Linear algebra
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Lecture 3.4
Linear Transformation and Matrices

So when we define linear transformations, we saw a lot of mattresses coming in as examples of linear transformations. In fact, we saw that any M cross N matrix can be thought of as a linear transformation from Rn into Rm, and it was vaguely hinted upon that we can associate a metrics to any in linear transformation in the right setting. So, in this video we will try to make precise what is meant by a matrix associated to a linear transformation, when what is called as the coordinate basis are specified. So, let us begin this video by defining what a coordinating basis is.

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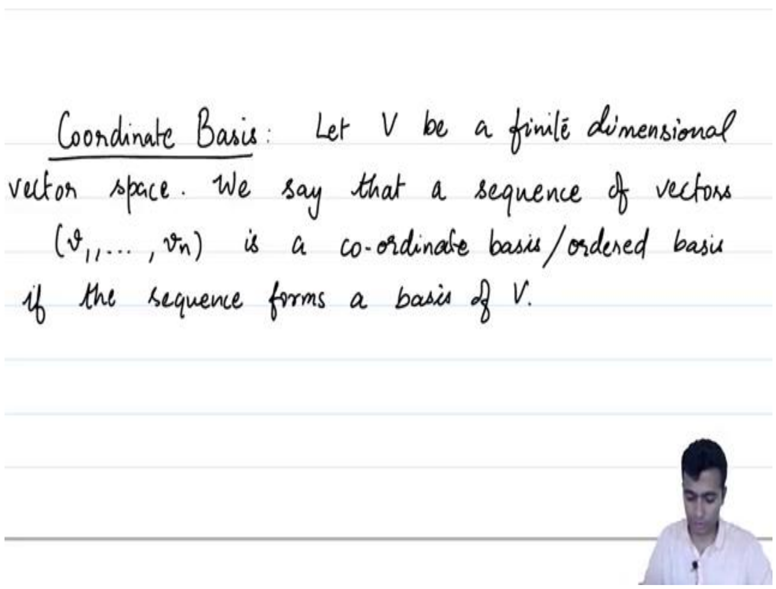

So, let us define what coordinate basis are, so, let us focus on attention on finite dimensional vector spaces here. So, let v be a finite dimensional vector space. Needless to say, all our vector spaces are over the real numbers. So, let beta, okay, let us not define it that way. We say that a sequence of vectors, that is all those vectors, sequence of vectors v1 to vn, we say that a sequence of vectors is a coordinate basis. Let us also give alternate name to it, it is also called ordered basis. If this sequence forms a basis of v.

So, let us just try to understand what this definition says because it does not seem like it says much, but that is not the case. We start off with a set say v1, v2 upto vn, which is both linearly independent and a spanning set it turns out to be a basis. When we speak about an ordered basis or a coordinate basis, we are concerned about the order in which we are considering the basis elements. So, in this particular definition you should observe that we are always considering a sequence of vectors v1, v2 up to vn.

So in other words, we can talk about the first vector, the second vector, third vector, nth Vector and so on. Which we could not have done when we just talked about a basis because there was no inherent order which we were referring to in the definition of our basis. So, from your knowledge of real analysis, you would realize that if two elements in a sequence are interchanged, the sequence has technically changed, so that is exactly what will reflect here as well. If the order in which we considered the basis change, the coordinate basis changes even though the vectors need not have change. So, let us give a few examples to illustrate all that we just saw.

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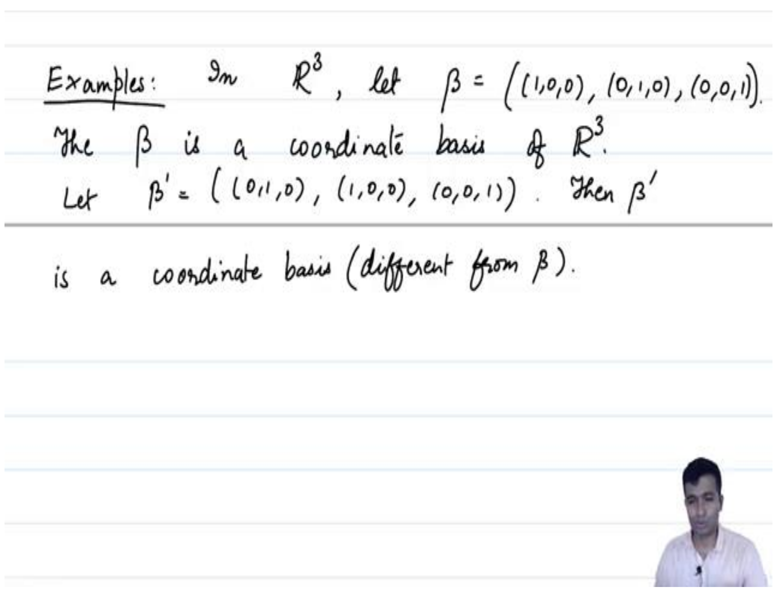

So examples, the simplest example should be in say, let us go to R3, R2 will be too simplistic to talk about the change in order. So in R3 let beta be the, so sometimes the Greek alphabets are used to denote ordered basis or coordinate basis. So beta let this be the sequence of vectors, 1, 0, 0, 0, 1, 0 and 0, 0, 1. From whatever we have seen till now, we know that the vectors 1, 0, 0, 0, 1, 0 and 0, 0, 1 form a basis and in this order beta is a coordinate basis of R3.

As pointed out just a couple of minutes back, if we change the order, let beta prime be a new collection or new sequence which is obtained by say interchanging the first and the second vector. So, let this be 0, 1, 0, 1, 0, 0, 0, 0, 1. Then beta as a set, the collection of these three vectors are no different from the collection of vectors which we get from beta. However, we are not just looking at data sets, we also do how an order of the vectors that is being considered in mind and therefore beta prime, then beta prime is a coordinate basis.

It is not just a coordinate basis, it is a coordinate basis which is different from beta, we just changed the order and we still treat it as a different ordered basis. So why are we doing all this? So, this does not seem like a property which we should keep notice of with so much care that is how it might sound in the first site. However, what we will see now, we will illustrate that there is a reason why all this is being done. So, this is a coordinated basis.

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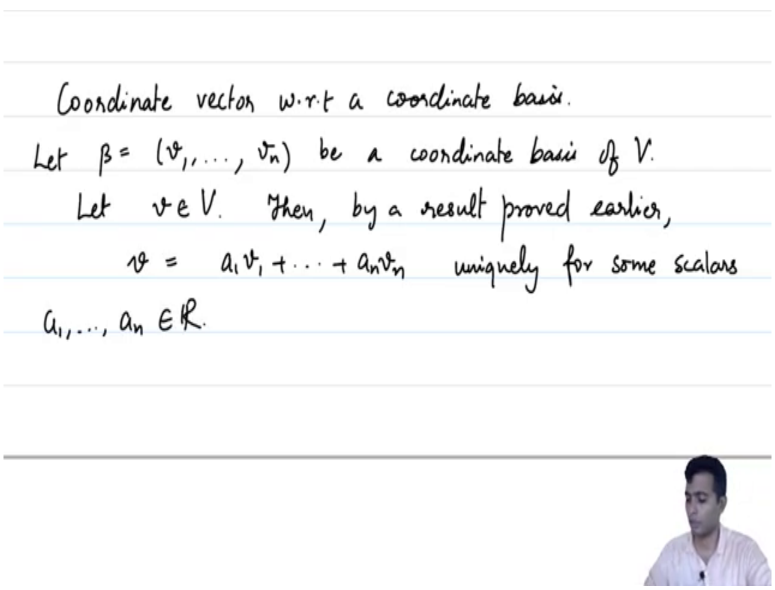

So, let us also know look at what is a coordinate vector corresponding to a with respect to a basis coordinate basis, then we will proceed. Coordinate vector with respect to a coordinate basis. So, let v again the same setup above v be a finite dimensional vector space. Let us say it is of dimension n and let beta be equal to v1 to vn be a ordered basis or a coordinate basis. So ordered basis is an equally popular term in literature of capital V of finite dimensional vector space.

Then, every vector in so let v so the small v vector in capital V and by one of the resellers we have early to that, we know that every vector in capital V can uniquely written as a linear combination of v1, v2 up to vn. Then, by result proved earlier, v can be written as v a1v1 plus dot, dot, dot anvn uniquely for some scalars a1, a2 up to an in R.

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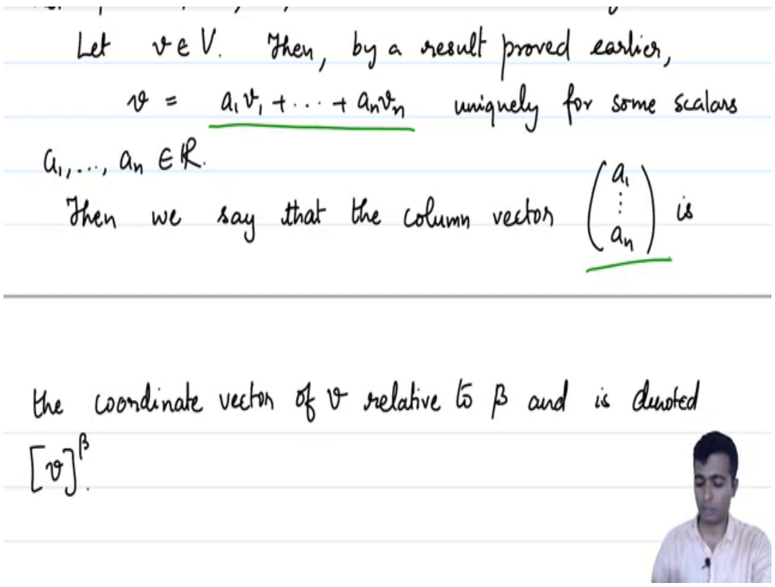

Then, we say that the column vector a1, dot dot dot up to an is the coordinate vector. Coordinate representation of v with respect to our relative to beta and is denoted T sorry, it is not T, v and a superscript beta. So, the moment we have a, ordered basis there is a v1, v2 up to vn in that order, then because it is a basis we will be able to write v as a unique linear combination of v1, v2 up to vn. And let us say this is the linear combination that we have a1v1 plus a2v2 plus up to anvn. In the column vector a1 to an is called the coordinate vector of v relative to the coordinate basis beta and it is notation is this v beta.

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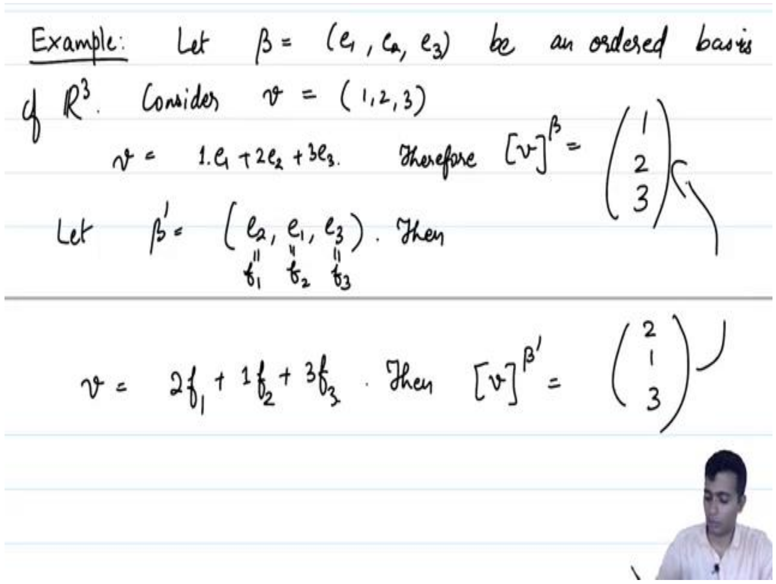

Example, again, let us go back to the example we have been considering. Let Beta be equal to the let me write it as e1, e2, e3 for the standard basis be an ordered basis of R3, recall that e1, e2 e3 are vectors with one in the, say for even it is one in the first coordinate and zeros in the other coordinates ej similarly is always defined as one in j coordinate and zero in the remaining coordinate.

So if it is R3, we are looking at three, three, tuples three vectors are three coordinates rather. So, consider a vector v, let us arbitrarily take a vector v and let us look at the vector 1, 2, 3 simplest vector that we can not the simplest vector but as a very simple vector. What is v in terms of e1, e2, e3? V is then equal to one times e1 plus two times e2 plus three times e3.

This is v, as a unique linear combination of v standard basis, and therefore v beta is represented by one, two and three. Suppose, we change the order, let beta be equal to e2, e1 and e3 like we had done earlier. Then what do we know about v? Then v, so let us call it now this to be f1 let us call this f2 and let us call this f3. Then v is equal to two times f1 plus one times f2 plus three times f3. I have just changed beta to beta prime here because beta was already taken. Then, our coordinator presentation, coordinate vector of v with respect to beta prime will then be given by 2, 1, 3. Note that this and this are not the same. So the columns which represent our given vector with respect to the two basis, two coordinate basis, have not changed and that is the main reason why we are interested in the order. The order is not the only thing that might change.

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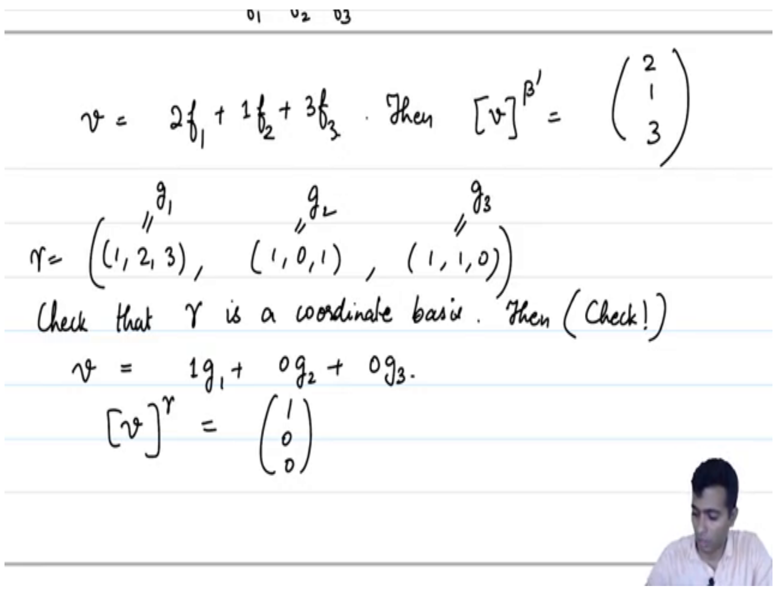

For example, what is the vector does the vector we start with? It was 1, 2, 3. Now let us complete 1, 2, 3 into some basis. It will certainly have a 1,, 0, 1 and 1, 1, 0 these three you can check is a basis for sure in this order. Let us consider gamma to be in this order.

Then check that gamma is a coordinate basis or just a basis, not just a spanning set it is also linearly independent. If it is not needed to check both, because it is a set which contains three elements in a dimension three vectors space and therefore if it is either linearly independent or linearly or spanning set, it will be a basis, because other will be naturally satisfied.

So v can then be written as then again check that v which is our 1, 2, 3 is nothing but, so let us call this ef, let us call g1 this is g2 and let us call the g3. I should I just call it B1,B2 and B3, It does not matter, anyway. This is going to be one times g1 plus zero times g2 plus zero times g3 and therefore the vector v 1 is in terms of this gamma coordinate vector of v relative to the coordinate basis gamma will nothing be, will be nothing but 1, 0, 0. So as you can see based on the choice of a basis, the coordinator vector of the given vector with respect to the basis will turn out to be quite, varying actually quite varying. In this case it was 1, 2, 3 it was 2, 1, 3 in the second case when we picked gamma to be 1, 2, 3, 1, 0, 1 and 1, 1, 0 it turns out be 1, 0, 0, so changes quite a lot. So the order and the basis that is contributed a lot. So let us look at more examples.

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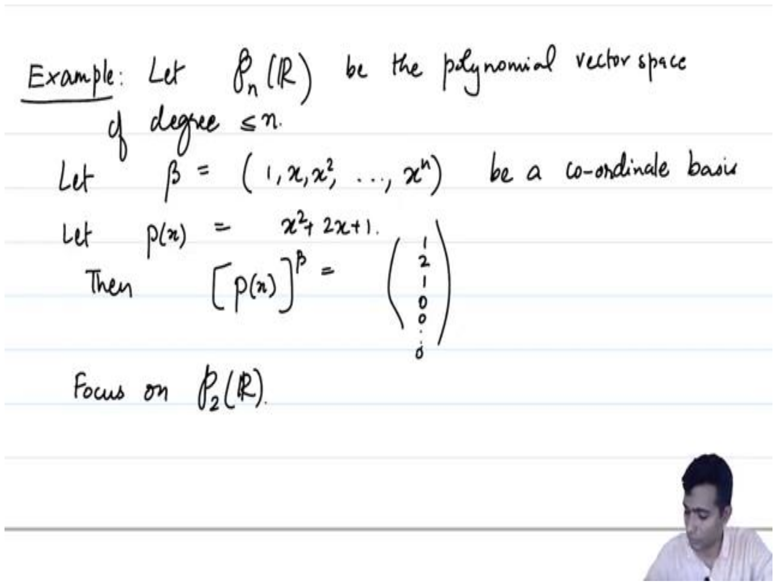

Another example, so coordinate basis is a simple way to have more complicated vector spaces expressed. Like in the case of Rn. So this following example will indicate that so let Pn of R be a polynomial ring of degree less than or equal to n with coefficients in R be the polynomial vector space of degree less than or equal to n and we know a particular basis or this particular vector space. Let beta be 1, x, x square to x to the power n be a coordinate basis in this particular order. If you consider it is a coordinate basis, for coordinate basis of Pn of R. Let p of x be a polynomial.

Let us say this is a x square plus 2x plus 1, then p of x will have a coordinator, representation, coordinate vector of p of x with respect to beta. We will just turn out to be equal to 1 2 1 0 0 dot dot dot 0. The n plus 1 cross 1 vector in a the right here. This is the coordinator plus the coordinate vector of p of x with respect to, but let us just make it a bit more simple. Let us focus on n equal to say 2, focus on P2 of R.

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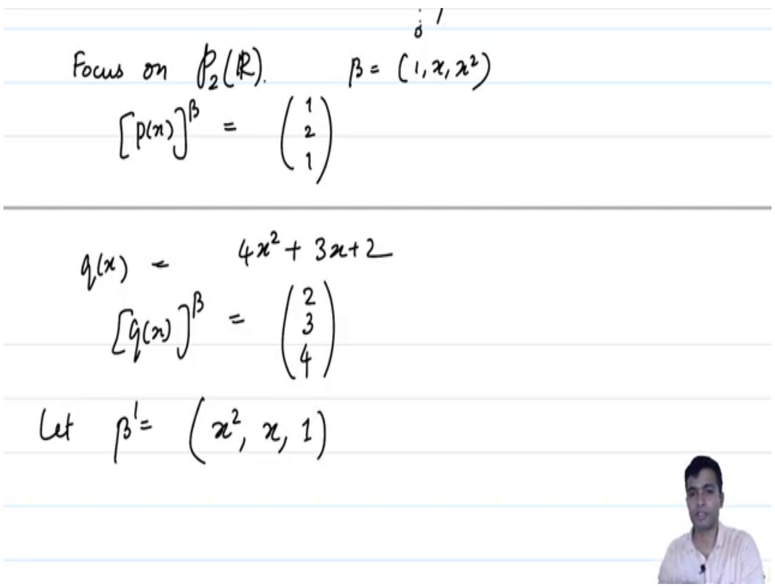

Then, p of x beta will just turn out to be equal to what was of our p of x, x square plus 2x plus 1 and beta, if it is 1x, x a square, this is going to be x square plus 2x plus 1 will be 1 2 1, okay. Let us now consider q of x to be equal to 4x square plus 3x plus 2 some other polynomial. That is a symmetric polynomial.

I do not know how to work with. It is a symmetric looking polynomial, at least with respect to this basis. What is q of x with respect to our beta? This is 2 3 4, you should check this out. Now, beta there is no natural order in which we should be considering these vectors. What if let us beta prime be a polynomials x square, x, 1 in this order. Basically, the reverse order as in beta. This could also have been the order in which we started looking at the basis.

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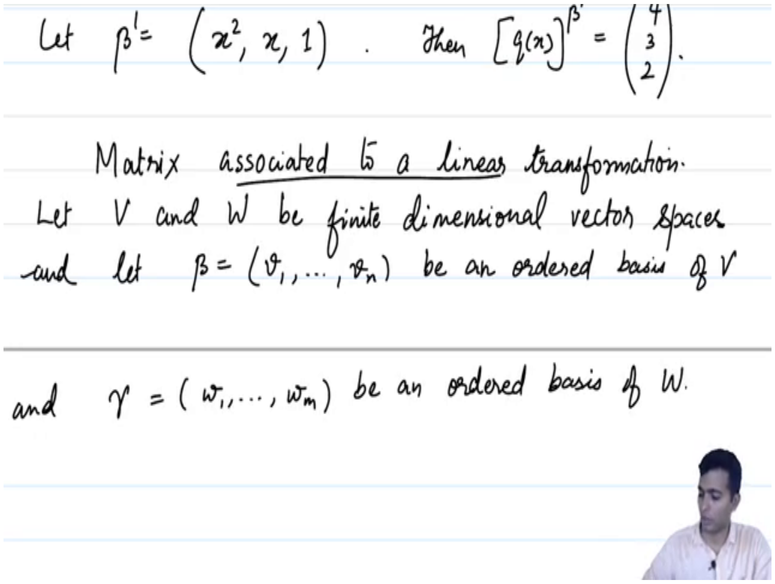

And then what would be q of x with respect to this beta prime the coordinate vector of q of x with respect to beta prime will just turned out to be equal to 4 3 2. So as you can see again, there is also illustrates, this example illustrates two things that p of x can be realized as something like Rn given a coordinate basis. Once you fix a coordinate basis, we can then represent every element or every vector Pn of x as a vector as a coordinate vector which is a column vector.

And the second thing is that again, if you change even the order in which the basis elements are being considered the coordinate basis changes and the entire representation, everything is once we fix a basis, alright. So we have spent some considerable amount of time trying to develop the concept of coordinate basis. Now let us move over to a what is meant by a matrix, which can be associated to a linear transformation.

So metrics associated to a linear transformation.
So let a V and a W be finite dimensional vector spaces. Dimension of V be say equal to N and dimension of W be equal to N and let us also fix some ordered basis here. So, and let beta equal to v1, v2 up to vn be a ordered basis. Let me use the word order basis just for the sake of familiarizing that term that is same as a coordinate basis of V and gamma, which is say w1 to if it m dimensional vector space that will be m vectors in every basis. So, the gamma be an ordered basis of w.

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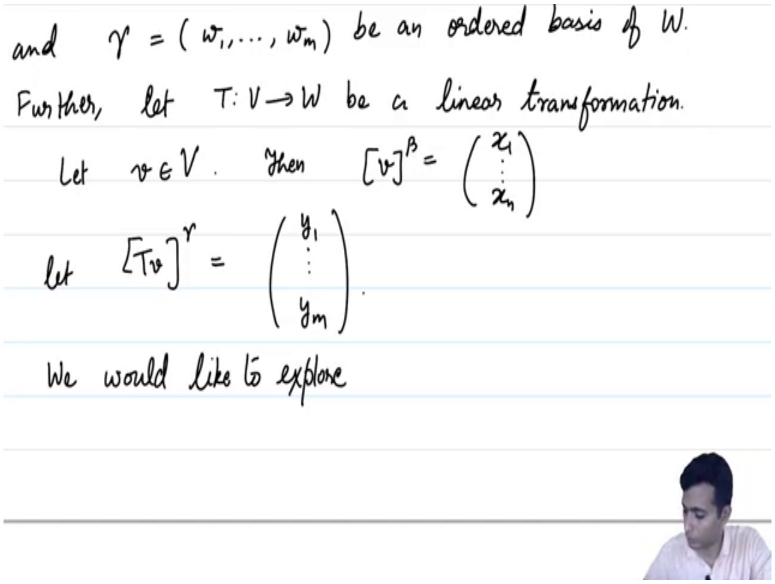

Further, let T be a map from v to w which is a linear transformation. Let T from v to w be a linear transformation. So now, we are curious about what would be the way in which the coordinate vector corresponding to a vector v in capital V and the coordinate vector corresponding to Tv in capital W with respect of gamma, how will they interact with each other? That is the question we would like ask. Let us write down the question more specifically. Let v be an arbitrary vector let fix a vector v in capital V.

Then every vector has a coordinate a presentation. The coordinate vector of v with respect to beta let it be equal to say something like x1 to xn. We in other words this, okay, we will come back to that. This is just telling us that v is x1v1 plus x2v2 plus xnvn. Let Tv with respect to the gamma again, the coordinate vector of Tv. Tv is remember vector in w and gamma is a coordinated basis here, coordinate basis in capital W of m vectors. Then this will be a column vector of size m cross n. So, let this be denoted by say something like y1 to ym. We would like to know how was the column vector x1 to xn related to y1 to ym.

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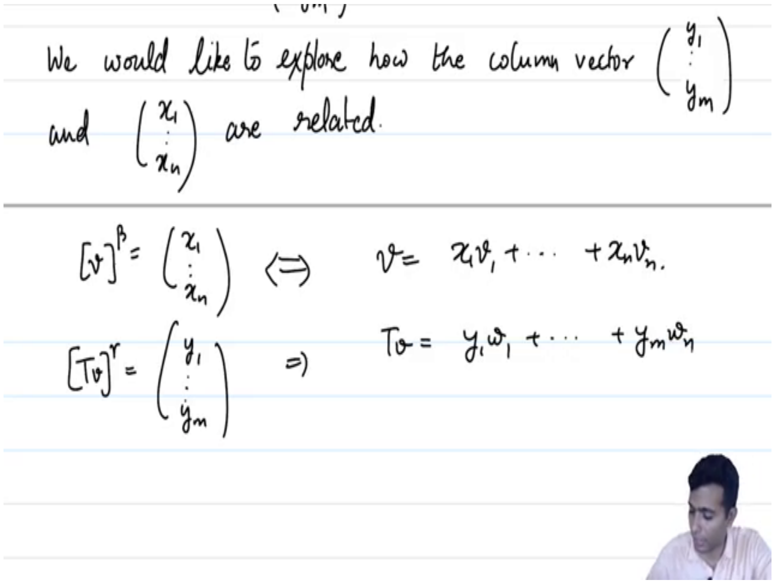

So, we would like to explore, how the column vector y1 to ym and the column vector x1 to xn are related, clearly the relationship has to be through our given linear metric, linear transformation T. Let us look at that again, so v with respect to beta being equal to x1 to xn implies it is statement, v is equal to x1v1 plus x2v2 plus up to xnvn.

We also know that Tv is y1 to ym with respect to gamma and that tells us that Tv is equal to y1w1 plus y2w2 plus up to ymwm, but we know that with respect to a given basis, the represent the linear combination of the basis will be unique to give us the vector Tv. If we can realize Tv as a linear combination of w1 to wm in some other manner, the coefficients have to be equal component wise. So, let us just give the first one a name star.

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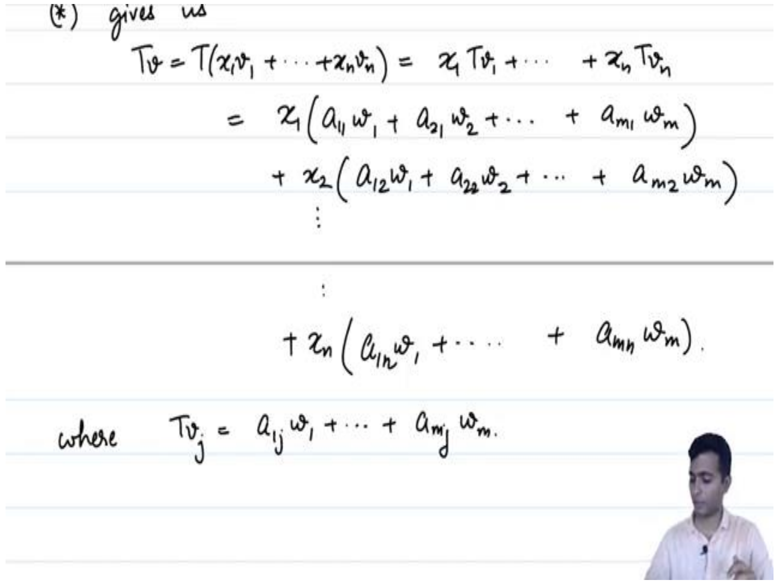

Star when we apply T to star, star gives us the following. Tv is equal to T of the right hand side which is T of x1v1 plus up to xnvn, but recall that T is linear transformation and this will just turn out to be x1 times Tv1 to plus up to xn times Tvn by the vary properties of a linear transformation, but then each of these Tvi are vectors in capital W, so let us just write it down as a linear combination of vector w1 to wm each of them. So, this will just turn out to be equal to x1 times let us write Tv1 in terms of the vectors w1 to wm.

So, this will just turn out to be a11w1 plus a21w2 plus am1wm. Observe that the second index corresponds to v1 and the first index corresponds to the running w1, w2 up to wm. Similarly, Tv2 can be written as a12 remember that this is corresponding to the second index corresponding to the vector v2. So this and the first one corresponding to w1. So this is going to a22w2 plus am2wm.

And x3 can be written, similarly finally xn times the first vector again, remember the index is corresponding to the w1 and the second one corresponding to vn and, so this will be n w1 plus dot dot dot amnwm. This is what? Tv can be, this is how Tv can be written as. So, were let me just write down, Tvj is equal to a1jw1 plus dot dot dot amjwm each of the vj can be written like this. Let me just maintain this because now we can rewrite this.

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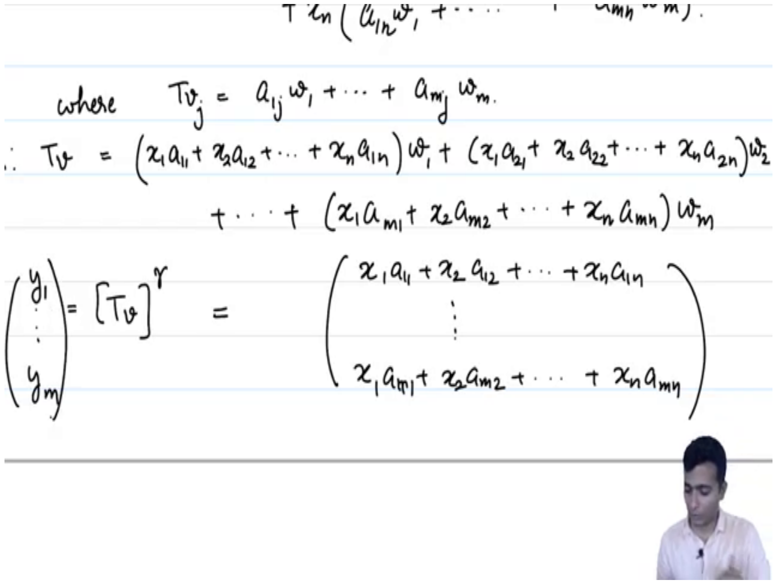

Therefore, Tv if you observed carefully after re-writing this, we will just turn out to be in terms of what would be the combination above in terms of w1, w2 up to wm, If you observe carefully, this is x1a11 plus x2a12 plus xna1n times w1 plus x1a21 plus x2a22 plus xna2n times w2 dot, dot, dot x1am1 plus x2am2 plus xn amn times wm. I had just re-written the above in terms of coefficients of w1, w2 up to wm, this is something which you should check just re-written it down.

And therefore Tv of gamma, gamma remember was w1, w2 up to wm.

This just can be written down as x1 a11 plus x2a12 plus xna1n dot dot dot xnamn, sorry x1a11 plus x2am2, I am making some unnecessary mistakes, this will be m1 plus dot dot dot xnamn and remember that this was nothing but y1, y2 up to ym. There we have the right hand side looks quite cumbersome but if we are to a recall our knowledge of how matrices multiply, we can write this as is this E metrics multiplication of a11, a12, a1n, am1, am2 up to amn.

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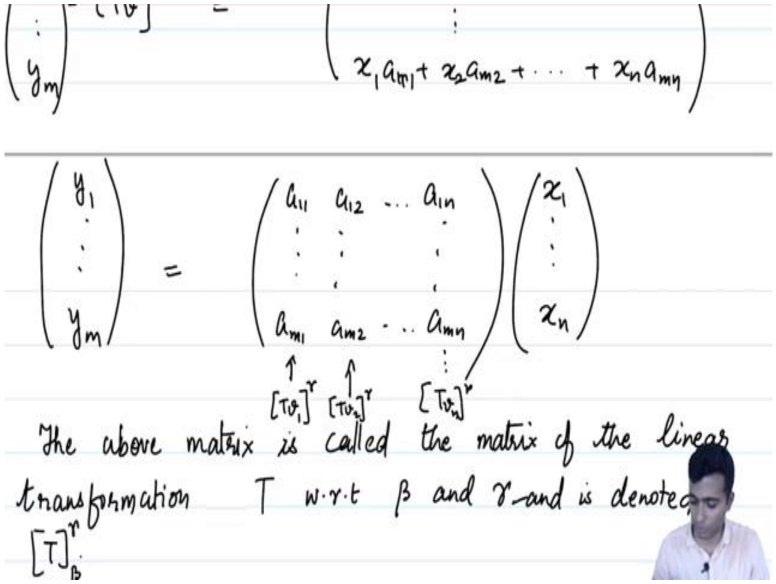

This times x1 upto xn. So this is exactly how the coordinate vector of Tv corresponding to gamma and the coordinate vector of, v corresponding to beta are related. It is related by a metrics multiplication. So, this matrix as you can see are having entries of what are a, if you recall, aij is where the coefficient of Tvi in terms of wj, the image of the vectors in the coordinate basis of v. When written in terms of the coordinate basis of w uses this particular matrix. So we denote, we call the above metrics. Okay, let me think about how to put it in words, okay.

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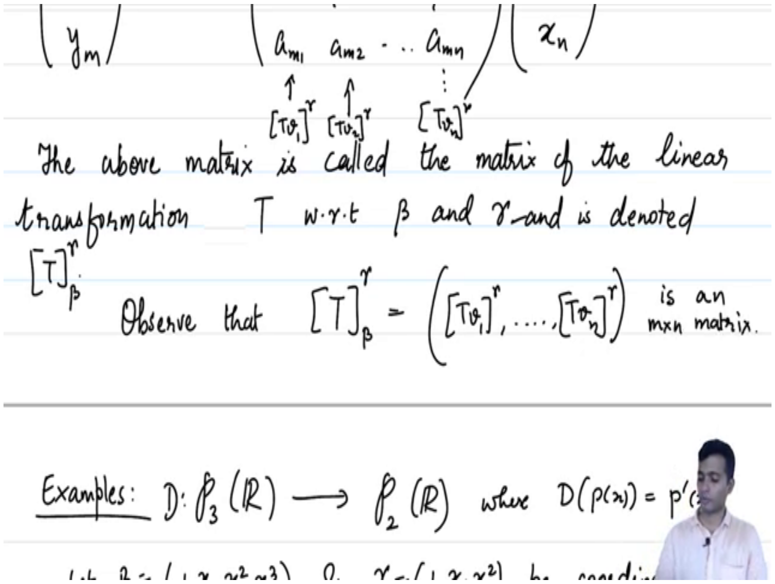

The above matrix, is called the matrix of the linear transformation T with respect to beta and gamma matrix of the linear transformation T. Observe that this matrix very heavily depends on what beta and gamma are, linear transformation T with the respect to beta and gamma. As I just mentioned, each of the coefficient here are obtained by a writing Tvj in terms of w1 to wm for each of Tj from w1 to wm. And this is also denoted T beta gamma.

So, the notation we are following has a name, it is called the Isen Einstein convention, which is used by physicists extensively. Let us not go into that, this is the notation, okay. So, we have defined what the matrix of a linear transformation is when the coordinated basis of the domain and the range of specified, remember that there is this dependence of beta and gamma extensively, this matrix will not make sense if you do not mention what the coordinated basis. So let us look at a couple of examples.

The example about would be, okay, example, it would be a good exercise to go back various examples of linear transformations. Pick your favorite basis on the left and a favorite basis on the right or in other words, a favorite coordinate basis of your domain and a favorite coordinate basis of your range or rather the target vector space and compute the metrics of the linear transformation with respect to the various basis you pick, you will see that the basis can look with the metrics, can look very different.

We will see how they are on related later but right now, let us again look at (())(36:45). One example which I would like to very easily we can show is the differentiation operator, which is from P3 of R2, P2 of R. Let D be a linear transformation where D of p of x is equal to p prime of x. So, what do we know about P3 of R2 and P2 of R. We know that P3 of R is four dimensional P2 of three dimensional, we also have at least the knowledge of one coordinate basis here.

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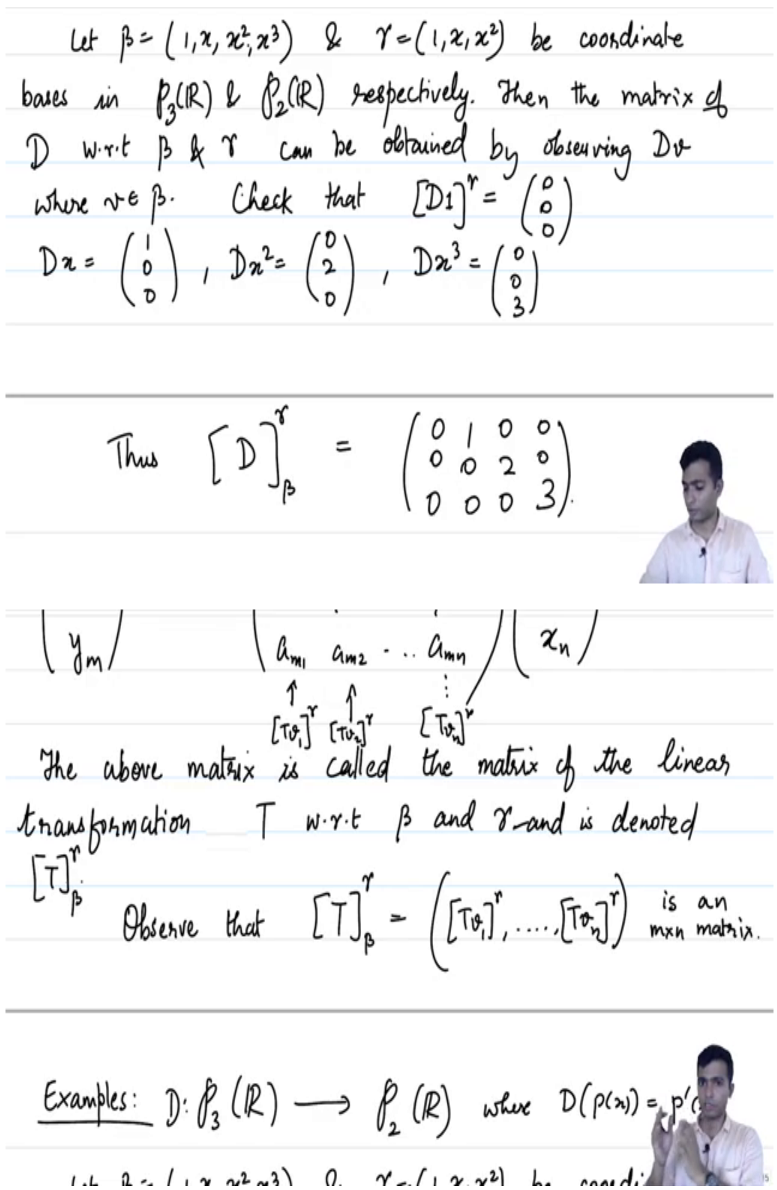

So let us beta denote the coordinate basis here, which is given by 1, x, x square and x cube and gamma. So this is 1, x, x, square. We are looking like similar problem is, but there problem is in different vectors spaces be coordinate basis in P3 of R and P2 of R respectively. So one point I would like to note here is that each of the columns if you observe carefully turns out to be the coordinate vector of Tv. So this is the coordinator vector of the Tv1 with respect to gamma. This turned out to be coordinate vector Tv2 with respect to gamma this turns out to be Tvn with respect to gamma.

So observe here, let me have something here observe that T beta, gamma is nothing but this particular matrix where the columns are, Tv1 with respect to gamma. Tvn with respect to gamma. So this is a m cross n matrix. So the columns turns out to be the coordinate vector of the basis vectors from the coordinate basis of beta of v mapped to the w and written in terms of a coordinate basis in w.

So to wright what the metrics of D, so then the matrix of D with respect to beta and gamma can be a obtained by looking at what is a D of each of the coordinate basis vectors in, sorry, by looking at what is the, what is D of the vector coordinate basis beta by observing D of v where v belongs to beta. So we can check that D1 with respect to gamma is 0 0 0 the constant is always killed by the differentiation Dx is equal to 1 0 0.

D x square is equal to the derivative of x square to x, which is 0 2 0. And what is the D x cube? This is going to be equal to 3x square, which is 0 0 3. Thus, matrix of D with respect to beta gamma will be 000, 100, 020, 003. Let us now check that this representation indeed is working. So, for example, let us take an arbitrary polynomial. In fact, let us take the general polynomial of p3 of R.

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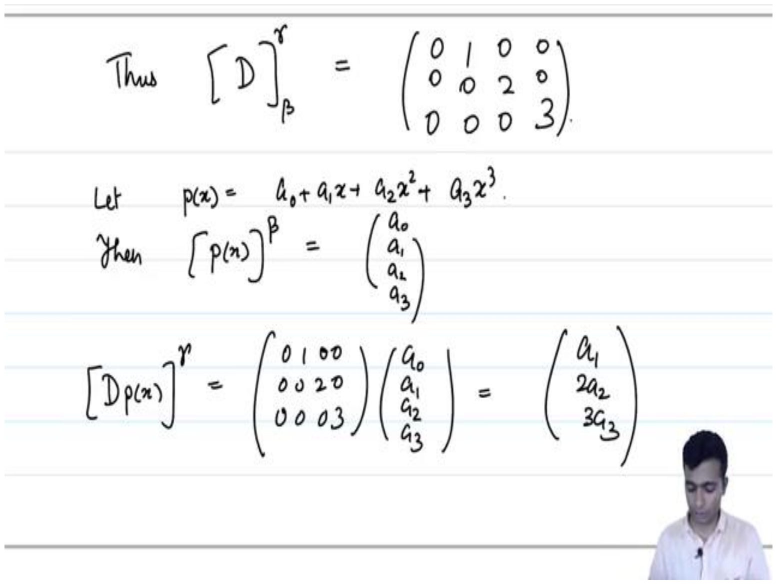

So let p of x be equal to something like say a0 plus a1 x plus a2 x square plus a3 x cube. Then what is p of x with respect to, so where is this? This is an element in p3 of r which has beta as it is coordinate basis which is one x, x square, x cube and this is going to be is a0, a1, a2 and a3. What is going to be D p of x with respect to gamma from whatever we have developed so far.

This is just going to be the matrix multiplication of the matric above this one, with the coordinate vector of p of x with respect to beta. And that will just be equal to 0100, 0020, 0003 times a0, a1, a2 and a3 which is going to be a1, 2a2, and 3a3 as it should be, tt should be a three cross one column vector because we are ending up in p2 of R which is a three dimensional vector space. And by the vary definition, what is this? Let us check whether what we have got is right.

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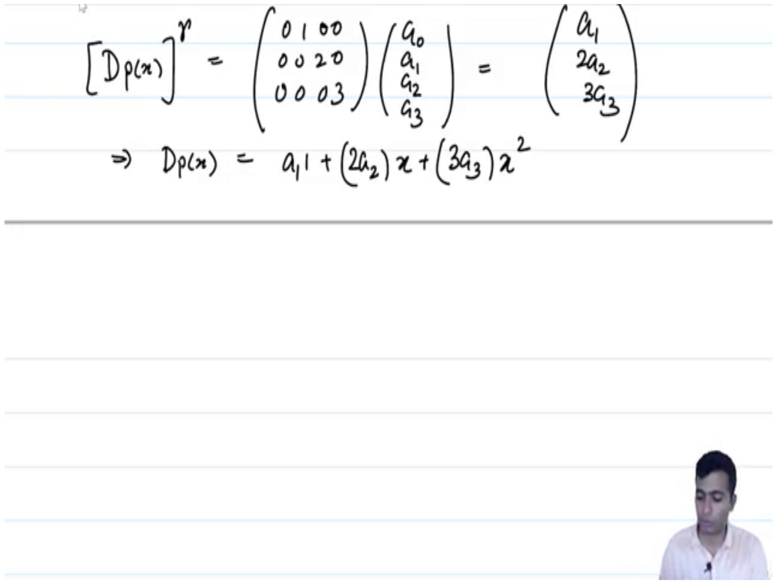

This implies D p of x is equal to this is with respect to gamma, which is a1 times 1 plus 2a2 times x plus 3a3 times x square, which exactly turns out to be our derivative of p of x. What is the derivative of p of x? It just turn of to be a1 plus 2ax square sorry, 2a2x plus 3a3 x square, which is what we have got here.

Yes, so this, that is work so many many times, we linear transformations that we are considering will not be so straightforward and it boils down to looking for a good basis with respect to which matrix will be nice and easy to work with. That will be a major part of this course. So we will come to all that later.

Right now, the thing to keep in mind is that even any linear transformation, if we have a coordinate basis in the domain and in the target vector space between finite dimensional vector space, of course, then we can talk about a metrics corresponding to these coordinate basis and the linear transformation from v to w, will just turn out to be the matrix multiplication of this matrix with the coordinate vector representing our given vector v with respective corresponding basis to get back the coordinate vector of Tv with respect to the basis of the beta vector space.