# Linear Algebra
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**Lecture 3.3**
**Linear Transformation and Basis**

In the last video, we saw what the rank of a linear transformation is, what a nullity of a linear transformation is and how they are related through the dimension theorem. In the proof of the dimension theorem, we explored how the linear transformation helps in describing the relationship of vectors in w in terms of vectors in v and vice versa.

So, there were a couple of arguments which were inbuilt into the proof of the dimension theorem, which we would like to extract and make it into a theorem here. So, let me start this lecture with a proposition which was actually covered once in the proof of the dimension theorem but nevertheless, it is a very good exercise to make it precise. So, let us look at a proposition to begin with he. So, mostly this lecture will deal with how a linear transformation interacts with the basis.

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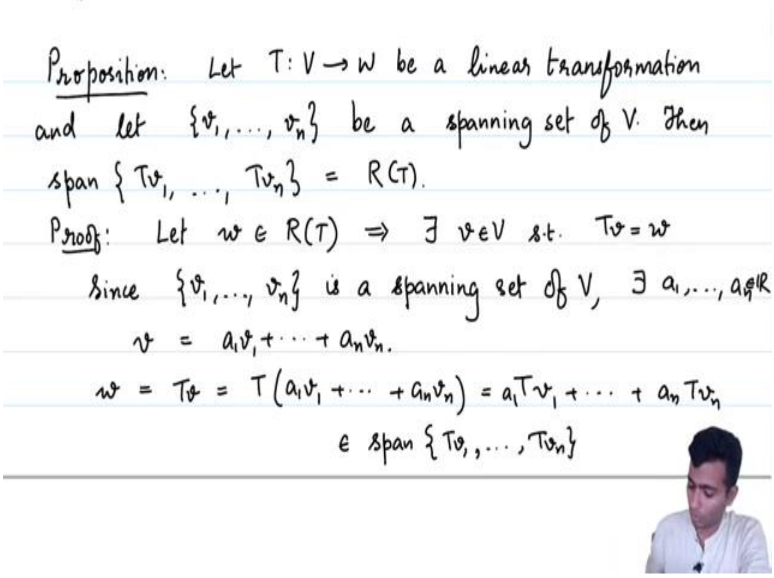


Let us see what is coming up. So, a proposition to begin with, so let T from V to W, be a linear transformation and let v1 to vn be a spanning set of V. The proposition says that the image of v1, v2, up to vn is a spanning set of the range of T. Then span of Tv1 to Tvn is equal to R of T. We had given a proof of this in the course of the argument restored, described in the course on proof of dimension theorem, let us single that out.

So, let us give a quick proof, what do we have to show to establish that span of Tv1 to Tvn is equal to R of T. We have to take an arbitrary element in R of T and show that it is in the span. So, let us start with an arbitrary element. Let w be an element in R of T, what does it mean to say that an element is in the vector is in the range of T? This means there X is a vector of v in capital V such that Tv is equal to w, but what do we know about v1 to vn? v1 to vn is a spanning set of V.

So, since v1 to vn is a spanning set of V, we can write any vector of V as a linear combination of v1 to vn in particular, we can write v as a linear combination of, there existt real coefficients a1, a2 up to an such that v is a1v1 plus a2v2 plus up to anvn, so our vector V is a linear combination of v1 to vn, not necessarily a unique linear combination because we do not know anything about linear independence of v1, v2 up to vn to begin with, but we certainly have at least one linear combination in this manner.

Now, consider T of v, what do we know about T? T is a Linear transformation. So T of v, which is T of a1v1 plus up to anvn by using an induction argument is equal to a1 times Tv1. We have already discussed this once in the proof of dimension theorem this will turn out to be equal to a1Tv1 plus a2Tv2 plus up to anTvn, which is an element of the span of Tv1 to Tvn but let us be a little more careful, what is Tv? Tv is nothing but our w as you can see, the choice of V was exclusively to see that you can see I am putting an underline here in green, the very choice of v was such that Tv  is equal to w.

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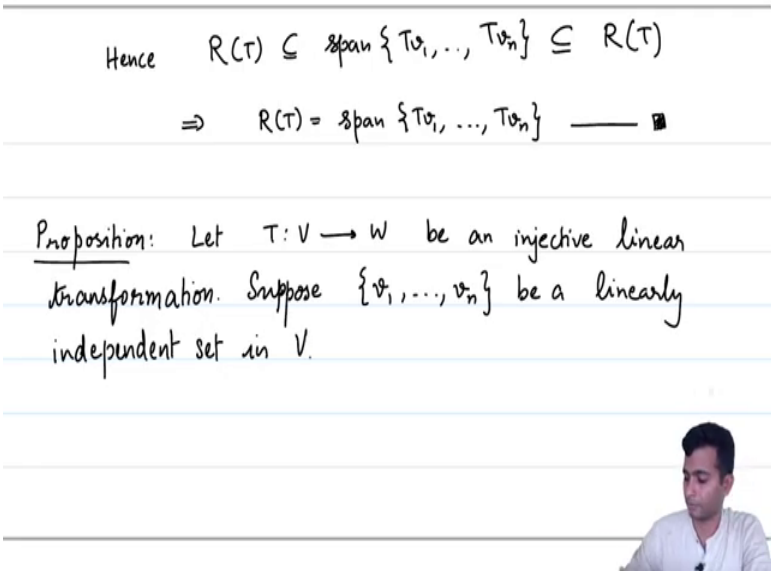

So what we have established now is that w is in this span of T1, Tv2 up to Tvn, but our choice of w was arbitrary. Hence, what we have established is that hence R of T is contain in the span of Tv1 up to Tvn, but Tv1, Tv2 up to Tvn each of these vectors are in w and a span of Tv1, Tv2 up to Tvn is the smallest subspace that contains these vectors. This was one of the reasons we proved in one of the videos earlier.

And therefore, any vector subspace which contains these vectors should also contain this span particular, this is also contained in w, sorry, in R of T, yes. So, earlier I said w but what I meant is Tv1, Tv2 up to Tvn are vectors in R of T,\. R of T is a subspace as we know and because span of Tv1 to Tvn is a smallest subspace which contains these vectors, these is contained in v subspace R of T, and this implies that R of T is equal to span of Tv1 to Tvn as first be established.

So, as you can see, this was an argument which we did use in the proof of the dimension theorem, but it is worthwhile to keep a proposition exclusively to capture this, it is a important proposition. We will prove one more similar statement, and a this is going to be a little different, but the style of the argument was similar or is similar to what was done in the dimension theorem. So, let T from V to W be an injective linear transformation. Suppose we start off with a linearly independence set, suppose v1 to vn be a linearly independent set in V. Then, as you should be guessing by now.

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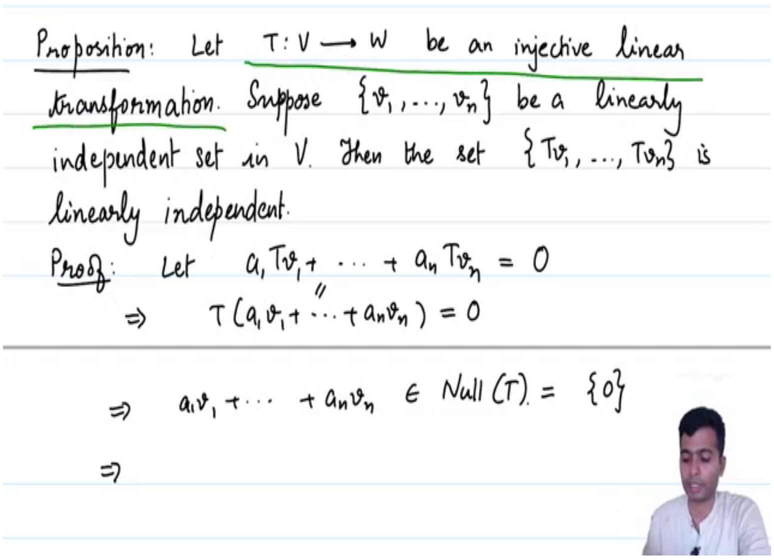


The set, conclusion is that the set Tv1 to Tvn is linearly independent. Let us have a quick look at the proof and you will see the similarities. So, to check that something is linearly independent, what do we do? We take a linear combination which is equal to the zero vector and we then check whether the coefficients are forced to be zero.

So let us start off with a linear combination. Let a1Tv1 plus dot dot dot plus an Tvn be equal to the zero vector of W. Remember that Tv1, Tv2 up to Tvn are vectors in w, so this is a linear combination in w. By the properties of a linear transformation, we can write which gives T of a1v1 plus a2v2 plus anvn is equal to the zero vector, why is that the case? Because this is equal to this, the vector a1Tv1 plus up to anTvn is equal to T of a1v1 plus up to anvn.

We just use the properties of the two properties of a linear transformation to establish that they are equal and this is equal to zero vector of w. But what does it mean to say that T sense vector to zero vector? It means that the vector is the is an element of the null space of T. So, this implies a1v1 plus anvn belongs to the null space of T, but in the previous video we saw a proposition which a stated and we proved the proposition which said that T is a, an injective linear transformation if and only if the null space of T is the zero vector space. So, this in particular is just the zero vector space.

So, this implies why is this is a zero vector space? Remember that our proposition has the assumption that this is then injective linear transformation and that is precisely what we will be using here. Null space of T is hence zero and this implies that this vector is equal to the only vector in the null space of T which is zero vector, but now let me bring your attention back to our hypothesis, which I am now again underling in green. We have start up with the hypotheses at v1, v2 up to vn is an independent set, linearly independent set.

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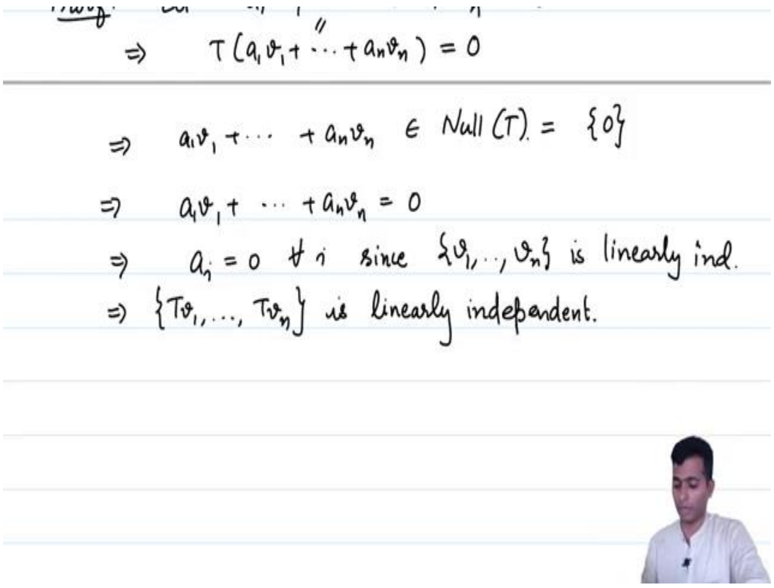


And therefore, this forces these coefficients to be equal to 0 ai is equal to 0 for all i  since v1 to vn is linearly independent and that is precisely what we had set out to prove, remember we started off with a linear combination of Tv1, Tv2 up to Tvn which is equal to the zero vector and we are now established and that are the coefficients are zero. Tv1 up to Tvn is linearly independent.

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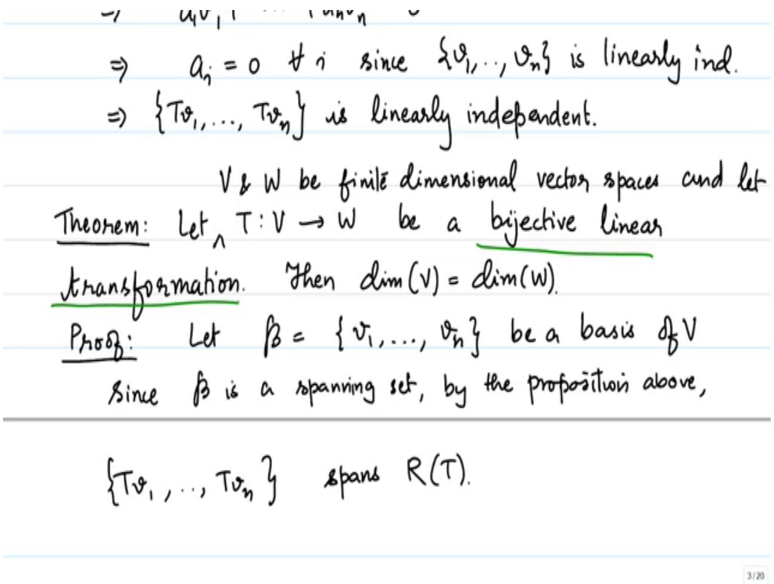


A combination of these two propositions gives this something quite nice. So, let us prove now one more theorem. So, let T from W to W be a bijective linear transformation. Then, dimension of V is equal to the dimension of W. So bijective if you recall is a function which is both injective and surjective at the same time. So, if you have a linear transformation which is both one to one and on to injective or surjective, then dimension should necessarily be equal.

So, let us give a quick proof of this, it is a remarkable statement because the moment we have a bijection, we have that the dimensions are preserved.  So, let us start with I should have added one, assumption here, so let us start off with that V and W be finite dimensional vector spaces. So, this is an assumption which I should be adding because we are not assuming that our vector spaces are finite dimensional, but this theorem we should be careful in stating it because this is a theorem for finite dimensional vector spaces and let T from V to W be bijective linear transformation.

So, since v is finite dimensional let the dimension of V, be equal to small n and let us pick a basis. So, let B equal to v1 up to vn be a basis of V. What did the first proposition tell us? The first proposition told us that, if we have a spanning set v1, v2 up to vn, then Tv1, Tv2 up to Tvn is a spanning set of R of T. Since B is a spanning set by the proposition above, the first proposition above which we just proved Tv1 to Tvn spans R of T, but what was the assumption on our T? T was assumed to be a bijective linear transformation. So, in particular our T is a surjective , it is on to w.

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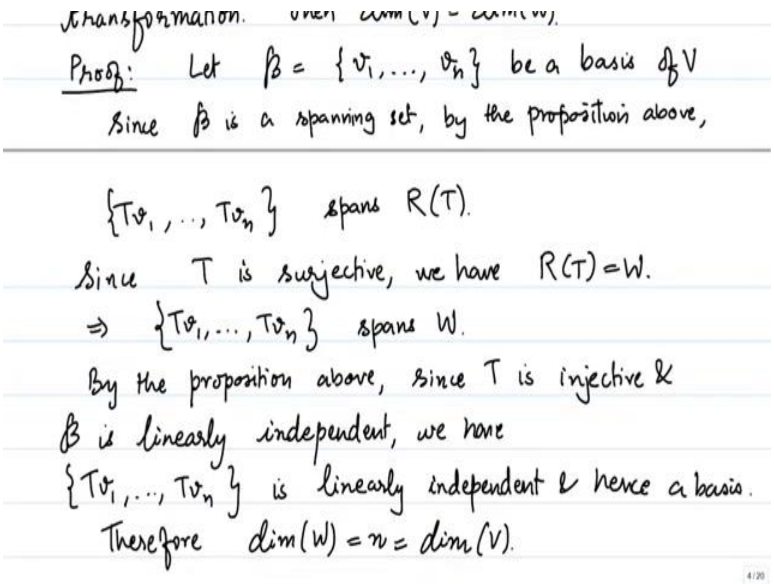


What is the meaning of map being surjective, a linear transformation being surjective? It means that the range of P is equal to w. So since, R of T, since T is surjective we have R of T is equal to W. Now, R of T is equal to W means that Tv1 to Tvn spans W . So we have established one aspect of Tv1, Tv2 up to Tvn being a basis of W. There is something else which is to be checked for to be a basis, namely linear independence.

But T is also an injective linear map and by the first, by the second proposition, sorry, by the proposition second proposition above, since T is injective and B is linearly independent, we have Tv1 to Tvn is linearly independent and hence a basis. Therefore, dimension of W is equal to n which is exactly same has the dimension of V.

So, I would like to bring your attention to another aspect in the proof of this statement. Not only have we established that the dimension of v is equal to the dimension w, but rather we have also shown that a basis is always mapped to a basis, if you have a bijective linear transformation. So, the converse of this statement is also true. So what does the potential converse? The converse says that if we start off with a linearly independent set Tv1, Tv2 up to Tvn which spans R of T actually which spans w then T, then v1, v2 up to vn should necessarily be a basis.

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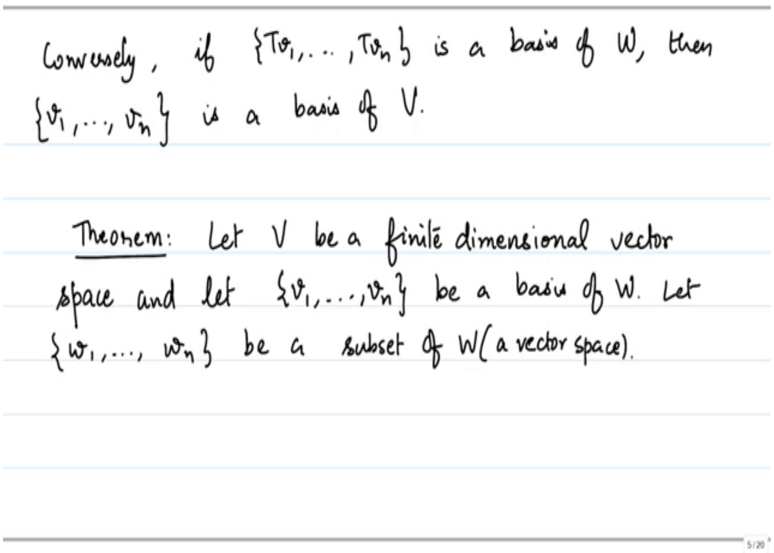


So just, I will just note that as a converse is quite straightforward, conversely, if Tv1 to Tvn is a basis of w, then v1 to vn is a basis of v. Typically, when we want to define a function from a set to another set, we have to deal with what the value of the function is at every point of the domain. So, if f is from say x to y to describe f, we will have to talk about f of x, for every x in capital X. What we will now do is that, in the case of linear transformation, we will see that that is not necessarily needed. In fact, that is not needed at all, we have to only specify the value of T at a basis, and that uniquely fixes what will be the linear transformation will be the entire vector space.

So, it many times reduces our effort to bothering about what our linear transformation is on many times a finite set if the vector space is finite dimensional. So, let us look at the statement to the theorem more carefully. So, let me state it down as a theorem. So let V be a finite dimensional vector space, and therefore it will have some dimension, say n and a let v1, okay. So let T be a linear transformation,  no, no. I want to say that the linear transformation can be defined at a basis.

And we can talk about the transformation being defined on me. So, let me fix a basis and let the v1 to vn be a basis of v. Let w1, w2 up to wn be a subset of capital w another vector space. So, I will not write down that w is a, so let us we do not need w to be  finite dimensional that is why I did not write down,  and w a vector space, okay. So, I will just write it out in bracket a vector space. What is the theorem? The theorem states that, then there exists a unique linear transformation from v to w such that Tvi is equal to wi.

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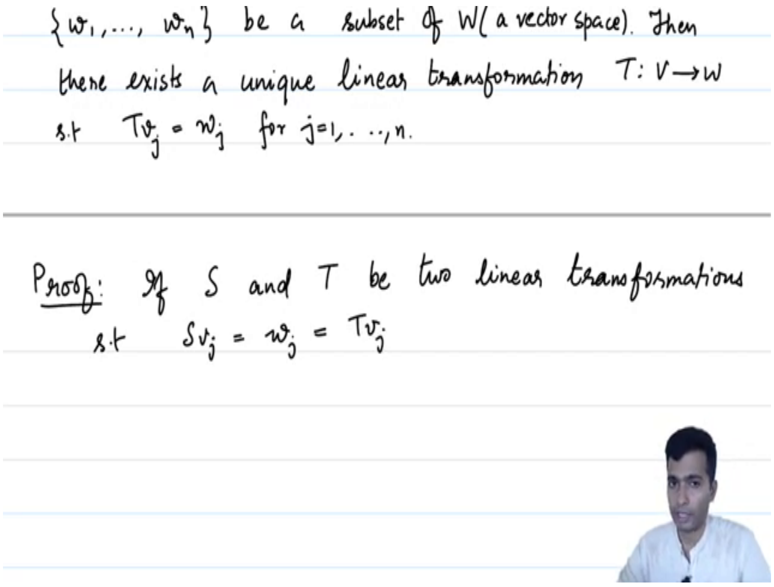


Then there exists a unique linear transformation T from V to W, such that Tvj is equal to wj for j is equal to 1 to n, okay. Let us give a proof of this theorem. So, there are two things to be shown. First being that, okay, not necessarily in that order, but we will do it in this order. First we will show that the linear transformation is unique if at all it exists. Then we will show that their exist is at least one such linear transformation.

So, let us look at if a T let s and T let us say s and t be two linear transformations. Of course, we do not even know whether there is even one linear transformation. We will come to that in a moment, but before that, let us look at the hypothetical situation where we take hold of two different linear transformations, which a map vj to wj and then we will show that S and T should necessarily be the same.

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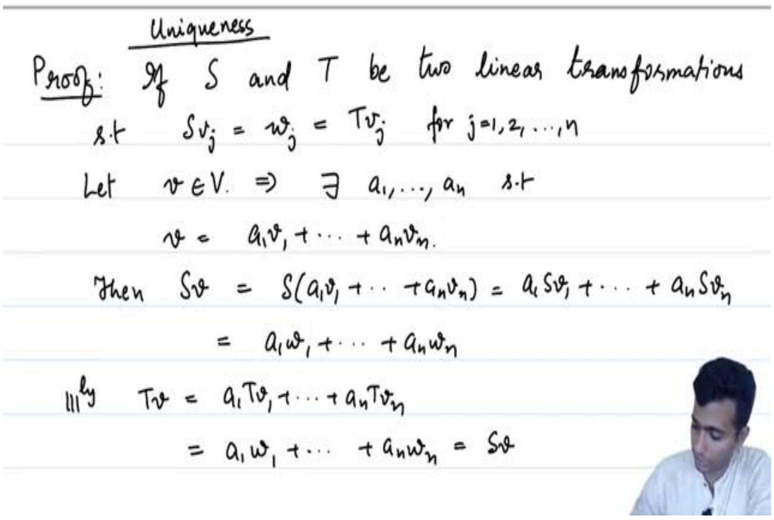


So let us T be two linear transformations such that S of vj is equal to wj which is equal to T of v. We will show that for every vector v in capital V as of v is equal to T of v. So let v be an arbitrary factor in capital v. What do we know about v1, v2 up to vn? We know that v1, v2 up to vn, is a basis of capital V. We just recall that part for you, it is in the hypothesis, so basis of, okay, so please make a correction here.

Good, that I came back, so this is not w this is v, so v1, v2 up to vn in the basis of v, and therefore we can write a vector v as a unique linear combination of v1, v2 up to vn. This implies that v there exist a1 to an in fact uniquely their exists a1 to an, such that v is equal to a1v1 plus anvn. What is going to be S of v? Then Sv is nothing but S of a1v1 plus anvn which is equal to a1sv1 plus ansvn but our assumption here which now I am again underlining in red tells us that for j equal to 1 to n, S of vj equal to wj.

Let me write here for j equal to 1 to n and that implies that this is equal to a1w1 plus up to anwn. Now let us try to see what will be T of v. Similarly, T of v will also be a1Tv1 plus up to anTvn which again by what I am, by this part of the statement is will be equal to a1w1 plus up to an wn, which is exactly this expression for S of v. And therefore, if at all their excess linear transformations which send vj to wj, then it should necessarily be a unique linear transformation that cannot be two such linear transformations. So, we have established uniqueness here, now let us establish existence.

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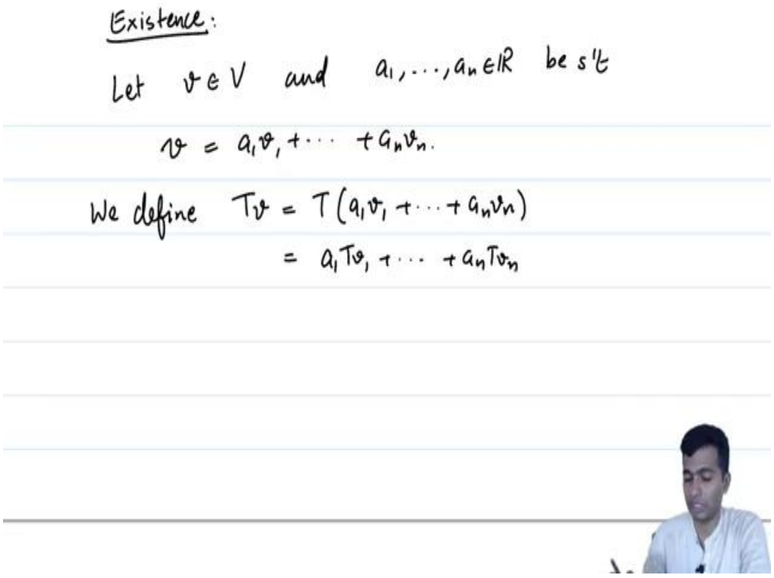


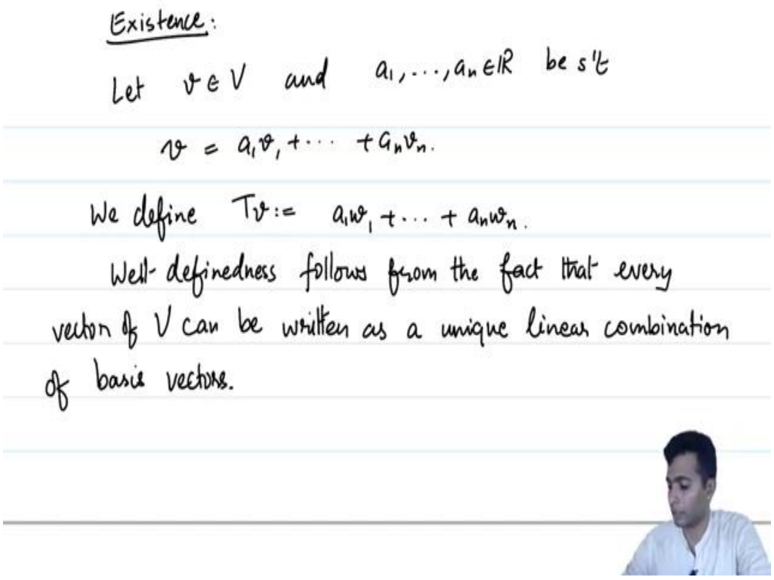


Let us bother about existence. Well we know what a T should be on each of the basis vectors. So, given a vector v in capital V by a theorem, we have proved earlier there exists a unique linear combination of v in terms of v1, v2 up to vn. So, let v be a vector in capital V and a1 to an in R be such that v is equal to a1v1 plus up to anvn. Now, we define T of v to be equal to, what should be T of v if at it is a linear map , we know how it is going to behave on the right hand side T of.

This will be exactly this, and we know that linearity forces it to be a1Tv1 plus anTvn and we know that vj is being sent to wj. So let me rub all these things and write down directly what our definition should be. Our definition should hence be a1w1 plus up to anwn many, many questions will come up. First question, namely first question, meaning whether T is a well defined map at all. Answer is yes, because of the theorem we proved earlier, namely that if v1, v2 up to vn is a basis of a given vector space v.

Then every has a unique linear combination of every vector has a unique expression as a linear combination of a basis vectors. So, in particular the choice of a1, a2 up to an. Therefore, this map is a well defined map, well defined as a clear from one of the theorems before, but that is not enough we have to check. Let me just write well definedness follows from the fact that every vector of V can be written as a unique linear combination of basis vectors.

Next however, we have to check that T is a linear transformation. So, let us just a quickly check that so, let v1, v2 be a vectors in capital V, we would like to check that T of v1 plus v2 is Tv1 plus Tv2.

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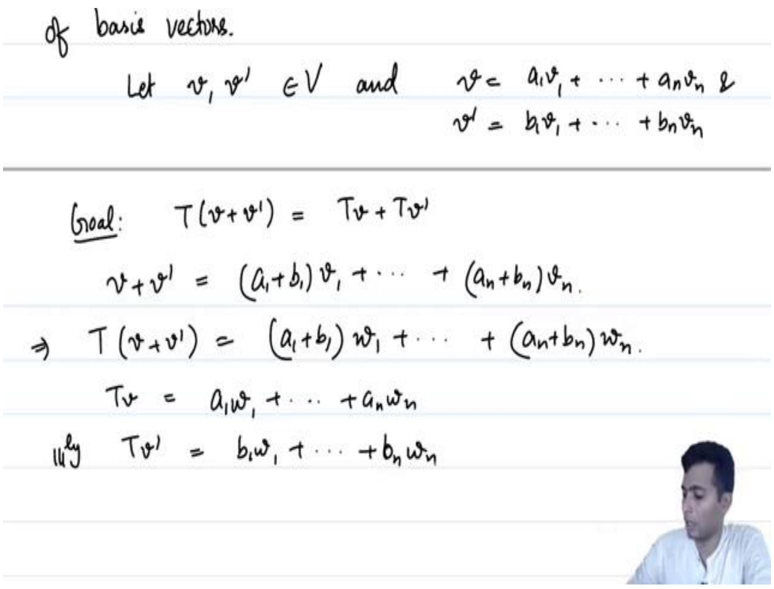


And let us give some expressions for this. v, v prime works well, so let v be equal to a1v1 plus up to anvn and v prime be equal to b1v1 plus up to bnvn. So, we would like to check that goal, let us just write it down as goal we would like to establish that T of v plus v prime is equal to Tv plus Tv prime. Let us look at the left hand side here. What is T of v plus v prime to ,before we do that, what is v plus v prime? Let us find out what is v plus v prime, this you should check is equal to a1 plus b1, v1 plus up to an plus bn, vn and therefore T of v plus v prime after all the manipulations will be a1 plus b1 times w1 plus up to an plus bn times wn by the very definition of our map T. Now, let us look at what is Tv plus Tv prime. So, just to see what Tv looks like, this is going to be T of a1v1 plus a2v2 plus up to anvn let me just quickly write it down this is just going to be a1w1 plus up to anwn. Similarly, T of v prime is b1w1 plus up to bnwn.

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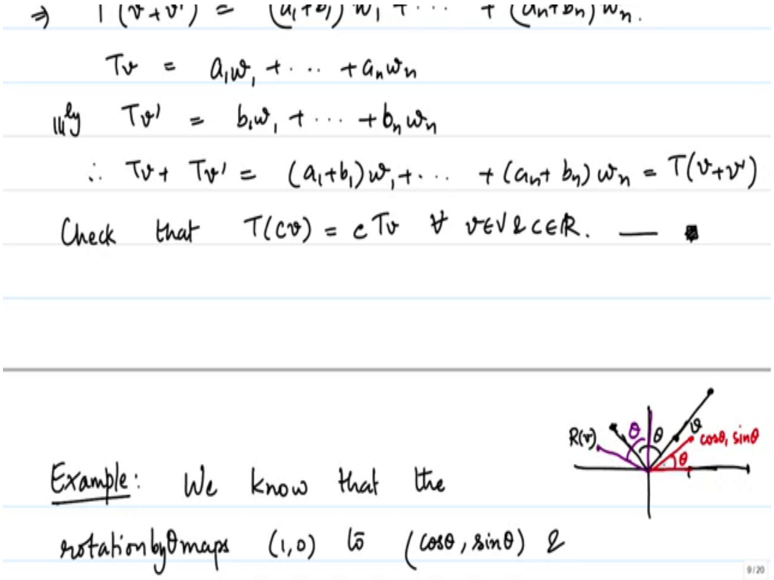

And therefore, Tv plus Tv prime after all the manipulation, this is going to be a1 plus b1, w1 plus up to an plus bn, wn. I am not going to write down the which is equal to T of v plus v prime. I am not going to write down the proof for scalar multiplication, the corresponding argument, which I will leave as an exercise,check that T of cv is equal to c times T of v where all v in capital V and c in R, so I will leave that as an exercise, but with this we have found out a well defined, unique linear transformation T from v to w, which maps vj to wj and that completes the proof.

So, maybe one more example is in order, so let us look at an example, from your knowledge of geometry you will be knowing what a rotation map is, right? So you take a vector v and a rotation by theta in the clockwise direction will be something like this, this is what rotation. So, if this is v this vector is what is going to be R of v, rotation around the origin. So, we would like to define this map very rigorously, rigorously, certainly yes, but observe that the map will certainly be a linear transformation.

Because if you add two vectors, v1 plus v2, this is say v1 and if you add v2, v2 will be like this, the corresponding vector which gets added will also be v1 plus v2, and similarly with the dilation. So, R which is a rotation will be, we know what it is, a geometry, let us know, try to make a precise definition of the rotation map in a concrete manner. So, the theorem which we just proved tells us that we do not need to bother defining our R on every vector.

If we know what it is on basis vectors, we can extend it to a definition on the entire vector space on here, in this case it is R2. So the easiest thing which we can think of where we should be able to talk about the rotation is on the X axis, the unit vector on the X axis and on the Y axis. So we know that the rotation of a the unit vector along the X axis is taken to what is this, this is say, so maybe I should draw with red here.

This is the vector that is being sent to say this is why just two lines say theta and we know that this point is going to be cos theta, sin theta. Basic trigonometry tells us that is the case. So we know that the rotation maps vector 1, 0 to cos theta rotation by theta, cos theta, comma sin theta. And we also know what it maps 0, 1, 2 one zero was mapped to cos theta, sin theta we also know what it maps 0, 1 to 0, 1 to.

So let us see this is say theta, sorry, this is our 0, 1 it gets mapped to something like this, and this is just going to be pi by 2 plus theta cos pu by 2 plus theta , which is minus sin theta and sin pi by 2 plus theta which is cos theta.

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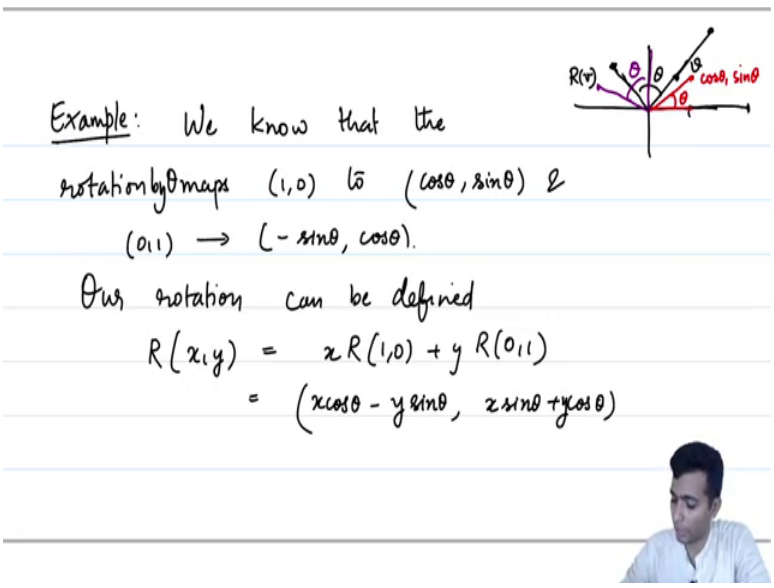


So you know that 0, 1 is mapped to minus of sin theta and cos theta. So our rotation on arbitrary vector x, y will be the following and be defined as by our previous theorem, R of x, y this is going to be x times R of 1, 0 plus y times R of 0, 1 and this is going to be x cos theta minus y sin theta,   x sin theta plus cos theta not cos theta it is y cos theta.

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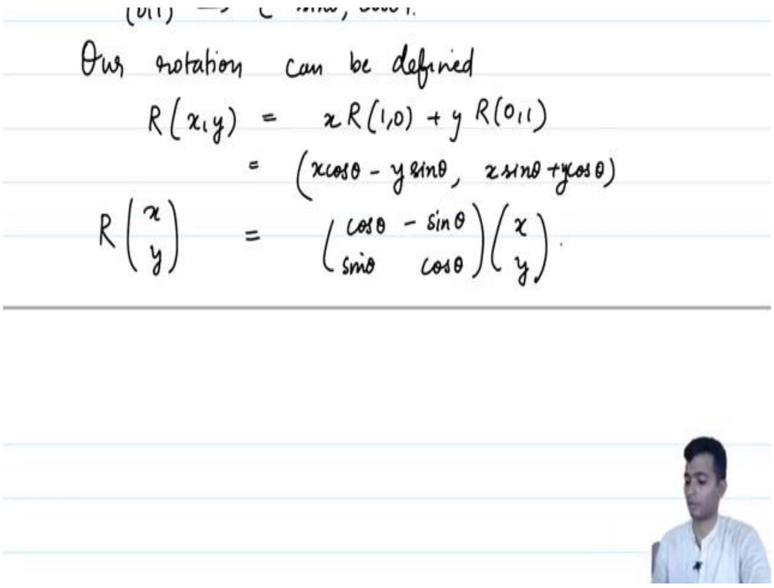


And if we are to write it in terms of row and sorry, column vectors x,  y is being sent to cos theta minus sin theta, sin theta, cos theta times be vector x,  y. So as you can see, I am somehow trying to get hold of some metrics to represent our linear transformation R. And we will very soon see that this can be speed up. All right.