk equal to 0 implies that degree of pk is not equal to bk.

But we know that degree of pk is equal to dk and therefore, bk cannot be 0 and therefore, ak has to be necessarily 0. Ok so let us get back to our star. We have just established that ak is 0.

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where b_k = co-eff. of x^{d_k} in $P_k(x)$.

$\Rightarrow a_k b_k = 0 \Rightarrow a_k = 0$ (since $b_k = 0$ implies that $deg(P_k) \neq d_k$).

$\Rightarrow a_1 P_1 + a_2 P_2 + \cdots + a_{k-1} P_{k-1} = 0$

By a similar argument $a_{k-1} = 0$

By repeating the above process, we have $a_k = a_{k-1} = \cdots = a_1 = 0$

$deg(P_2) = d_2$

After finitely many steps, we have

$S = \{P_1, P_2, \ldots, P_k\}$ s.t. $deg(P_i) = d_i$

and $d_1 < d_2 < d_3 < \cdots < d_k$.

Suppose $\underbrace{a_1 P_1 + a_2 P_2 + \cdots + a_k P_k}_{= q} = 0$ — (*)

Co-eff. of $x^{d_k} = a_k b_k$

where b_k = co-eff. of x^{d_k} in $P_k(x)$.

This implies that a1p1 plus a2p2 plus up to ak minus 1, pk minus 1, this is equal to be 0 polynomial. Because ak is 0, the last polynomial does not contribute but by a similar argument we can conclude that ak minus 1 is also equal to 0 and so on.

And by a similar argument, now once ak minus 1 is 0 inductively you can say that now a1 a2 up to, by repeating the process about not inducting, let me just say repeating the above process. We have a1 or rather ak equal to ak minus 1 equal to up to a1 equal to 0. But that is precisely what we wanted to prove, if you notice if there is a linear combination of p1 p2 up to pk equal to 0, we have just established that the coefficients are necessarily 0.

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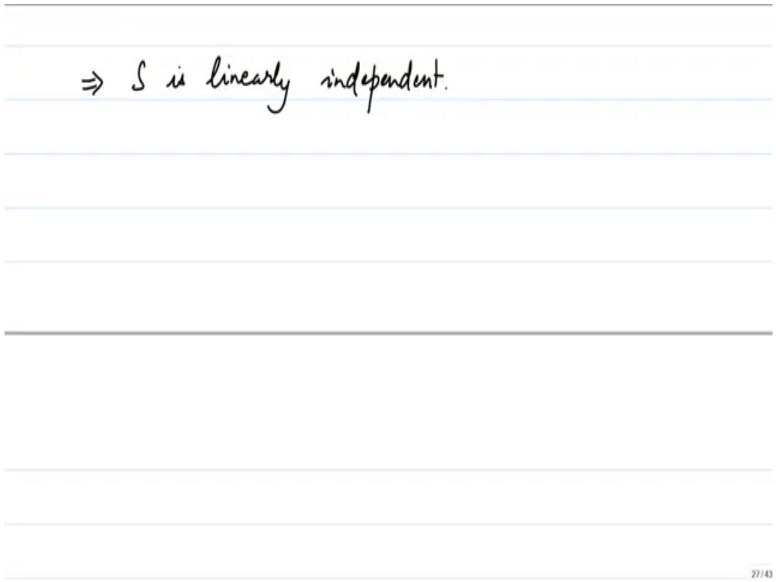

This implies that S is linearly independent. So, the next problem is about proving whether a given subset is a basis of the given vector space.

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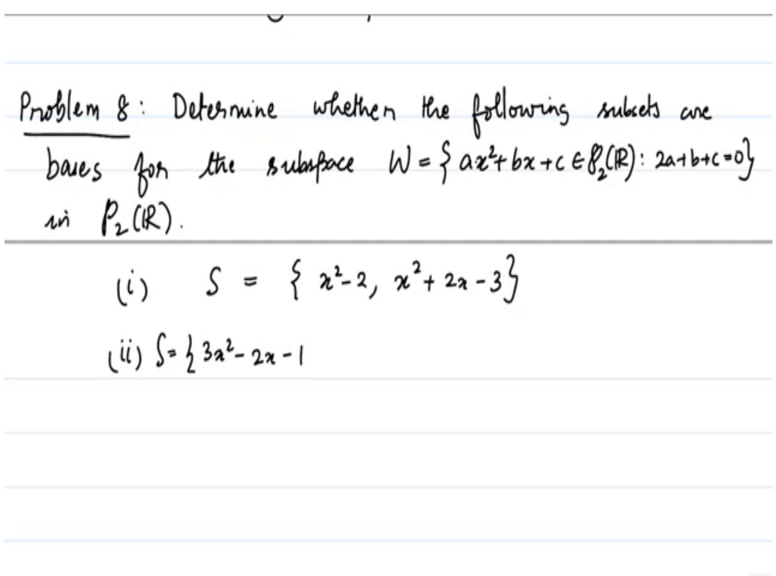

Problem 8: Determine whether the following subsets are bases for the subspace $W = \{ax^2+bx+c \in P_2(\mathbb{R}) : 2a+b+c=0\}$ in $P_2(\mathbb{R})$.

(i) $S = \{x^2-2,\ x^2+2x-3\}$

(ii) $S = \{3x^2-2x-4,\ x^2-x-1,\ -2x^2+x+3\}$

(iii) $S = \{x^2-2,\ -2x^2+x+3\}$.

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So, this is problem 8. So, determine whether the following subsets or following sets are basis for the vector space or subspace W, a of P of R such that 2a plus b plus c is equal to say 0. So, where is this a subspace, subspace w in P2 of R. So, we are ok so what are the sets let us see, the first set is S which is given by x square minus 2, x square minus 2x or rather plus 2x minus 3.

2, S is equal to 3x square minus 2x minus 1. So, this is 2a plus b minus C. So, S is equal to 3x square minus 2x minus 4, x square minus x minus 1, 2x square or rather minus 2x square plus x minus 2, 4 plus 3. And third one would be S to be say x square minus 2, minus 2x square plus x plus 3. Let us take these three. Ok, so we need to check whether the following subsets are basis of W. What was W? Let me just put W in your view, W is the set of all those polynomials in P2 or R whose coefficients satisfy some relation.

So, what do we need to check, whether something is indeed the basis of W? We need to check that first condition is to see whether it is linearly independent. And the second condition would be to check whether the span of the given set. So, let us do that.

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(iii) $S = \{x^2-2,\ -2x^2+x+3\}$.

Solution: (i) $S = \{x^2-2,\ x^2+2x-3\}$

Let $a, b \in \mathbb{R}$ s.t $a(x^2-2)+b(x^2+2x-3)=0$

$\Rightarrow (a+b)x^2+2bx+(-2a-3b)=0$

$\Rightarrow 2b=0 \Rightarrow b=0$

$a+b=0 \Rightarrow a=0$

Hence S is linearly independent

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$\Rightarrow S$ is linearly independent.

Problem 8: Determine whether the following subsets are bases for the subspace $W=\{ax^2+bx+c \in P_2(\mathbb{R}) : 2a+b+c=0\}$ in $P_2(\mathbb{R})$.

(i) $S = \{x^2-2,\ x^2+2x-4\}$

(ii) $S = \{3x^2-2x-4,\ x^2-x-1,\ -2x^2+x+3\}$

(iii) $S = \{x^2-2,\ -2x^2+x+3\}$.

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So, solution. So, the first problem, let us do both. Let S be x square minus 2, x square plus 2x minus 3. So, linear independence, so let a, comma b be scalars such that a times x square minus 2 plus b times x square plus 2x minus 3 is equal to 0, the 0 polynomial.

So, that would imply a plus b times x square plus 2bx plus minus of 2a minus of 3b is equal to 0. But what does it mean to say that a polynomial is 0 or its coefficients are 0. This implies 2b is equal to 0 and hence b is equal to 0. And the first one a plus b is equal to 0 implies a is also equal to 0, which is inconsistent with minus of 2a plus minus of 3b equal to 0. So, yes, this forces a and b to be equal to 0.

Hence, S is linearly independent. Yes. So, the first thing to note, which I just skipped is to check whether s is a subset of W, x square minus 2 if you if you notice x square minus two will just satisfy 2a minus 2a plus b plus c should be 0 so 2 minus 2 is 0 and yes and 2, 2a is 2, 2 plus 2 minus 3. Yeah. So, this is a problem. This cannot be in. Oh, no, this is bad. 2 plus 2 4 minus so this should have been something like minus 4 or something.

So, let me just do one thing, let me tweak it so that you know, I gain what I want so it has to be for 4 notice that if it is not 4 that vector will not be in W.

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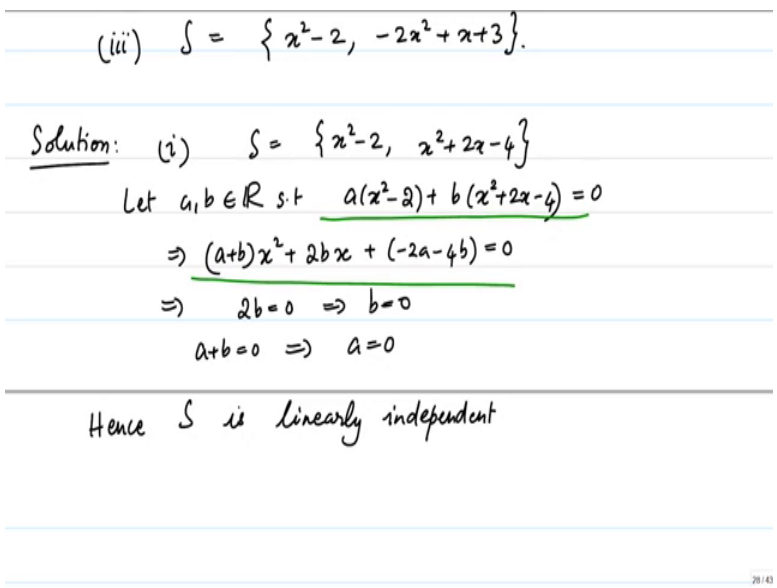

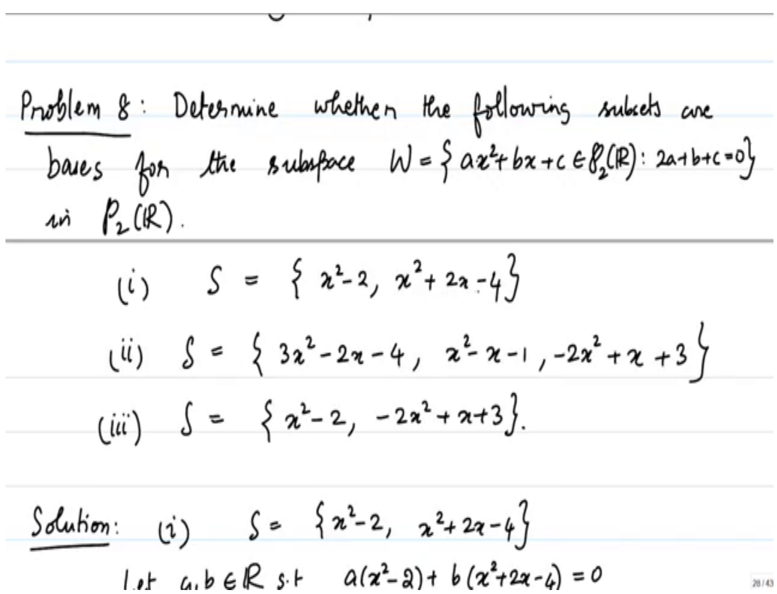

So, let me come down and slightly change everything. Well, I would like to have that this particular vector is in W because otherwise we can straightaway say that these vectors are not in W and hence it cannot be a basis, but yeah, so at least let us make it a little more challenging by taking two vectors in W and checking for whether it is a basis.

So, yes, this is now in this case let us check once more x square plus 2x minus 4 will satisfy 2 plus 2 plus minus 4, which is 0. So, this is a subset of W and this argument which would not have been which is not disturbed at all. So, let me, let us go back once more and check line by line. Suppose, you have a linear combination which is equal to 0 and grouping for the coefficients we get a plus b is 0, 2b is 0, minus of 2a minus 4b is 0 or rather a plus 2b 0.

But all the, the first two forces, both a and b to be 0 and it is consistent with the third. Therefore, a and b are forced to be 0. Therefore, S is a linearly independent set, but we are only halfway through. We have just shown that S is a linearly independent subset of W and the question is whether S is a basis. So, we have to check for whether it is a spanning set. Let me not check brute force that this is a spanning set. I would rather use an indirect argument to conclude that it is a spanning set.

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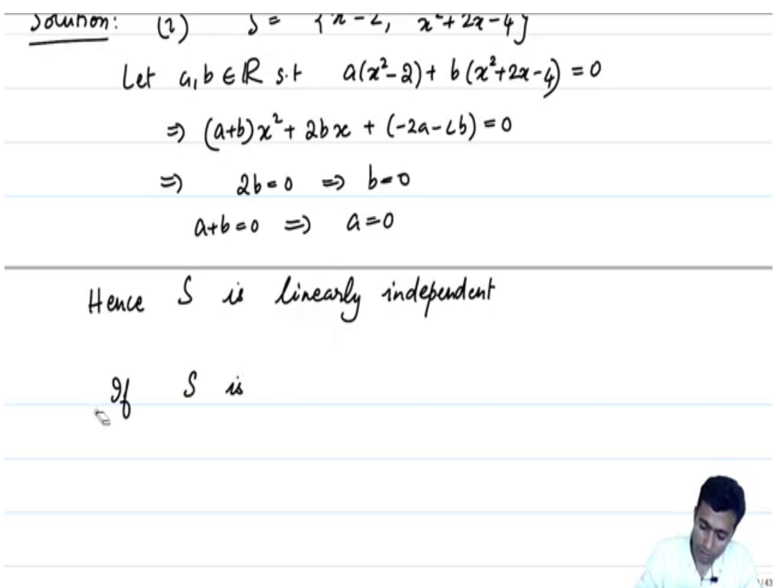

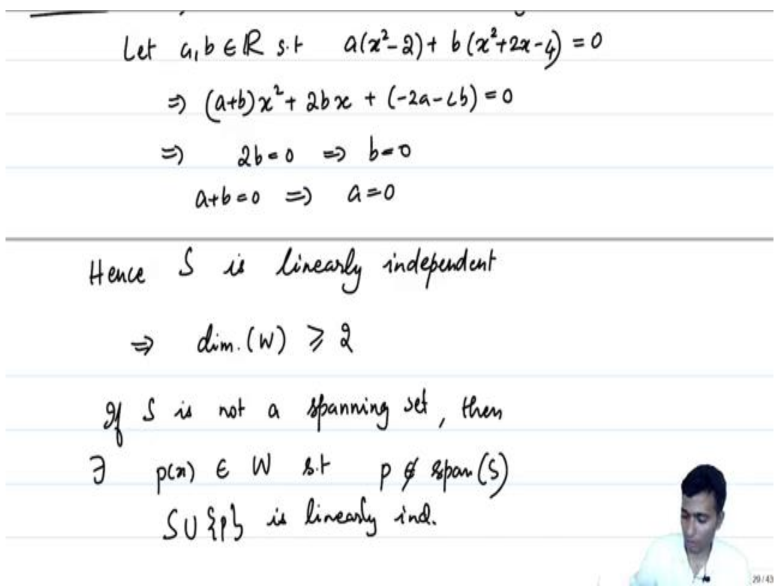

So, if S is, so what is the meaning of S being linearly independent that means that dimension of W is greater than or equal to 2. Why is that the case? That is because every linearly independent set is contained in a basis. And therefore, S being linearly independent means that there exist a basis of W which contains S, and therefore it should have at least size 2, so dimension of W should be greater than or equal to 2.

If S is not a spanning set. So, if S is not a spanning set then there exists some polynomial P in capital W such that P is not in span of what were the two polynomials or span of S. Let me just call it S, span of S. That is what it means. And therefore, S Union P is linearly independent.

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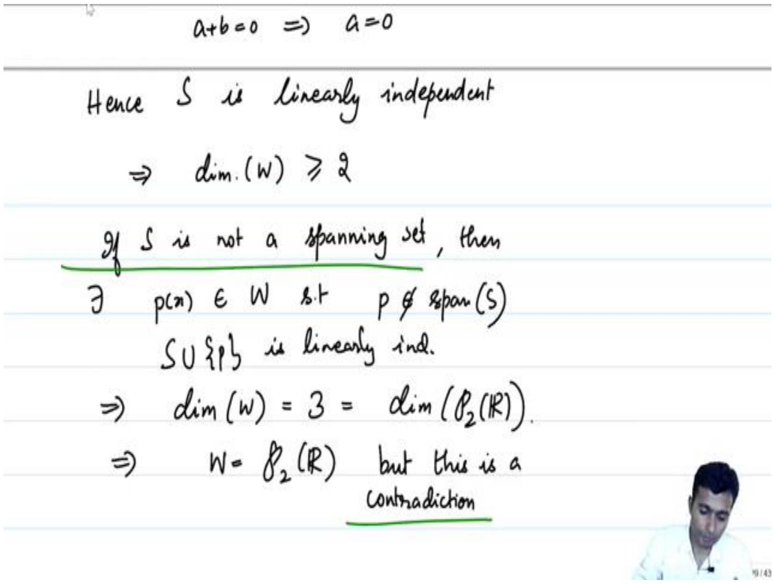

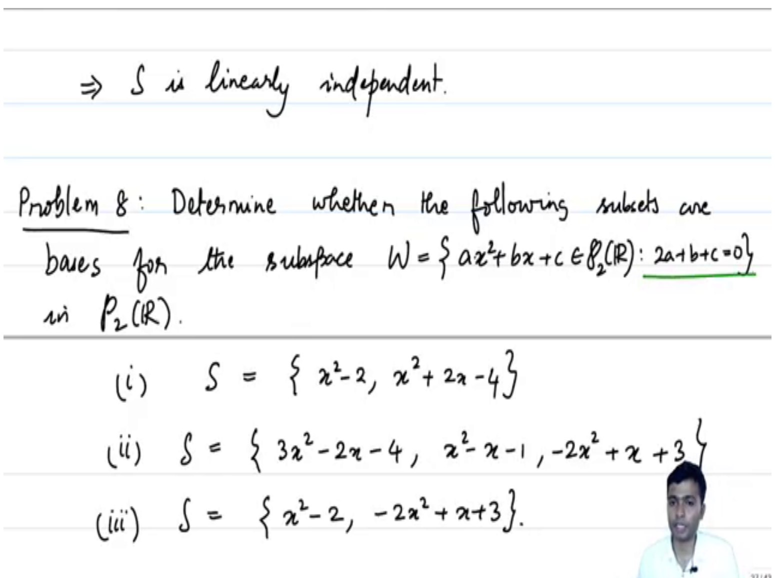

And that would imply that dimension of W is at least 3, but it cannot be more than 3 because P2 of R has dimension 3 which is equal to the dimension of P2 of R. And what do we know about (dimen) subspaces of P2 of R, which has dimension equal to 3. It has to be necessarily equal to P2 of R. This implies that W is equal to P2 of R. But this is a contradiction. Why is this a contradiction because every element in P2 of R does not satisfy the equation of the coefficients, which I am now underlining in green.

For example, you look at 2x square. 2x square has coefficients 2 for x square and 0 and 0 so the 2a plus b plus c will give us 4, which is not equal to 0, so P2 of R every element of P2 of R does not satisfy this relation when it comes to coefficients and therefore what was our assumption? So, we proved this problem, which apparently looked very numerical by a contradiction argument. So, what was the, yeah. So, if S is not a spanning set, this was the assumption we started of it and we arrived at a contradiction.

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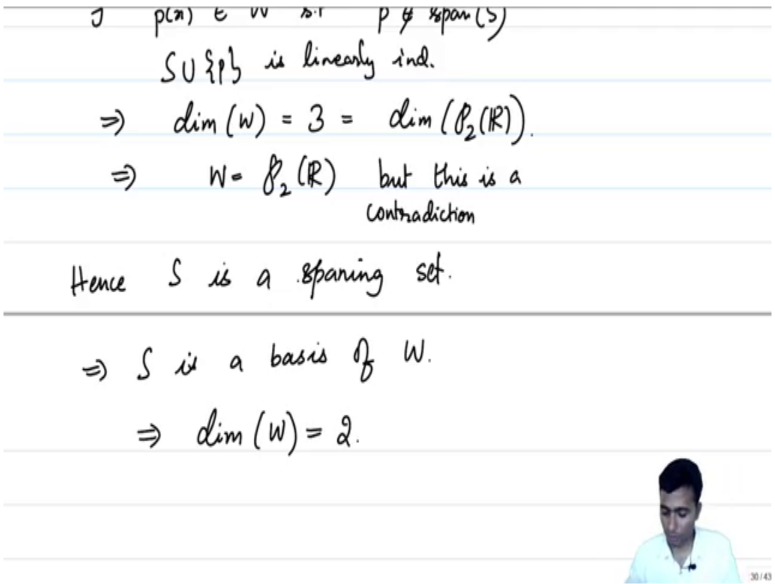

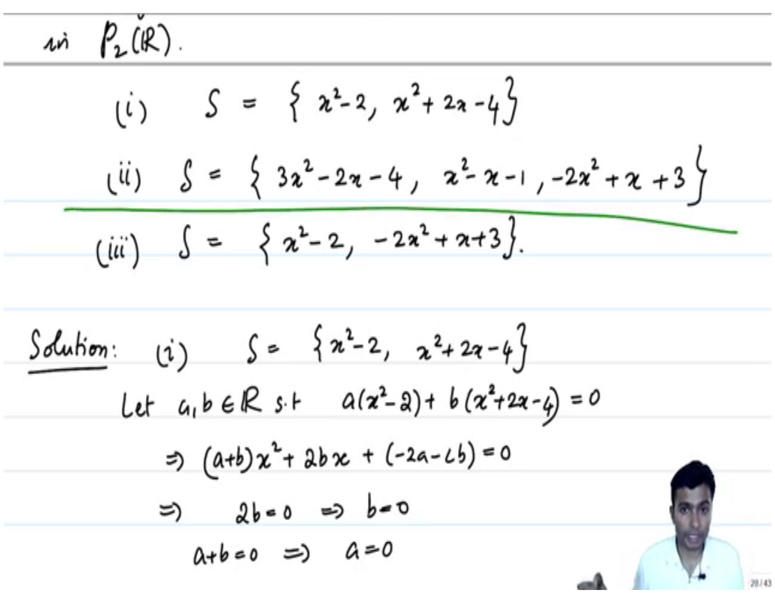

And therefore, hence, S is a spanning set that means S is both a spanning set and a linearly independent set, which implies that S is a basis of W. All right, so we know have ok we can conclude something more. This means that a dimension of W is equal to 2. Dimension of W is equal to 2. Ok good because if you look at 2 which I just underlined in green S is a subset of, I hope it is a subset of W.

So, notice that S has three elements from W and we know that dimension of W is equal to 2 and therefore, you take any subset of W which has more than two elements. It should necessarily be linearly dependent by one of the consequences.

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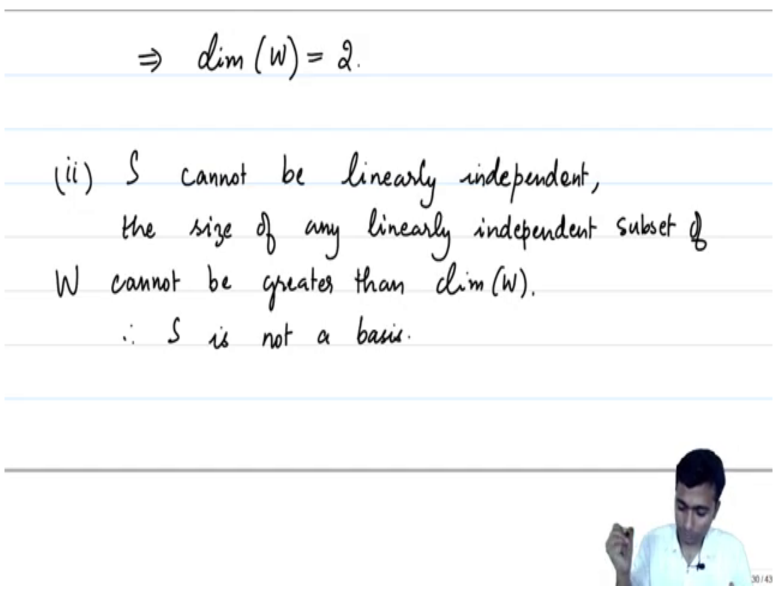

So, let me say this, S cannot be linearly independent since by one of the consequences of the replacement theorem, the size of any linearly independent subset of W should be less than or equal to or let me write it like this cannot be greater than the dimension of W. And this has 3 elements, which is more than the dimension of W. And therefore, S is not a basis.

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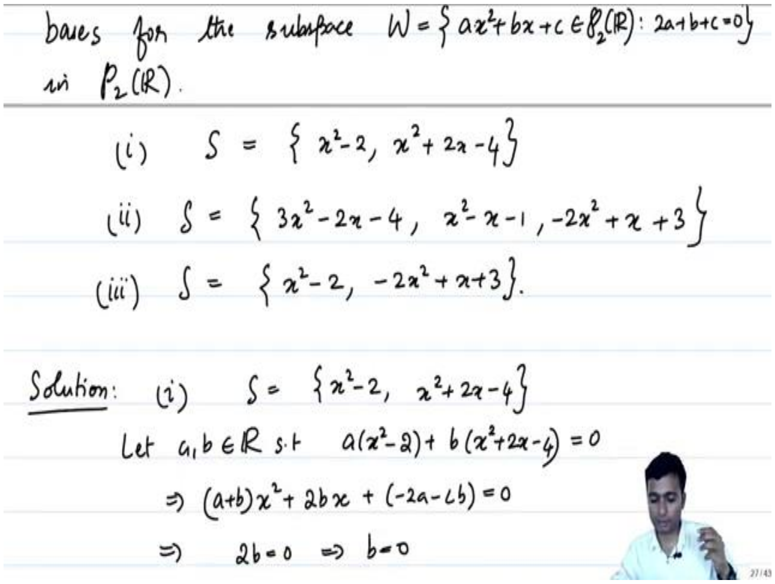

Ok so now let us come to problem number 3. Problem number 3 has two vectors x square minus 2 and minus 2x square plus x plus 3 minus 2x square minus 4 plus 1 plus 3 is 0. Yes. So, this is a subset of S it consists of two vectors. We would like to see whether it is a basis again by one of the consequences of replacement theorem.

It is enough to check whether a subset of W of size 2 is linearly independent or rather let me put it this way, to check that S is a basis of W it is enough to check for either linear independence or the spanning property.

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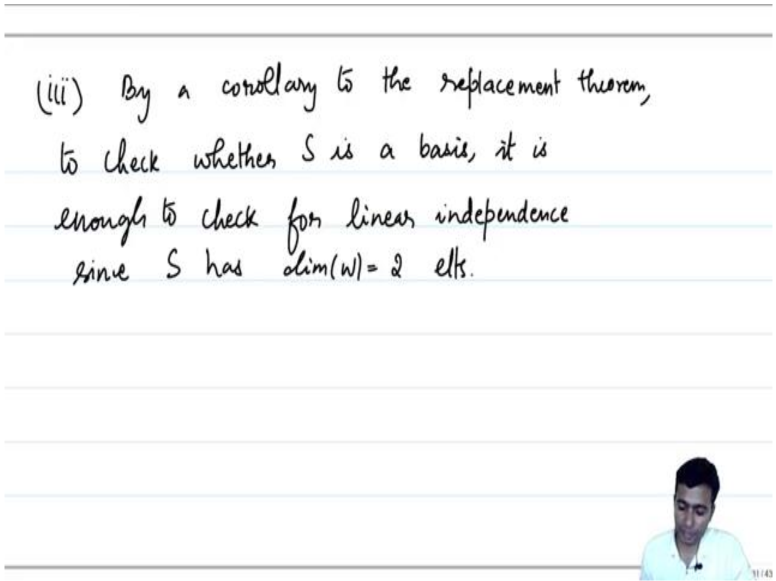

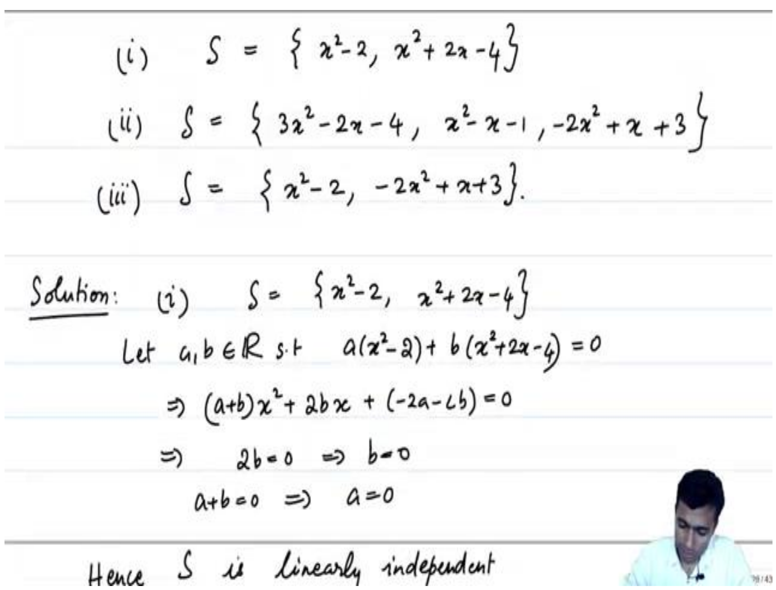

So, (())(84:03) I just said by a corollary to the replacement theorem to check whether S is a basis, it is enough to check for linear independence. And why is that the case? Because since, S has dimension of W equal to 2 elements.

So, if this set is linearly independent its necessarily a span set. We could have done the other way also. We could have just checked for spanning property and any spanning set which has dimension of W elements should necessarily be linearly independent set. But we will just check for linear independence. What was the set S?

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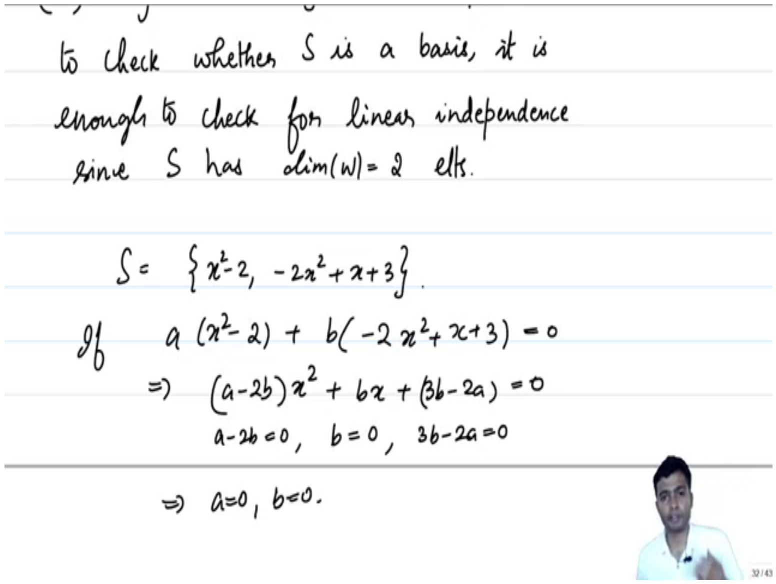

S was x square minus 2, minus 2x square plus x plus 3 and the check for linear independence is straightforward. If a times x square minus 2 plus this b times minus 2x square plus x plus 3 is equal to 0. This would imply a minus 2b times x square plus bx plus 3b minus 2a is equal to 0. And from this we have a minus 2b is equal to 0, b is equal to 0, 3b minus 2 is equal to 0. The second equation already tells us that b is equal to 0. And along with the second equation, both first or third, which is just consistent tells us that a is equal to 0 and b is equal to 0, therefore, we have established that it is linearly independent.

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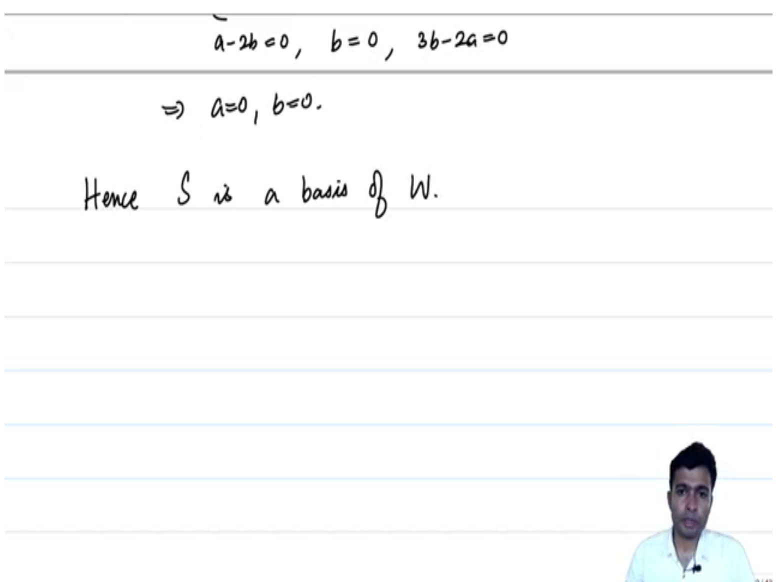

And therefore, it is automatically a spanning set and which makes it into a basis of W. So, the next problem demands that we determine whether what the dimension of R given subspace is ok.

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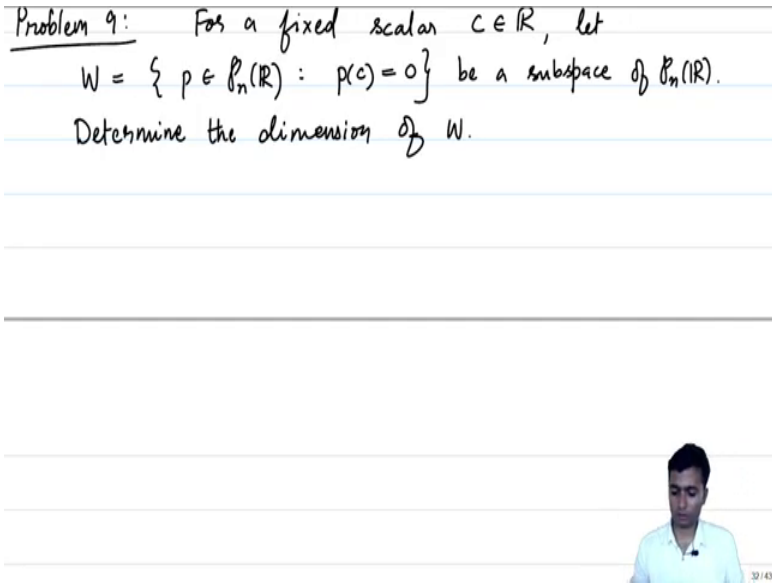

So, problem 9. For a fixed scalar a or rather c for a fixed scalar c, let W be the set of all polynomials in Pn of R such that P of C is equal to 0. Check that this is a subspace, be a subspace of Pn of R. Find or determine the dimension of W.

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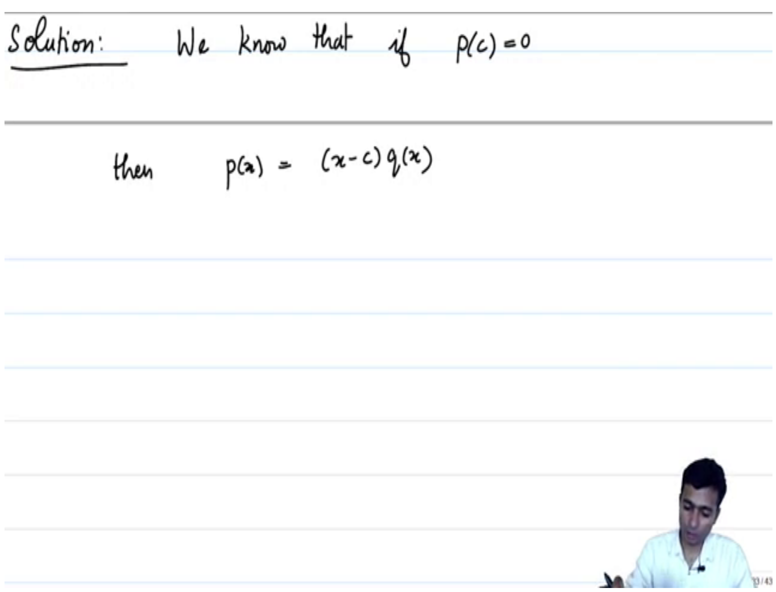

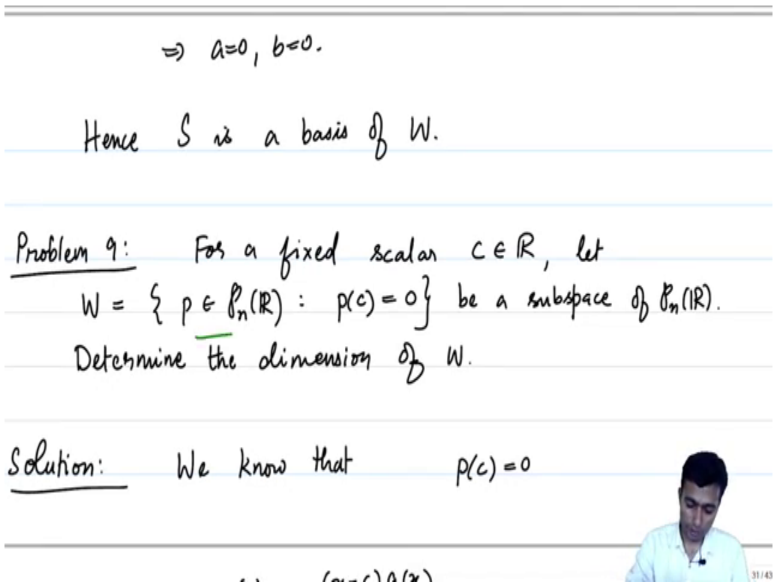

Ok, so let us look at the solution. So, of course one needs to check that W is indeed a subspace even though it is being said that it is a subspace. It is our job to check that that is a subspace because otherwise there is no question of finding dimension. But yeah, so let us as of now, let me leave that as an exercise for you to take that it is indeed a subspace. Our goal in this problem, however, is to determine the dimension of W.

Ok so what is W precisely? W is the set of all polynomials such that P vanishes at C. So, we know from our basic algebra. We know that if P of C is equal to 0 then P of X is what? This is going to be X minus C times Q of X where degree of Q of X is degree of P of X minus 1. So, this is exactly what our P of X will look like. So, in particular, if Q of X if you look at any polynomial of the type X minus C times Q of X that will satisfy the condition that it vanishes at C.

So, it is if and only if it is in some sense, P of C is 0. We know that let me put it this P of C is 0 if and only if this is the case. So, if you look at Q of X varying over all polynomials with say degree 1 to, degrees 0 to n minus 1. So, this is Pn of R which is up to degree N. In particular, if Q of X is any polynomial degree and minus 1, we should be able to get hold of this. So, that gives as a natural candidate.

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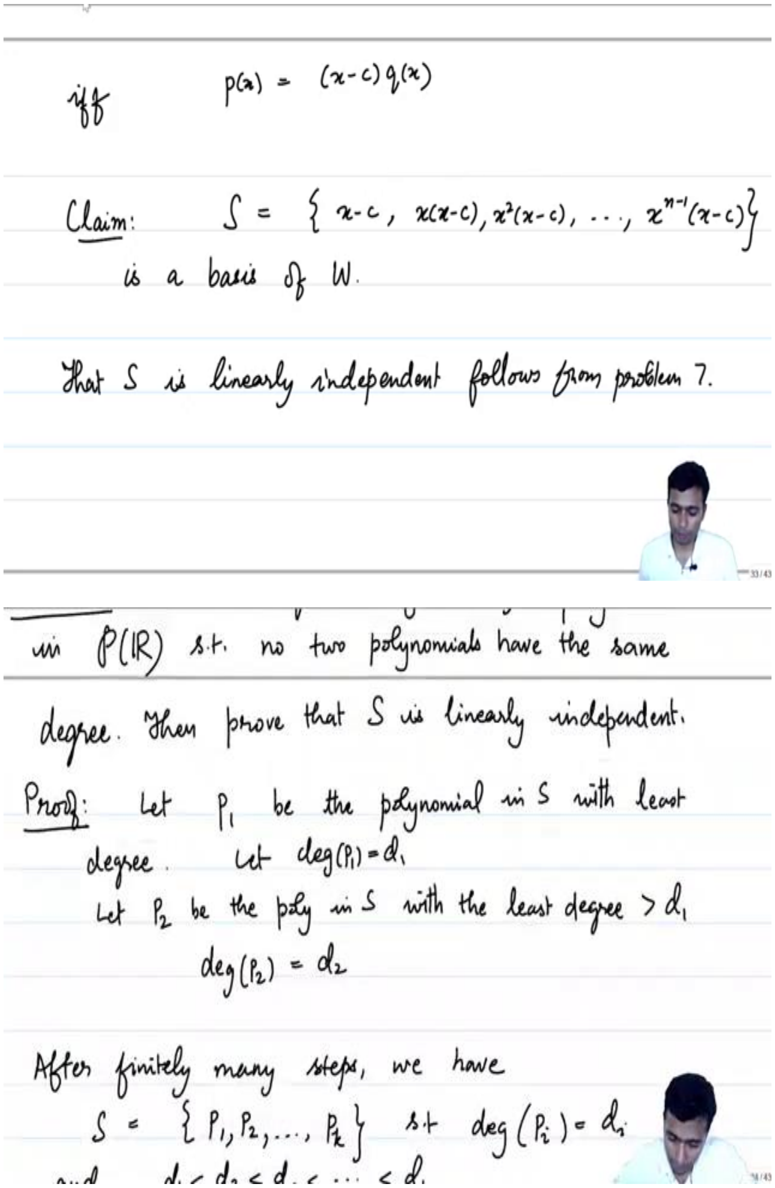

So, let me put as a claim. S be equal to 1 times X minus C, which is X minus C, X times X minus C, X square times X minus C, X to the power n minus 1 times X minus C. This is a potential candidate is a basis of W such as described the motivation to S or conjecture that this particular set will be a basis.

So, the fact that S is independent follows from one of the problems we have just proved that S is linearly independent follows from problem 7. What is that problem 7 said that if there are if there is a subset S, which consists of polynomials which have different degrees, no two of them have the same degrees, same degrees, then they should be linearly independent.

So, notice that we have, we can. We could have done this. The same argument goes through in Pn of R as well. So, let me write it. I think problem 7. So, problem 7, why? Because this is a degree 1, this is a degree 2 this is degree 3 and this is degree n all these have different degrees, so they are necessarily linearly independent.

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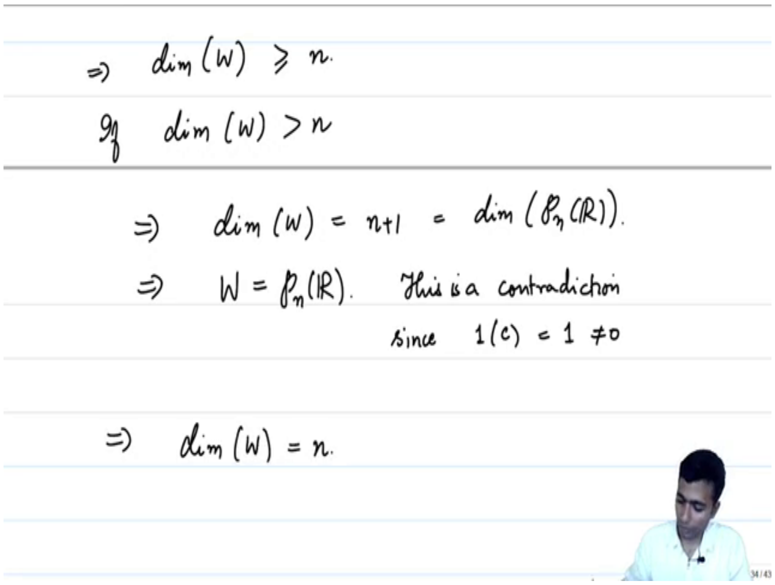

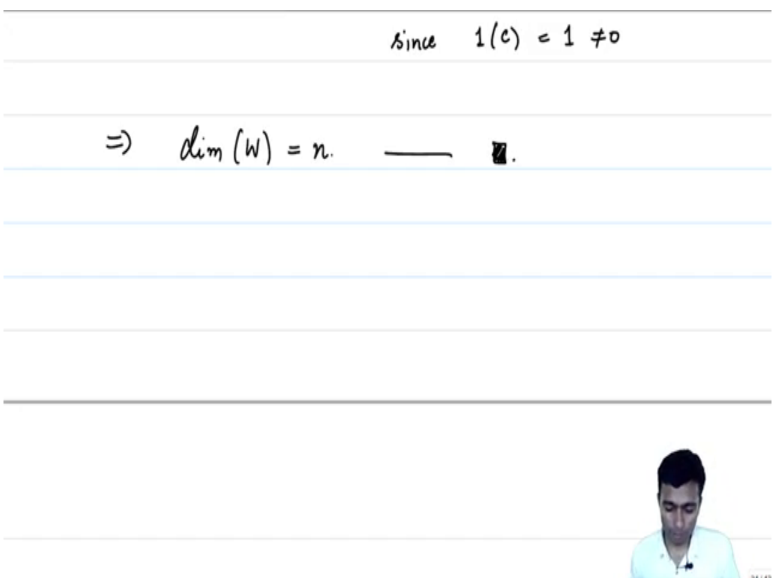

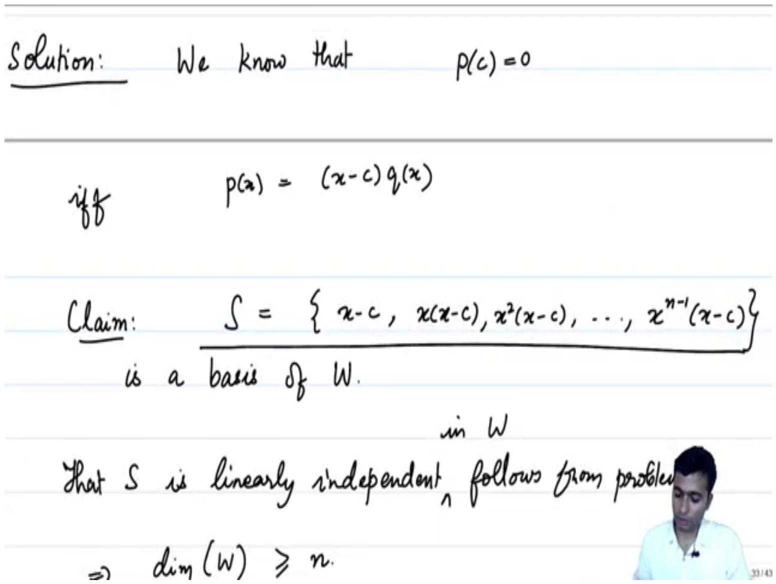

And what is the size of n so that means dimension of w linearly independent where in W. So, all these are vectors in W and they are linearly independent, therefore dimension of W should be greater than how many of them are there? This is 0, 1, 2 up to n minus 1 so there are n of them greater than or equal to n.

So, if dimension of W is not equal to n, if dimension of W is greater than n, it has to be n plus 1 because Pn of R has dimension n plus 1. This would imply dimension of W will be equal to n plus 1 which is equal to the dimension of Pn of R. We have used this type of an argument as well earlier in one of the problems that would imply w is equal to Pn of R only subspace of Pn of R which has dimension equal to 1 plus n itself.

But that cannot happen because this is a contradiction. Because what was our W? W was the collection of all those polynomials which vanishes at C. So, in particular, if you look at the constant polynomial, 1 at C is equal to 1, which is not equal to 0, constant polynomial, non-0 constant polynomials do not vanish at any point. And W should necessarily vanishes at every element in w should necessarily vanish at c vanish at c I mean P of c is equal to 0. So, this is a contradiction which implies dimension of W so this assumption which I just under which I just underlined cannot happen.

This implies dimension of W is equal to n. But, what do we have? We have now S which is equal to Oh, we already have just proved that dimension of S is equal to dimension of W is equal to n which is what our goal was. So, also notice that S is indeed a spanning set. So, yes this is our solution to the problem.