Linear Algebra
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Lecture- 3.2
Rank Nullity

So, now let us define the next notion of what is called as the null space associated to a linear transformation.

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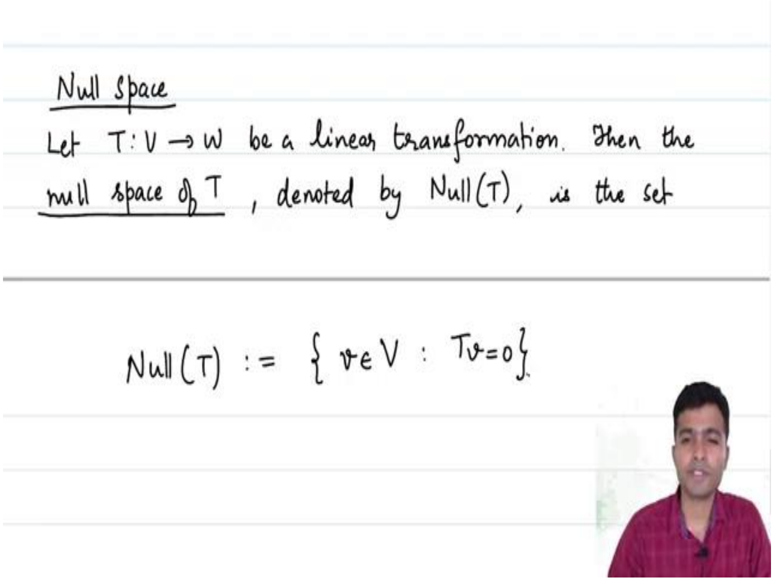

So this is our definition of a null space. So, let T from V to W be a linear transformation, then the null space of T, so let me just underline what is being defined. The null space of T denoted as denoted by null of T is the set, so let us see what the set is, null of T is defined to be the set of all v in capital V such that it is a subset of V in particular such that Tv is equal to 0.

So, the null space of T is the collection all those vectors which are killed by our linear transformation T or in other words, it is the collection of all those vectors in V which are mapped to the 0 vector of W by T.

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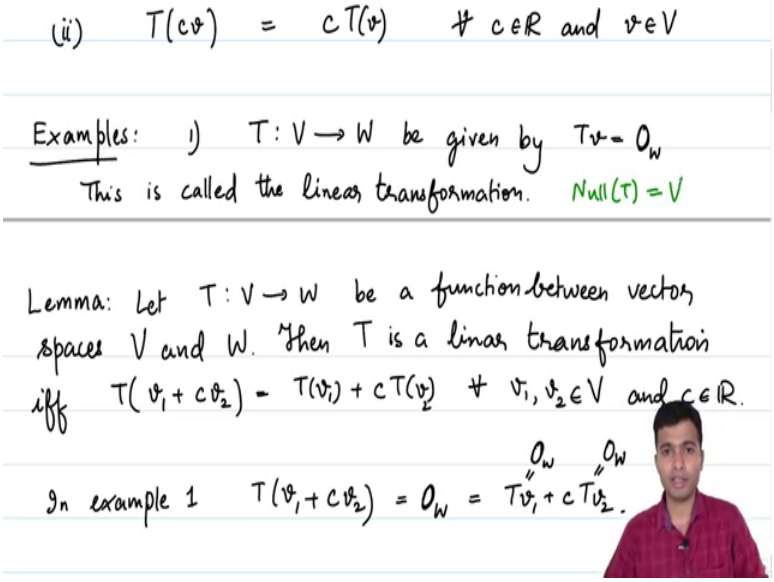

So, let us just scroll up look at our examples and see if we can say anything about the null space. So, what would be the null space here, so let me just mark it in green, so here the null space is the null of T will be the entire vector space V every vector is being saying to be a 0 vector here so, it is the entire vector space.

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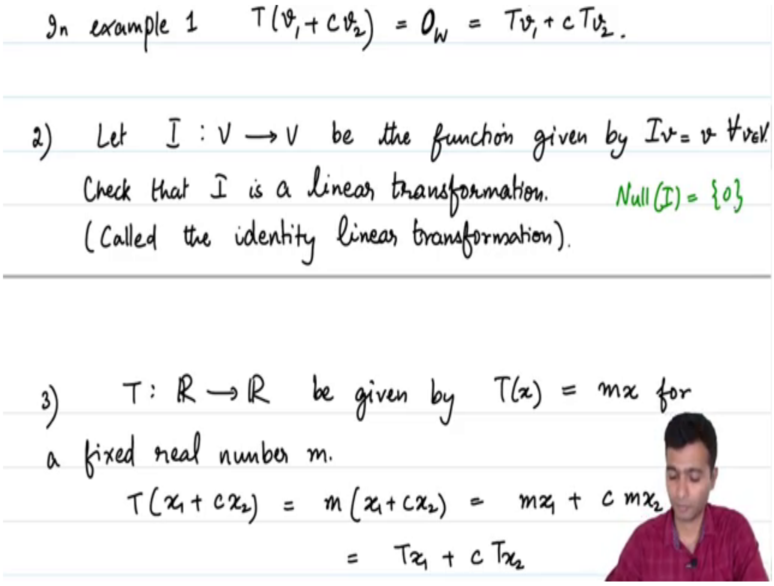

What would be the null space in the case of the identity map? So let us see. It is the set of all vectors v in capital V such that Iv is equal to 0, but Iv is equal to v so this forces v to be equal to 0. So, the null space here is just the 0 space, the 0 set.

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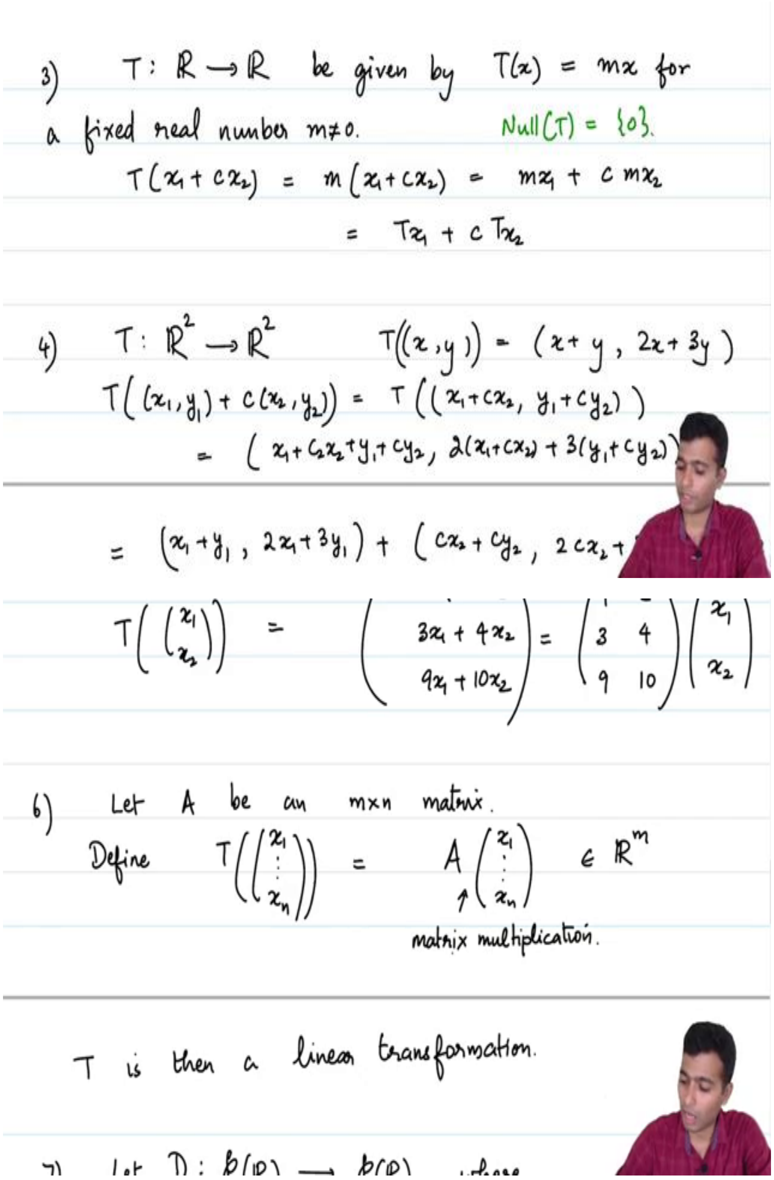

What would be the null space here? Null space will again be the collection of all elements x such that, m times x is equal to 0, but, if m times x is equal to 0 because m is a non-zero okay m not equal to 0, let us put this condition here. Because if m is equal to 0 it will just be our zero transformation-linear transformation where we have already seen that examples.

So, let us only look at non-zero linear transformations and this example. If m is non-zero, mx is equal to 0 forces x to be 0. So, here null of T will now again turned out to be just the 0 vector. The same would be the case you should check that is the same case in 4 as well, the null space is just the 0 space, 0 set. I am not I am using the word space I am coming to that in a minute. This will also be actually a linear transformation where the null space is just the zero space. It is not actually a maybe I should have given some more examples where this is actually is a good example, but I will not go into details here.

In null space of the linear transformation T which is given in example 6 will be all those vectors in Rn, which when multiplied by a given matrix is equal to the zero vector. So, we will see that later we will certainly revisit this example, we will see that, if A is an invertible matrix the null space is going to be the zero vector. If A is not an invertible matrix then null space will be some subspace in fact subset of Rn.

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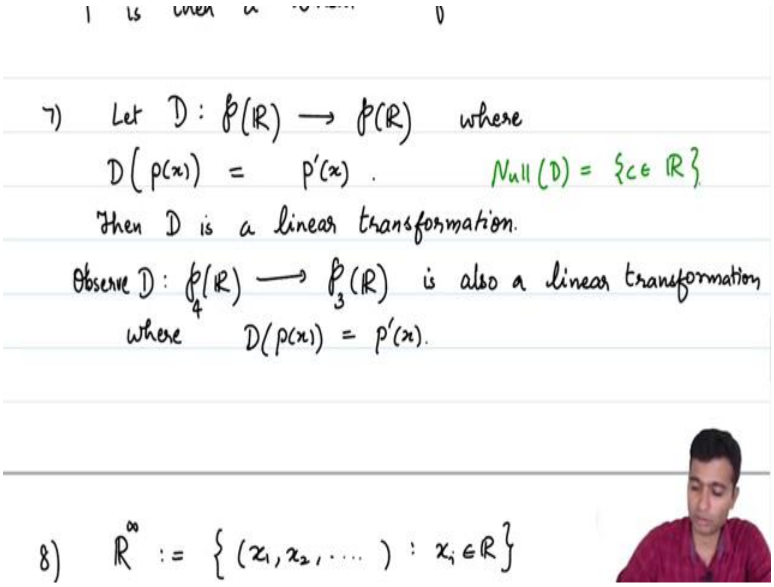

The case when we look at 7, the null space what are the polynomials which when differentiated give us the zero vector or the zero polynomial the constant.

So, the null space here is the set of all c in P of R, in c in R all those constant polynomials whatever the null space turns out to be. One thing we can certainly conclude is the following lemma.

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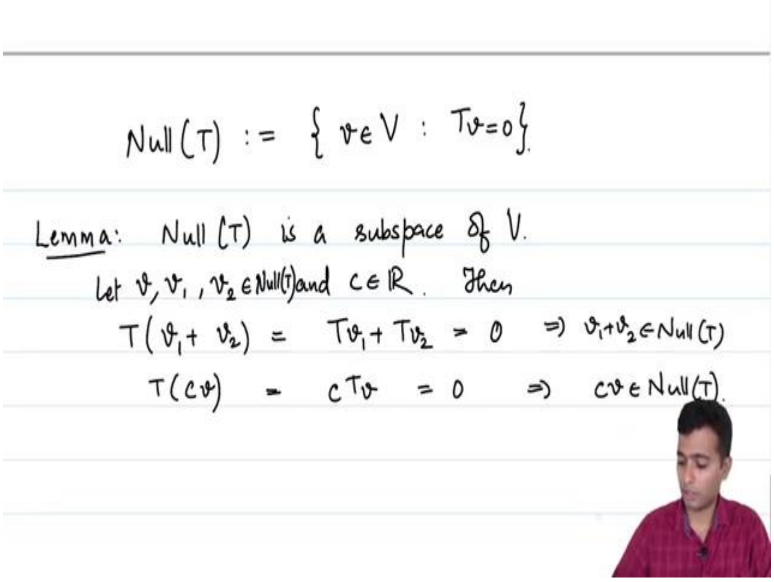

Null of T is a subspace vector subspace of V. This actually quite straightforward so let v1, v2 to be in capital V and c be in capital R then not in capital V. Let v1, v2 be in null of T and c be in capital R then, what is T of v1 plus cv2? We want to show that null of T is a just one minute let me not make unnecessary mistakes, so let v comma v1 comma v2.

T of v1 plus v2 will turned out to be T of v1 plus T of v2, but, we know that v1 and v2 belong to the null space of T and therefore, Tv1 is 0, Tv2 is 0 therefore, 0 plus 0 will give you back the 0. Similarly, look at T of cv this is equal to by the property of a linear transformation. So, the above one was also because of the fact that T is a linear transformation, this will turn out to be cTv. But, Tv is the zero vector and c times the zero vector is again zero vector. So, the first one implies v1 plus v2 belongs to null of T and this implies that cv belongs to null of T and therefore null of T is a vector subspace.

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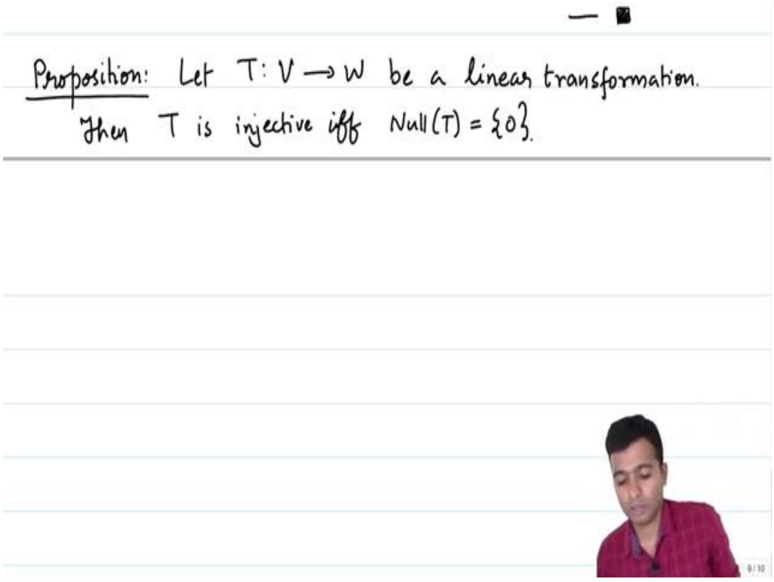

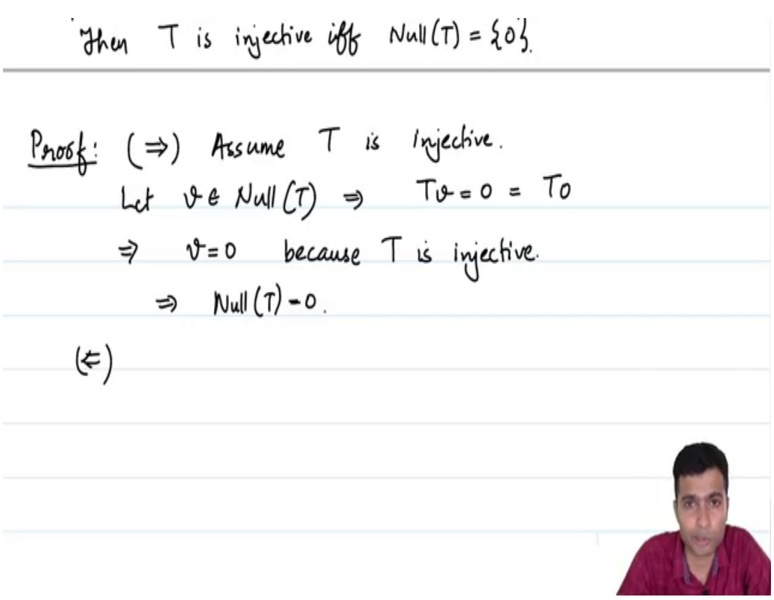

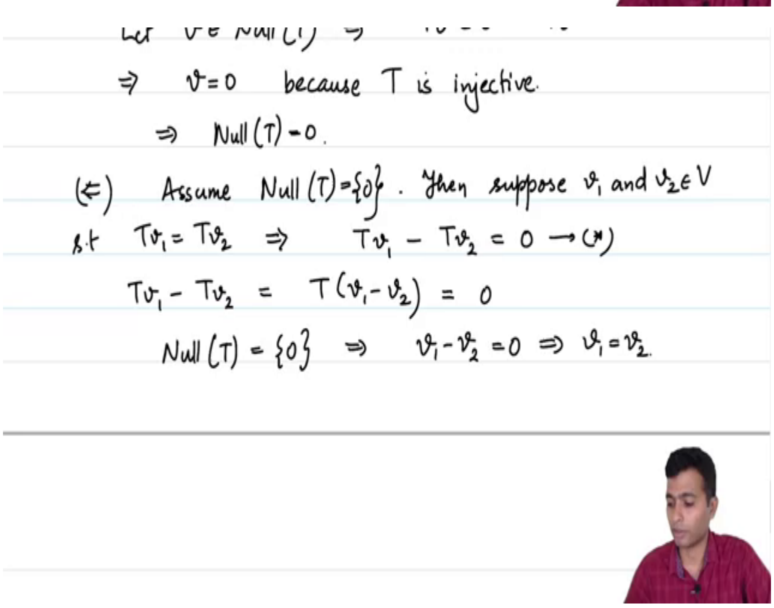

Let us prove the following proposition, a null space is not some arbitrary vector subspace, it is a subspace, which captures some information about the given linear transformation. So, let T from V to W be a linear transformation, so remember that V and W are vector spaces and W is not necessarily a subspace of V. So, T from V to W be a linear transformation, then T is injective if and only if, null of T is equal to 0. So, the null of T being 0 captures information about whether T is injective? So this is an if and only if statement, so let us give a proof of that.

In many of the examples that we gave above, we saw that the null space is zero. It is quite easy many times to check whether the null space is zero or not as compared to checking whether T is injective. So in many of the cases above, we saw that the null space was 0 in all those cases T turned out to be an injective map. So, let us give a quick proof of this. So, in this direction we will assume that T is injective, then let v an element in the null space of T. We will show that, v is necessarily equal to zero vector. What is the meaning of v being in the null space of T, this means that Tv is equal to 0.

But, we already know that the zero vector of v should necessarily map to zero, and therefore this is equal to T of zero. This implies v is equal to 0 because T is injective, so if we start off with some vector in null of T, we see that it has to be necessarily 0. This implies that the null of T is 0, that was quite straight forward. Let us look at the other direction as it turns out the other direction is also equally straightforward. So, assume that null of T is the 0 vector, then suppose v1 and v2 are two vectors in V such that, Tv1 is equal to Tv2 our goal is to show that T is injective.

So, we will take hold of two vectors v1 and v2 such that Tv1 is equal to Tv2, we will conclude that v1 is equal to v2 from that. This however implies that Tv1 minus Tv2 is equal to the 0 vector by adding the additive inverse of Tv2. Now, this is a an exercise for you to check that, this Tv1 minus Tv2 is nothing but T of v1 minus v2, just need to check that T of minus v is equal to minus of T of v which is the case, but, this is equal to our 0 vector as noted here in star and null of T we know is just the 0 vector here. This implies v1 minus v2 is the 0 vector which implies v1 is equal to v2.

Therefore, our T is injective, so this completes the proof.

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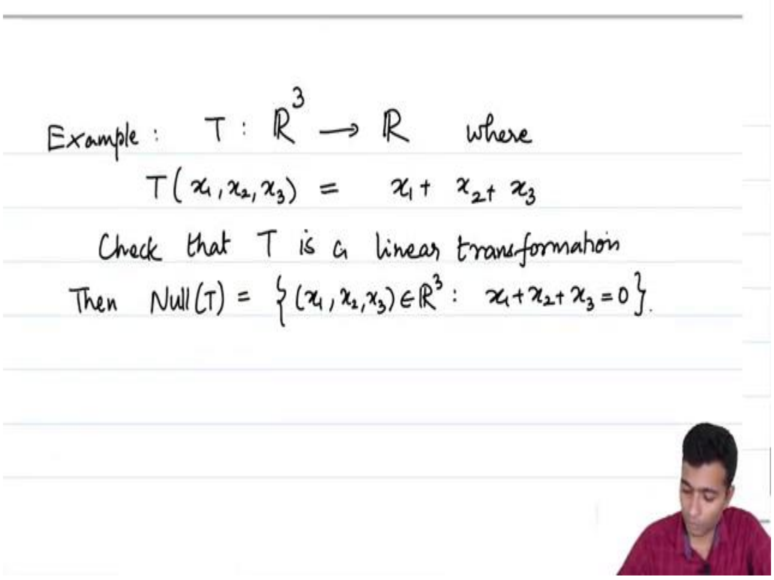

So, maybe I should give an example in Rn is actually quite straight forward to give some example. Let us look at a map say T from R3 to R where T of x1, x2, x3 is equal to x1 plus 2 x2 plus x3 equal to x1 plus 2 x2 plus x3. Then, observe that what is so this check that this is a linear transformation, so as I said we will keep giving more and more examples on the way check that T is a linear transformation. And okay maybe we should look at the example where this is x1 plus x2 plus x3 because we then know exactly what the null space would be.

Then null space of T is the set of all x1, x2, x3 in R3, such that x1 plus x2 plus x3 is equal to 0. If you go back to maybe two or three lectures back, we had actually explicitly studied this vector subspace of R3, in fact we had even computed the basis for this particular vector subspace.

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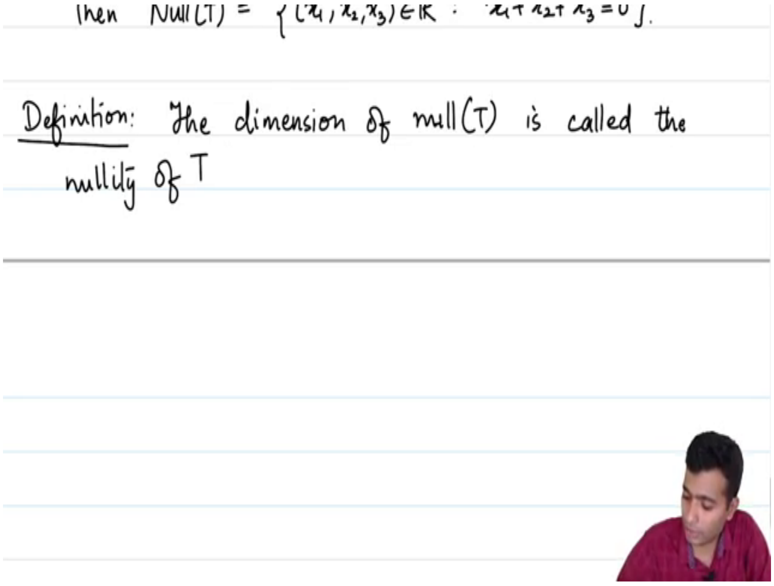

So, that is about the null space of T okay, one more definition. The definition states that, definition does not state anything, definition is of nullity of T. The dimension of the null space of T remember that null space is a subspace of the vector space V, and hence we can talk about the dimension of this particular vector space, vector subspace. So, the dimension of null space of T is called the nullity of T, let me just underline, no let me not underline it. So this is called the nullity of T.

So, let us look at more examples now, let us not look at any more examples let us move ahead. Just like in the case of, so if you recall when we talked about linear independents there was the notion of spanning set, a linearly independent set had a corresponding notion of a spanning set. Similarly, in null space of a given linear transformation also has a corresponding notion of what is called as the range of T. So, that is the next thing that we will be exploring.

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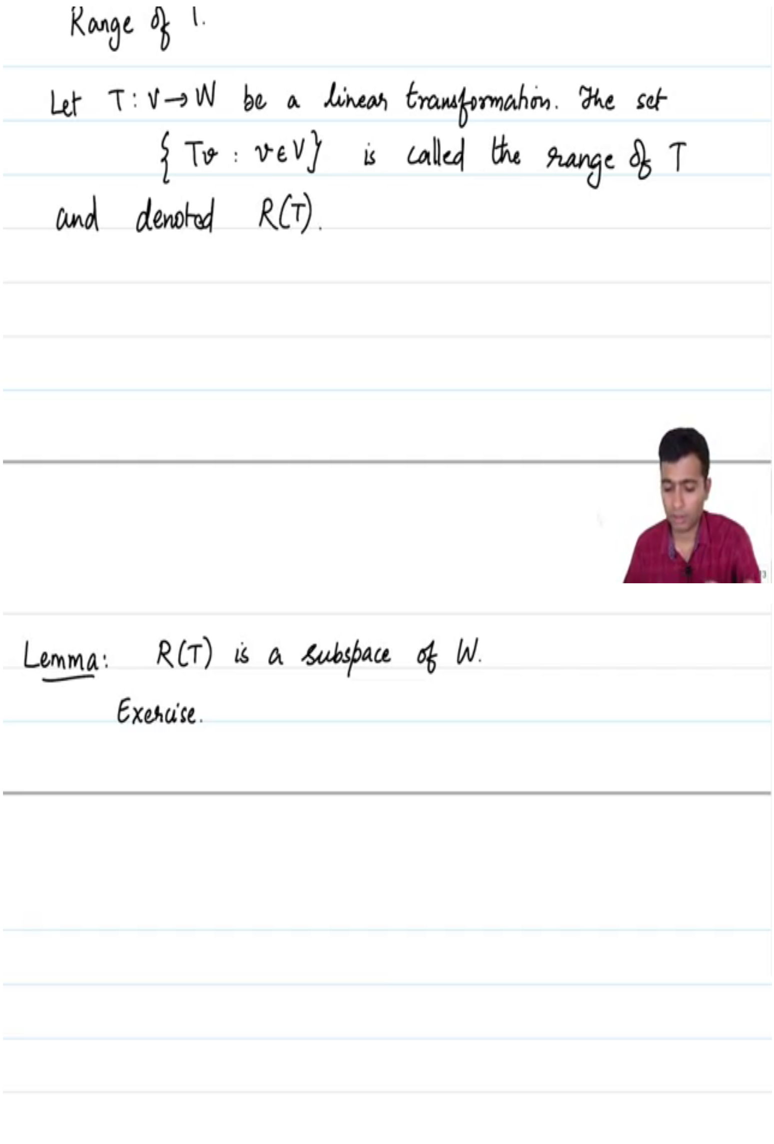

Range of a linear transformation T, so let T from V to W again remember here again V and W are arbitrary vector spaces be a linear transformation and as the name suggest, we are going to define what is called as the linear transformation, we are going to define what the range is, it is going to be the set-theoretic range of T. Then, the set, Tv where v belongs to capital V is called the range of T and denoted R of T. So, we might be wondering why the range is being brought out, it is just the set theoretic range, yes that is true, this is just the set theoretic range.

However, the properties of a linear transformation forces the range of a linear transformation to be a vector subspace. So, let me just write that in a lemma and I will leave this as an exercise R of T is a subspace, a subspace of what? Remember that the vectors are mapped into W, so this is a subspace, vector subspace of W. So, this is an exercise for you to check. I will not go into any examples you should really go back to each and every one of the examples of linear transformations we have seen till now. And try to calculate what the range of each of the vector, each of the linear transformations we defined is.

So, observe that range of T is equal to W is the same as demanding that T is a surjective linear transformation. Just like how the dimension of the null space of T had a name, the corresponding notion all of the range of T also has a name.

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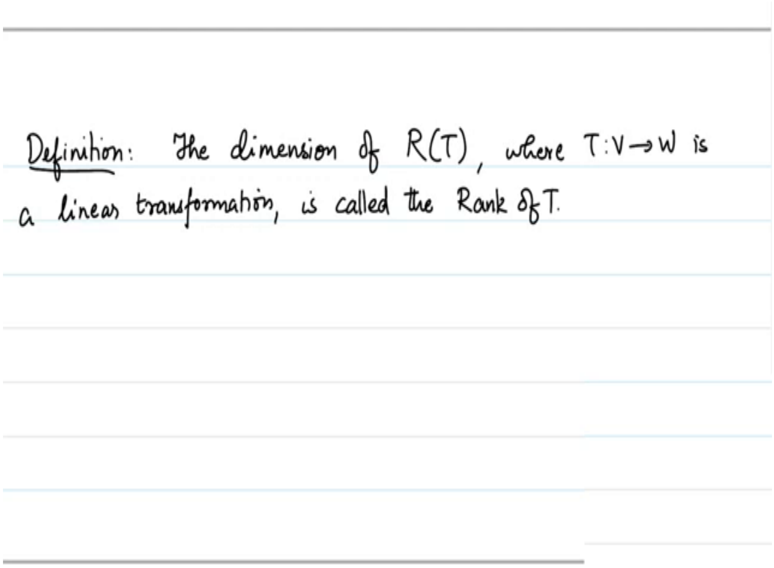

So, definition, the dimension of R of T, where T from V to W is a linear transformation is called the rank of T. So, from matrix theory you might have heard of the word rank and if you have heard of it, I should already break the suspense in tell you that this is related to that, we will come to that at a later date.

So, the dimensions of the null space and the dimension of V range space they are related, and that is captured in what is called as the dimension theorem. So, let us now state the dimension theorem and give a proof of it.

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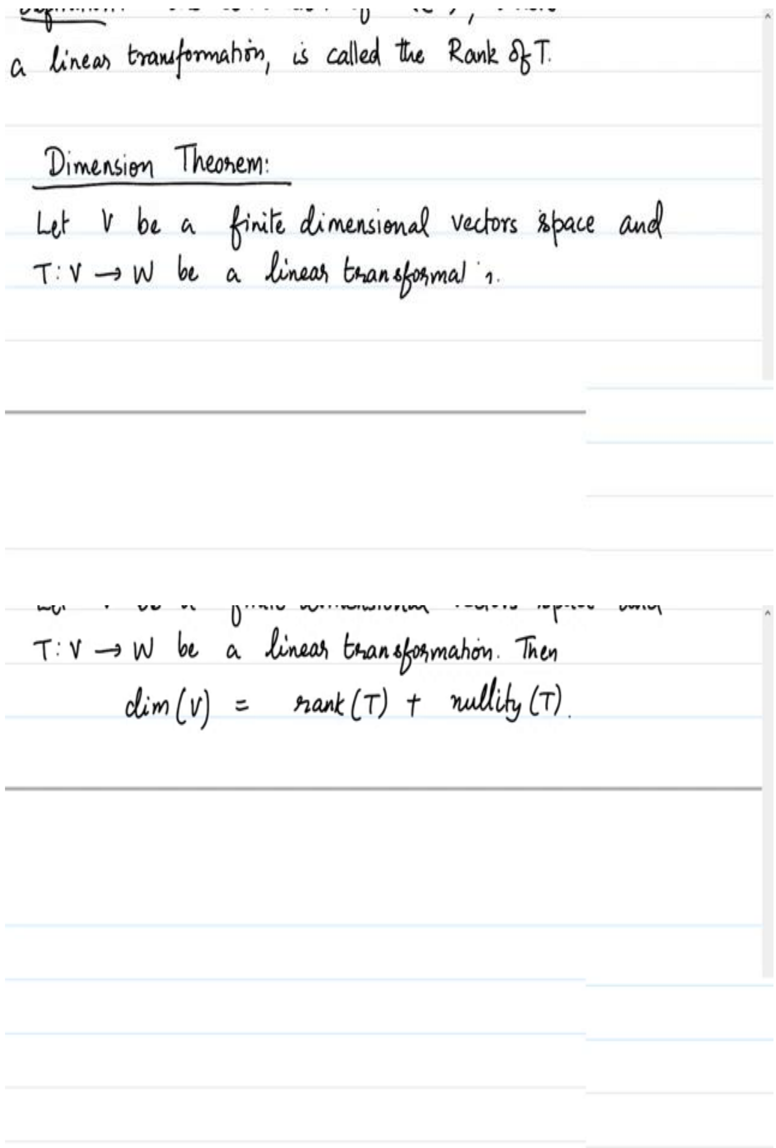

So, our next goal would be to state and prove the dimension theorem. So, observe one thing all this while, we were not demanding that our vector space is V or W should be finite dimensional or not. Ever as of now till now, we were giving a definition of a linear transformation of null spaces of nullity of range, rank all these things were being talked about without any reference to whether our vector spaces finite dimensional.

In this theorem however we will put a restriction on V, so let be a finite dimensional vector space and T be a linear transformation from V to W. Then the dimension theorem states that, the dimension of v is equal to rank of T plus the nullity of T.

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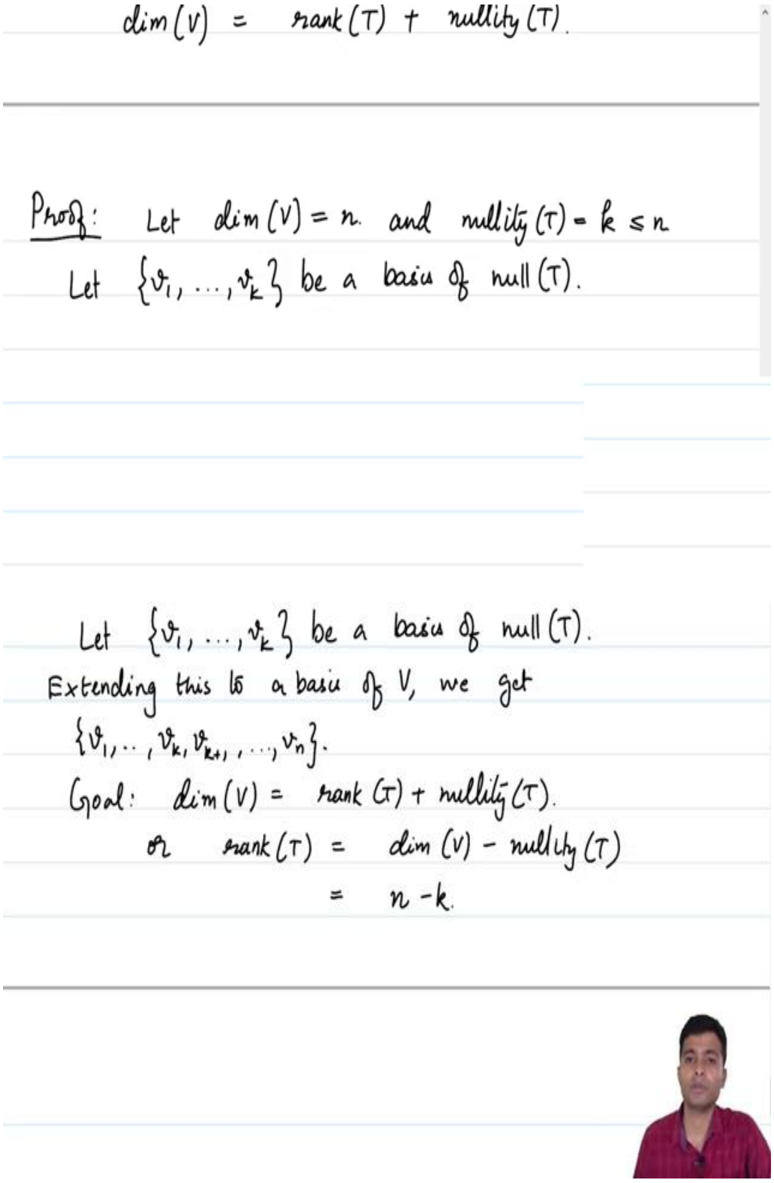

So, let us look at a proof of this theorem. This proof is quite instructor it really tells and it really shows us how the vectors V and W are interacting with each other through the map T. So, this is one of those first cases where you will actually look at such statements so let us look at a proof of this.

So, we know that dimension of V is finite so let dimensions give it a name. Dimension of V be equal to n, and we also know that null space of T is a subspace of V, so in particular the dimension of the null space will be less than or equal to be dimension of V. And let us call that k, nullity of T be equal to k and we know that this is less than or equal to n. If k is equal to n what is the meaning of that? It means that, the null space is the entire vector space which means that every vectors being mapped to the zero vector.

And therefore, we know that the range is the zero vector space and dimension of the zero vector space is zero. So, in that case it is quite straight forward to check that dimension of V is equal to rank of T plus nullity of T. We will not be doing this by induction but nevertheless it was quite straight forward to show that is why it was pinpointed. Let us try to prove this by a constructive approach. So, we know that null space of T has a basis which has k elements. By one of the corollaries, we know that there is a basis and that it has to necessarily by another corollary it should necessarily have k elements.

So, let v1 up to vk be a basis of null of T, we know that this is a vector subspace and we also know have a basis for that. We will extend this basis to a basis of V extending this to a basis of V, we get v1 to vk, vk plus 1 to vn, what does it mean to extend a linearly independent set? We could use the replacement theorem, we can start off with a basis of V and we have a linearly independent set of size k. Then we know that there is a subset of the basis of size n minus k with this particular set, it will be the spanning set.

Now, the spanning set which has size n should necessarily be linearly independent again, so this is a basis which can be obtained in that way. It is corollary which we had done I am just recalling the proof of that particular thing. So, we extend it to get a basis of V. Now, if you look at Tv1 it is 0, Tv2 up to Tvk each of them is 0, so my claim is so what do we have to check? We know that nullity of T is equal to k we know that dimension so goal just recall our goal so that we have a clear idea what to prove.

We have to prove that dimension of V is equal to rank of T plus nullity of T or in other words or rank of T should be equal to dimension of V minus nullity of T, but, we know what these things in the right are, this is equal to n and this is equal to k. So, we have to show that rank of T is equal to n minus k, but what is rank of T? Rank of T is the dimension of the range of T. So, in another words we need to get hold of n minus k vectors which will be a basis of our range of T and that is going to be our next end hour.

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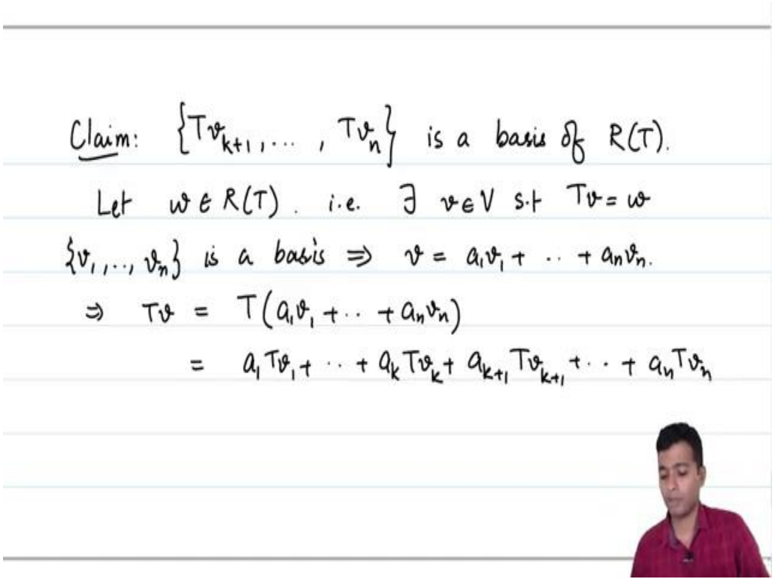

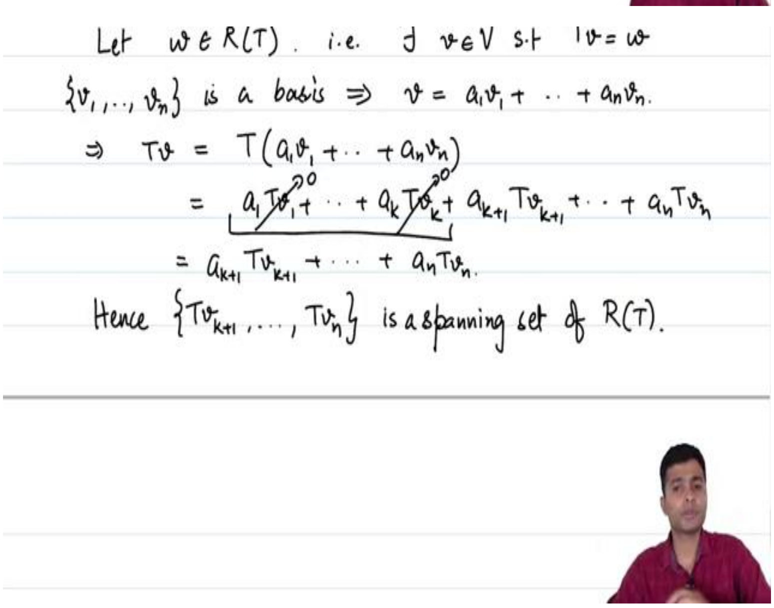

In fact, let me write down a direct claim, T of v1 sorry vk plus 1 up to T of vn those vectors which we added to the basis of null space to obtain a basis of v.

We look at the image of that. This is a basis of the range of T which I will write as R of T. We already then have our result because this set has size exactly equal to n minus k. So, how to go about with a claim, so we have this claim, we have to show that this is both a spanning set and a linearly independent set. So, let us start our work to show that it is a spanning set. So, let W be an element of R of T. So, we have to show that W can be written as a linear combination of Tvk plus 1 to Tvn.

But, what does it mean to say that V is in R of T, that means that it is the image of some vector V i.e. there exist a vector v in capital V such that Tv is equal to w that precisely our definition of what R of T is, but, we know that v1 to vn we should call it a okay let us v1 to vn is a basis and therefore, every vector of the vector space can be written as a linear combination of v1 to vn. In fact, uniquely by one of the lemmas or propositions we had proved. So, this implies that there exist a1 to an, such that v is equal to a1 v1 plus up to an vn.

But, then Tv is equal to T of a1 v1 plus up to an vn, in fact I should have written an ak vk and ak plus 1, vk plus 1 at the middle, I will do that in the next step. By the properties of linear transformation and an induction argument, you can show that this is equal to T of a1 v1 plus T of a2 v2 plus up to T of an vn. And then you can show that this is equal to a1 Tv1 plus a2 Tv2 plus up to ak Tvk plus ak plus 1 Tvk plus 1 plus up to an Tvn, but, what is Tv1, Tv2 up to Tvk to recall that v1, v2 up to vk is a basis of the null space of T.

So, this is our zero vector, Tv2 is our zero vector, Tvk is our zero vector all these are zero vectors. So, this is the zero vector of W and hence this is equal to Tak plus 1, Tvk plus 1 plus up to an Tvn. That is establishing the claim that or the fact that Tvk plus 1 up to Tvn, this set is a spanning set of R of T, and If I show that it is a linearly independent we have proved our theorem. So, how do we prove that it is a linearly independent? We take a linear combination which is equal to 0 and try to prove that it is coefficients are all equal to 0 and hence it is linearly independent.

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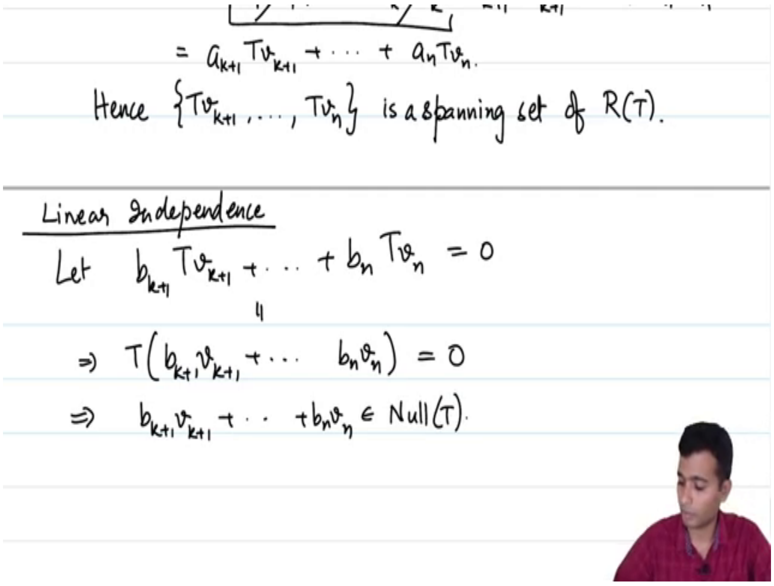

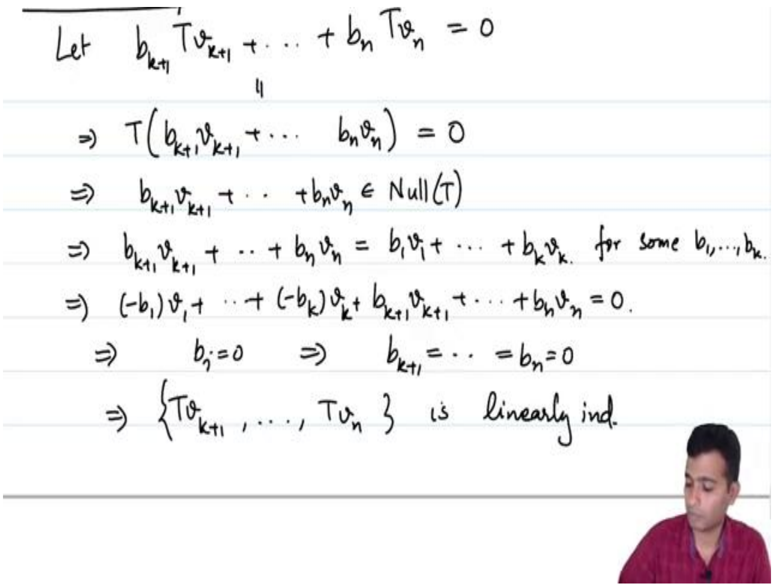

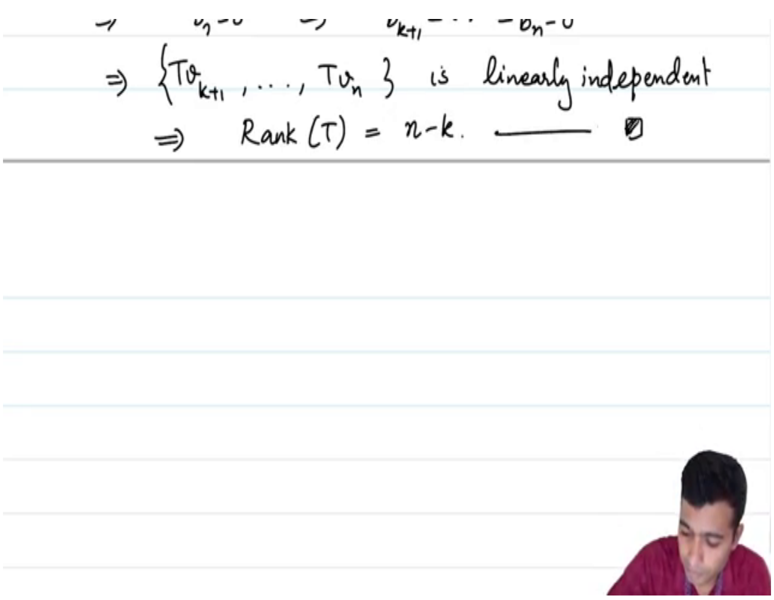

So, let us now use some other notations so that it is not confusing here. Let bk plus 1, so linear independent this what we are next trying to prove. Let bk plus 1 Tvk plus 1 plus up to bn Tvn be equal to 0 this 0 is a 0 vector of W and we have to show that bk plus one to bn each of these scalers is necessarily equal to 0. But, by the same argument above going in the opposite direction this implies that T of bk plus 1, vk plus 1 plus up to bn vn is equal to 0. Why is that the case? Because these two are equal that is what we know by the very definition of linear transformations so, this is equal to 0.

But, what is the meaning of some vector being being sent to the zero vector by a linear transformation, it means that it belongs to the null space of T. So, this implies that bk plus 1 vk plus 1 plus up to bn vn belongs to the null space of T by the very definition of a what null of T is, but, every vector of null of T can be written as a linear combination of it is basis vectors of, if you have a basis already which we have. So, this implies that there exist a scalers b1 to say bk such that this vector is equal to b1 v1 plus up to bk vk.

Any element has to be for some b has for some b1, b2 up to bk rewriting this we have minus of b1 v1 plus, minus of bk vk plus bk plus 1, vk plus 1 plus bn vn is equal to 0. So, remember that 0 here is the zero vector in V, the 0 here was the zero vector in W. So, I have stopped writing all the subscripts it is for you to understand from the context that, this is the zero vector in V here what we have just written. We know now that v1 to vn is a basis of capital V and this forces because of linear independence this forces each of these bi is to be zero.

In particular this implies that bk plus 1 up to bn is equal to 0 and that is precisely what we wanted see observe that, we wanted to establish that this forces our coefficients to be equal to zero. This implies hence that Tvk plus one up to Tvn is linearly independent and that implies rank of T is equal to n minus k and that completes our proof. So, this is one of the most beautiful examples of how linear transformation helps us take information from our domain to our range and back. As you can see the linear independence of v1, v2 up to vn was used to prove that the vectors Tvk plus 1 to Tvn which are vectors in W are linearly independent.

And similarly many-many such examples of how T takes properties of vectors in v2, the corresponding vectors in W can be seen here. So, this is a very powerful result actually just to give you an indicator of how powerful this is, let us look at this example.

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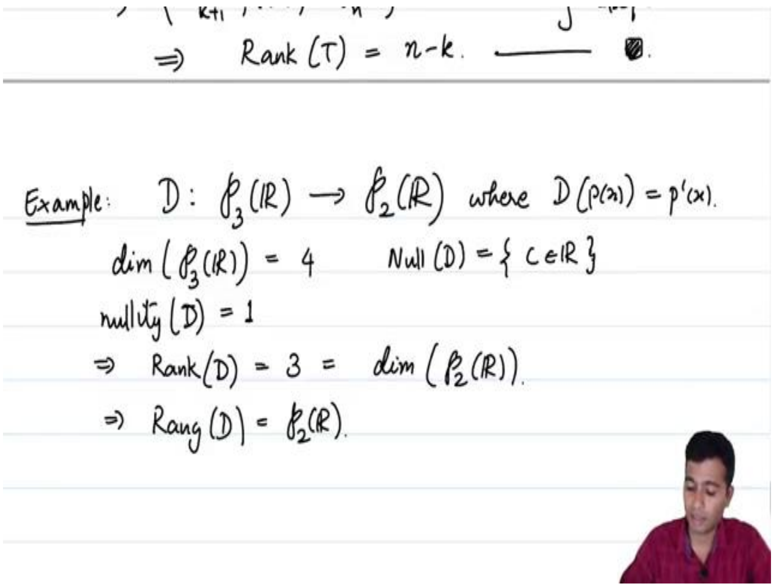

So, let D be the map from P3 of R to P2 of R, where D is our differentiation map, D of P of x is equal to P prime of x, then observe what is happening then dimension of our P3 of R we needed it to be a finite dimensional vector space, in this case this is equal to 4.

We already saw what the null space of the differentiation operator is, differentiation linear function transformation is. The dimension there will be equal to the dimension of the constant polynomials as a subspace. So, null space of D is the set of all c in R the constant polynomials and nullity, hence is equal to 1, R has dimension 1 over itself. So, you can check that this is basis is given by any non-zero real number and this implies our rank of T D in this case, sorry D is equal to 4 minus 1 which is equal to 3, but, observe that this is equal to the dimension of P2 of R.

Which indicates that D is a surjective (())(35:42) basically the range of D hence is equal to P2 of R entirely. So, what does it mean to say that? This means that every degree 2 polynomial can be realised as the derivative of a degree 3 polynomial. We know this by our information our knowledge from calculous, but this is a linear algebra way of obtaining something similar of course some calculous was used here.

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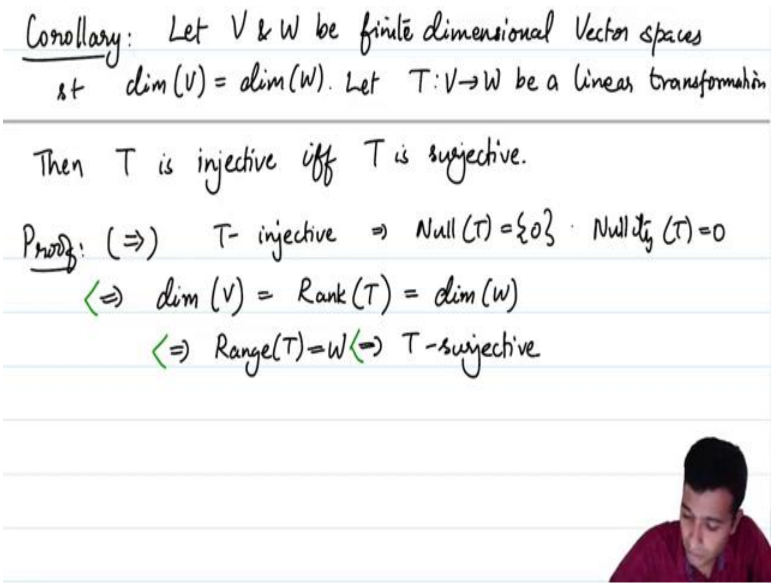

So, let me stop by giving another proposition which is a maybe not a proposition let me just call it a corollary. So, Corollary tool the dimension theorem, let V and W be finite dimensional vector spaces.

V and W be finite dimensional vector spaces, such that dimension of V is equal to the dimension of W, then suppose further let T from V to W be a linear transformation. The conclusion is that then, T is injective if and only if T is surjective, T is injective if and only if T is surjective. So, let us look at the proof. This actually is a direct consequence of our dimension theorem, so by the dimension theorem before we even enter into okay let us look at the direction. Let us assume that T is injective let us prove that T is surjective. T is injective implies that the null space of T is the zero space.

That was a theorem which we proved some time back and therefore the nullity of T is equal to 0. So, the dimension theorem tells us that, dimension of V is equal to the rank of T but, dimension of V is equal to the dimension of W by the very hypothesis of this corollary so this is equal to the dimension of W.

So, we have a vector subspace of W which has the same dimension as W and this implies, range of T is equal to W which implies T is surjective. So, can we do the same thing in the opposite direction, if I am to put a green arrow, what does it mean? So T surjective implies range of T is equal to the range of W.

Range of T is equal to the range of W implies that rank of T is equal to the dimension of W is equal to the dimension of V, but, that implies by the rank nullity the dimension theorem this implies that the nullity of T is equal to 0 which implies.