Linear Algebra
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Lecture 1.1
Vector Spaces

Hello, welcome to this course on Linear algebra. My name is Pranav, I am from Kerala School of mathematics. This is a 12 week course and a detailed description of this course can be found in the website of the course. The textbooks we will be following is called, the primary textbook that we will be following is called Linear algebra. It is by Friedberg Insel and Spence. Let me just note it down.

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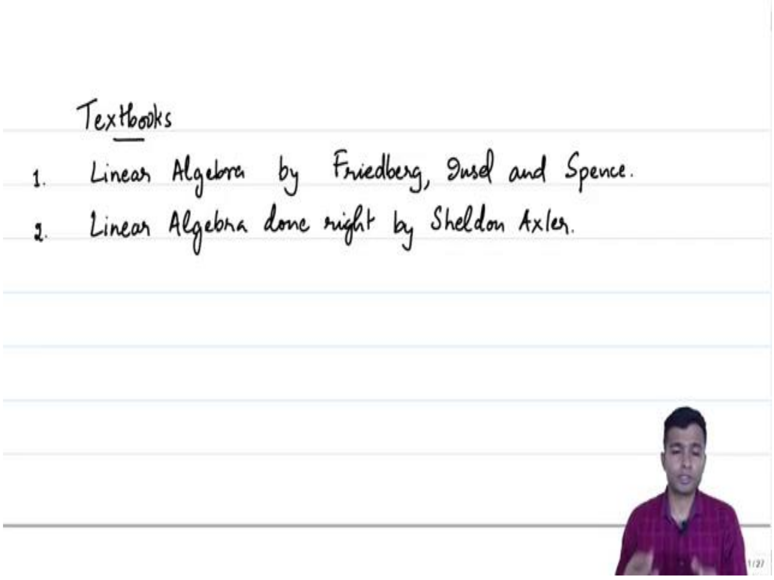

Primary textbook is called Linear algebra by Friedberg, Insel and Spence. This is a very classical subject. There are many beautiful textbooks written on the subject of linear algebra. I would also refer you to another book which is called Linear algebra done right, by Sheldon Axler. This is also a very elegant book. We will be focusing quite a lot on giving rigorous proofs and solving many problems. We will have a problem session every week and this will be supplemented by weekly assignments. So, you will be, you are very strongly encouraged to work on the assignment problems on your own. Solving problems is the most effective way to understand the concepts and the theory that we have developed including theorems that we prove in much greater depth.

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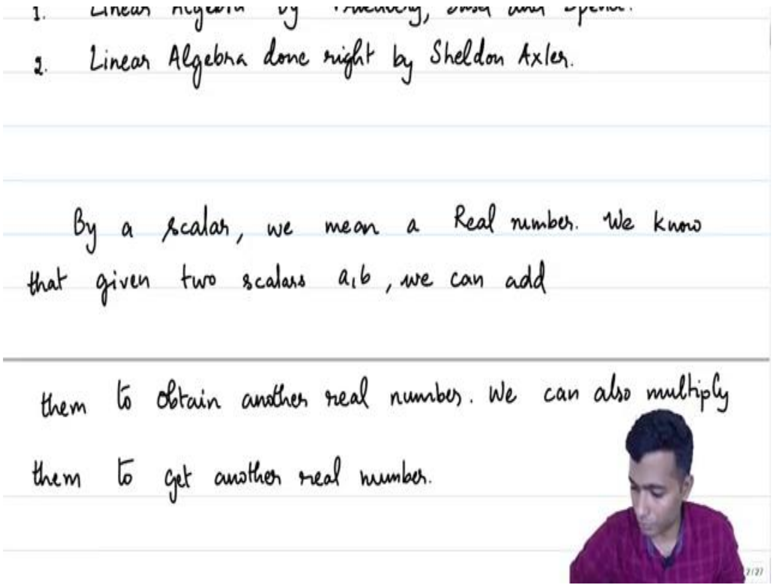

Alright, so that is all I have to say to introduce this course to you. Let us begin the study of linear algebra. Let us start by recalling the concept of scalars. In high school when we do physics, when we do Newtonian mechanics, a scalar is a terminology which is used to describe anything which can be described by a number. For example, the mass of an object or the speed at which a car is going or the distance somebody has walked and so on. So, a scalar very informally speaking is something which can be described by just a number. By a scalar in this course, we will just mean a real number, by a scalar we mean a real number.

We are all familiar with real numbers. Given two scalars, we know that we can add them and obtain a new scalar. You could also multiply them and get back another scalar. If say 2 and 3 is given, we know that 2 + 3 is equal to 5 a real number. 2 times 3 for example is 6 (which) is also a real number. We know that given two scalars or two real numbers a, b we can add and multiply them to obtain another real number. I will maybe say it separately. We can add them to obtain another real number. We can also say multiply them to get another real number. These are operations which we are very familiar with. Been using these operations for a long time now. What are the properties of addition and subtraction that we are quite familiar with? Let us just quickly note down some of these properties.

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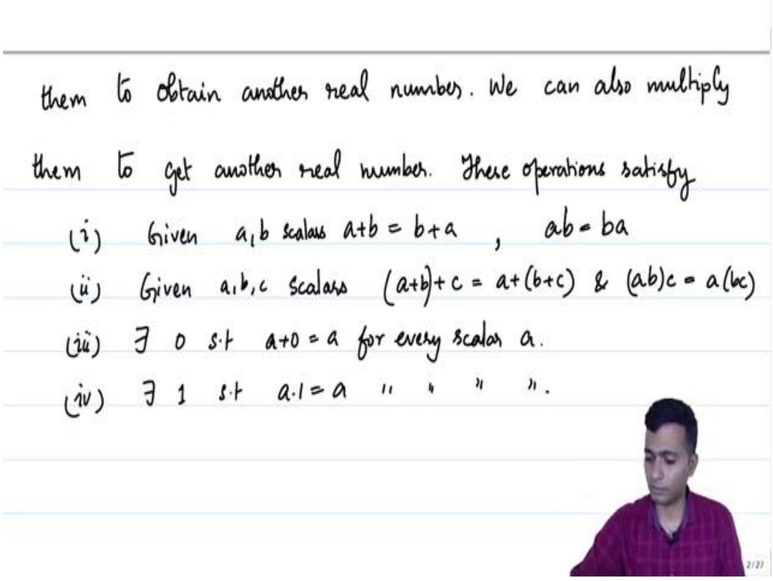

These operations: addition and multiplication operation, they satisfy a few properties. The operations satisfy the following properties. Given say a, b, a + b is equal to b + a. The order in which we add does not matter. If we add say 2 + 3, we get 5. If we add 3 + 2, it also gives us the same number 5. Similar is the case with multiplication: you multiply say 5 and 10 you get 50, and 10 and 5 if you multiply, again it is 50. These operations, they commute: the order in which the addition or the multiplication is done is not important.

Another property that we are familiar with is that if we take three scalars, say a, b, c are scalars, let us look at a + b + c. Now the question arises about whether a + b is added first or whether b + c is added first and then added to the other. This property tells us that the order in which we do (addition) is irrelevant, does not matter. The addition is called, what is called as Associative. And similar is the case with our multiplication. The order in which we multiply the three scalars is not important.

We also know that there is a 0 real number right? There is a real number called 0 which is very special. If you add any number to it you get back the same number. If you add say 3 to 0, we get back 3. So there exists 0 such that a + 0 is equal to a for every scalar a. We also have that given any scalar a, if you multiply it to 1, we get back a (correction). So just like 0 behaves for addition, 1 behaves as an identity for multiplication. So, there exists 1 such that a times 1 is equal to a for every scalar a.

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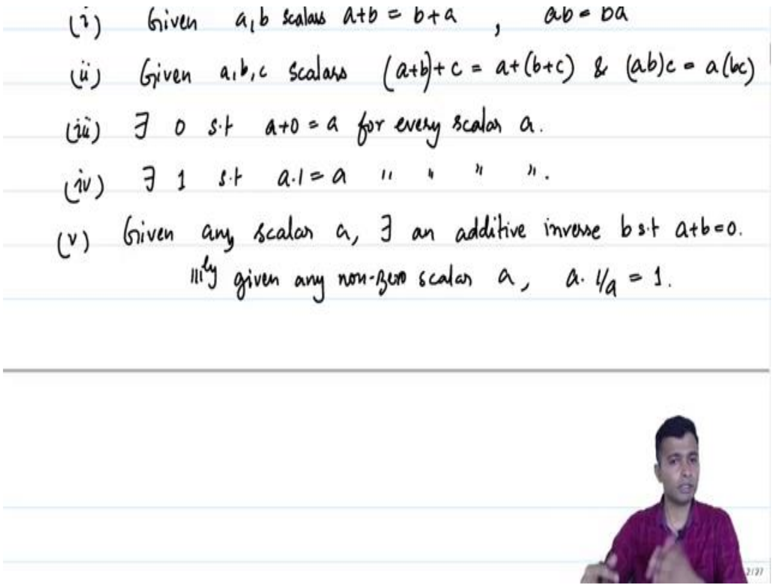

And we have more, given any scalar a we know that -a (exists). If 2 is given, we know that -2 is an additive inverse. If you add -2 to 2 we get back 0 right? There exists an inverse so let me write it as additive inverse. (There exists) b such that a + b is equal to 0 right. Similarly, given any non-zero scalar we have a multiplicative inverse. Given any non-zero scalar a, 1 by a times a (is equal to) a times 1 by a is equal to 1, right? If you consider 2, 1 by 2 is the multiplicative inverse of 2. Similarly b is this -a (above). So I could have just written it that way, it is ok does not matter. Every element will have both additive inverse and multiplicative inverse.

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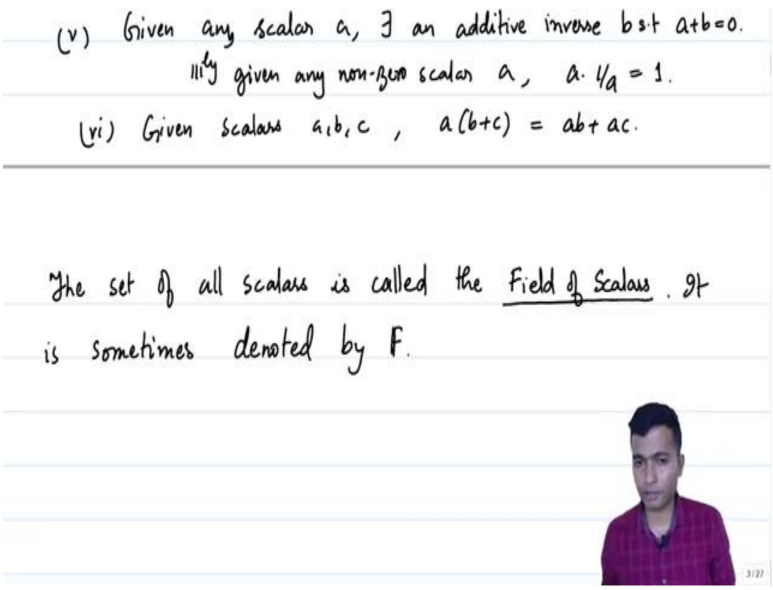

And the addition and the multiplication operation are not standing apart, they interact with each other. So given scalars a, b, c, a times b + c is equal to ab + ac. We have summarized some of the very common properties of real numbers we are quite familiar with. We know a lot more about real numbers, but let me just single out these properties because these are the properties of real numbers we will be using, properties of the scalars that we will be using in this course.

The collection or the set of all scalars is called the field of scalars. I will not write it in capitals, just underline it, (and is) called the field of scalars. It is sometimes denoted, sometimes denoted by F. In this course, our field of scalars will always be Real Numbers most of the time.

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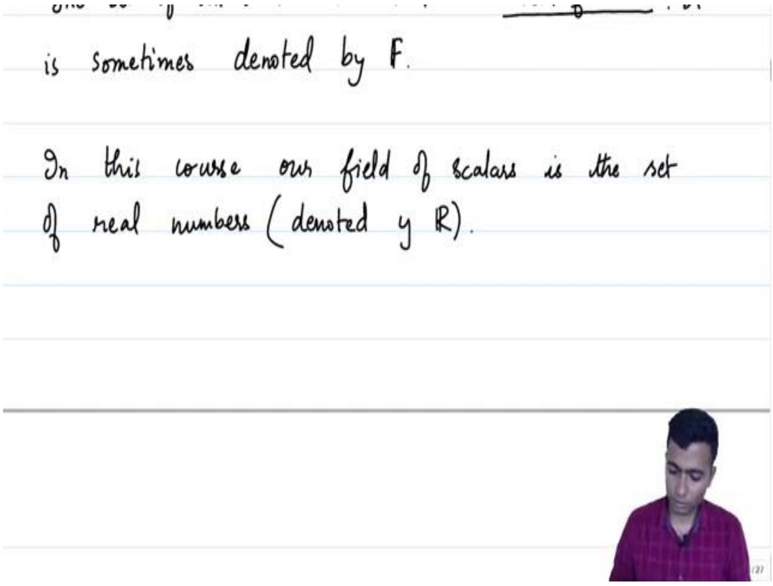

In this course our field of scalars is field is the set of Real Numbers, most of the time. Real Numbers are denoted by R. Alright. Let us now jump into the definition of what a vector space is. So what is a vector space?

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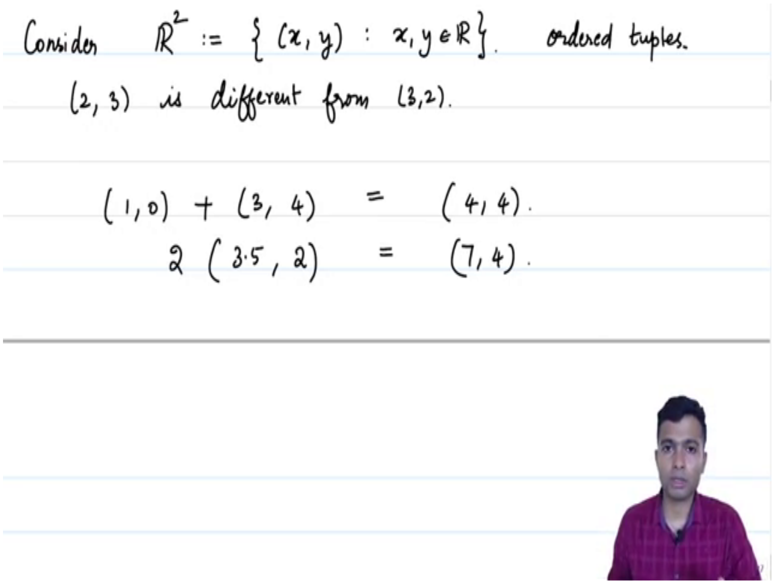

Let us discuss some vector spaces which are quite familiar and then we will give a formal definition. The next topic or the first topic rather is vector spaces. Let us start with something very familiar, something which we are all quite familiar with. Consider the Cartesian product R2 from coordinate geometry. What is this? This is just the set of all (x, y) such that both x and y are in R. They are ordered tuples (x, y) where each of the coordinates are real numbers. The order matters. So for example (2, 3) they are ordered tuples. Remember that. (2, 3) is different from (3, 2). Right? And we are also quite familiar with the notion of adding 2 elements in R2. So if say (1, 0) and (3, 4) is given to you, we know that we can write this as (4, 4). Right? We are also familiar with the notion of multiplying scalar to vector in R2.

So given say for 3 or 2 times, let us put something here 3.5 and 2, this we know is equal to 7 and 4. From coordinate geometry we know that we can do these operations on the space R2. Right?

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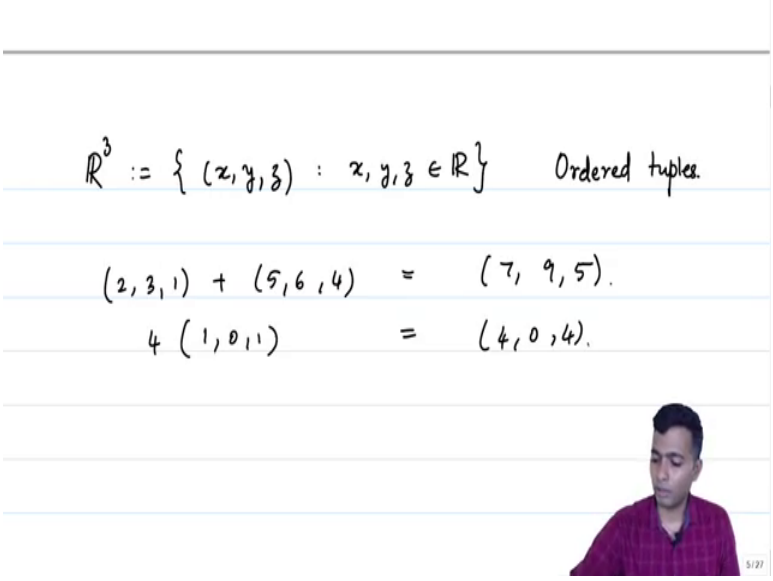

We also have considered one more space R3, in physics many times. This has been dealt with. This is the Cartesian product of Real Numbers with itself 3 times. This again is an ordered tuple where all x y z are in R. They are all real numbers so again ordered tuples. The order in which we are considering matters. And in a very similar manner, we have defined addition in R3. So, say (2, 3, 1) is added to (5, 6, 4) what was the answer? This will be just (7, 9, 5). Right? We know that it is taken component-wise and added. If you multiply four times (1,0,1) we know that this is (4,0,4). Again this is done component wise. These are operations which we are already familiar with, from our high school physics for example. R3 for example was used to describe the components of say the velocity or the acceleration for that matter. And when we add two velocity vectors, we just added it in this manner: x component, y component, z component and so on. All right.

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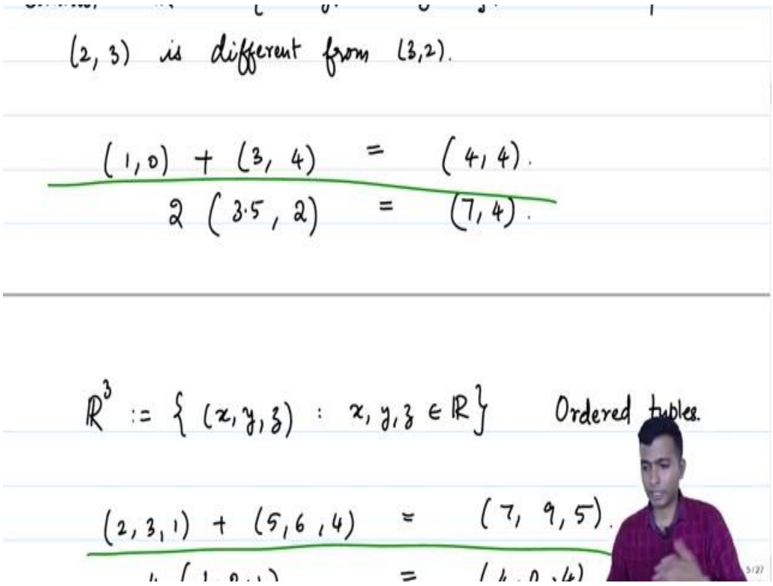

These two examples and the operations therein satisfy certain properties. Notice one thing before we go ahead. The addition here, that is being described, for example, this one is for two vectors in R2 and this one is for two vectors in R3. We cannot take a vector in, an element in R2 and an element in R3 and add. Right? It does not make sense to add an element in R2 to an element in R3. Right? So, these operations of say, addition and multiplication by a scalar, they satisfy certain nice properties. Or let me put it in a different manner: there are certain properties which are satisfied by these operations, which are of great interest to us, which has the potential to abstractify. It can be put in a more general setting and we could study those objects which satisfy similar properties and that is where vector spaces come from.

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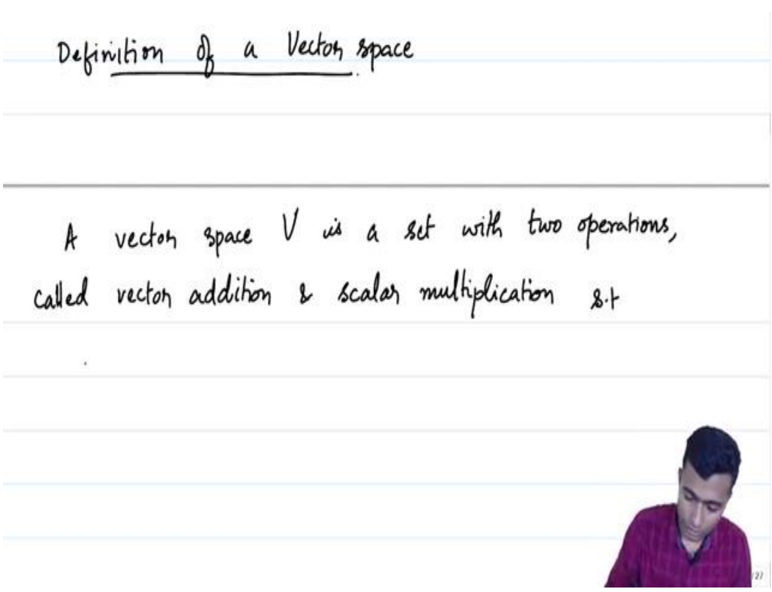

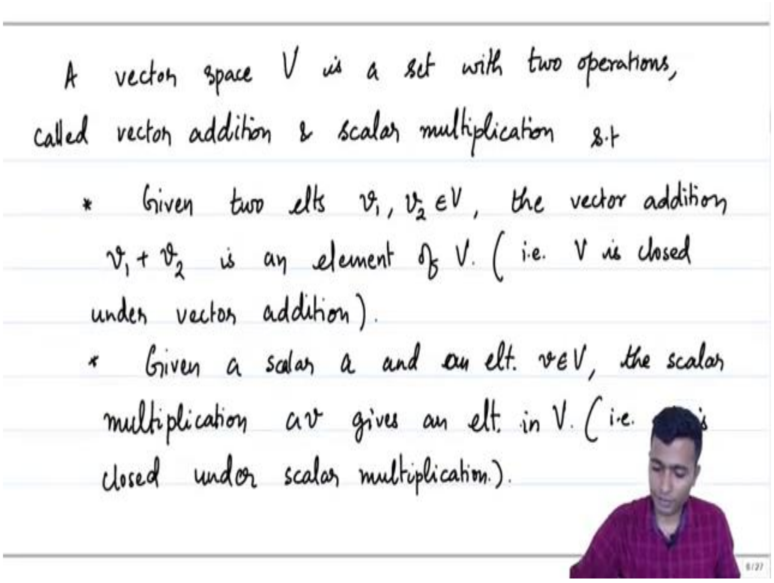

Let us now give a formal definition of a vector space. So, definition of a vector space. A vector space is a set with two such operations which satisfy a few properties. Let us list all the properties down. Vector space, let us denote it by V, V is a set with two operations which are called say vector, they are called vector addition and scalar multiplication which satisfy the, ok. The vector addition and scalar multiplication they satisfy certain properties such that....... Let us just see what addition satisfies, given two elements v1 and v2 in V, their vector addition is also an element of V. The vector addition v1 + v2 is an element of V. We also say, the alternate way of saying this is that V is closed under addition. Let me just write it down in bracket i. e., V is closed under vector addition.

And similarly, the scalar multiplication, so given a scalar and a vector, we can talk about scalar multiplication and the scalar multiplication is also similarly closed... sorry... the set V is closed under the operation of scalar multiplication. Let me just write it down: given a scalar, recall that the scalar multiplication involved multiplication of element by a scalar. Right? Given scalar a and a vector and an element let me write elt in short for element, v in V, scalar multiplication av gives an element in V. Or in other words, V is closed under scalar multiplication.

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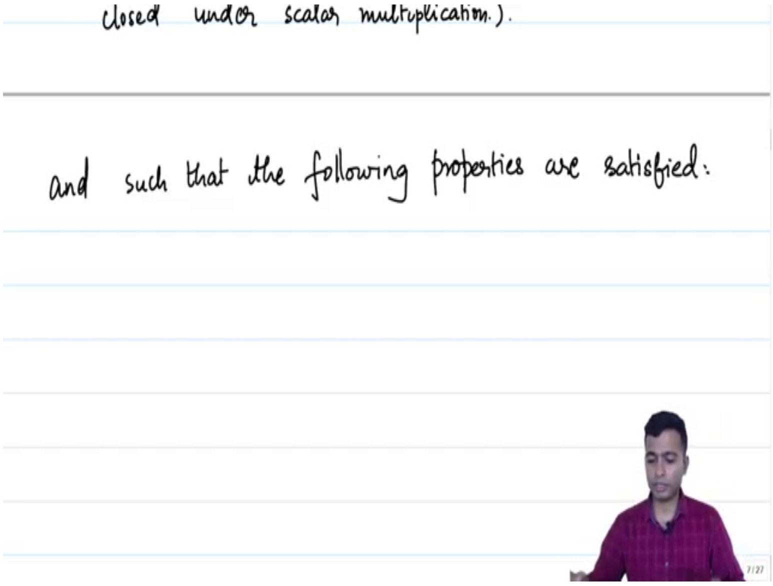

So, we start with a vector space as a set with these two operations, such that these conditions of the space being closed under these operations are satisfied and such that the following properties are satisfied. The following properties are satisfied. Let us see, let us go back to our example.

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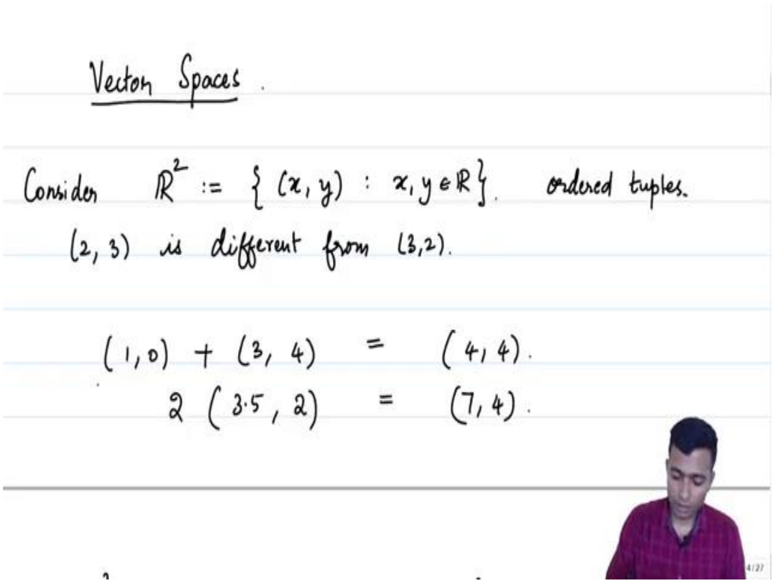

So in the example of R2 we defined vector addition and we also defined a scalar multiplication even though we know already, we did not define it, we used the existing knowledge and what we would like to see what are the properties that are satisfied by these two operations in R2. The first operation, the first property is that the order in which you look at.... Let us focus on vector addition temporarily. The order in which we add two elements in R2, again, it does not matter. If you add say (1, 2) to (2, 3), and if you add (2, 3) to (1, 2) it should be the same. Right? That is one property which we would like to generalize.

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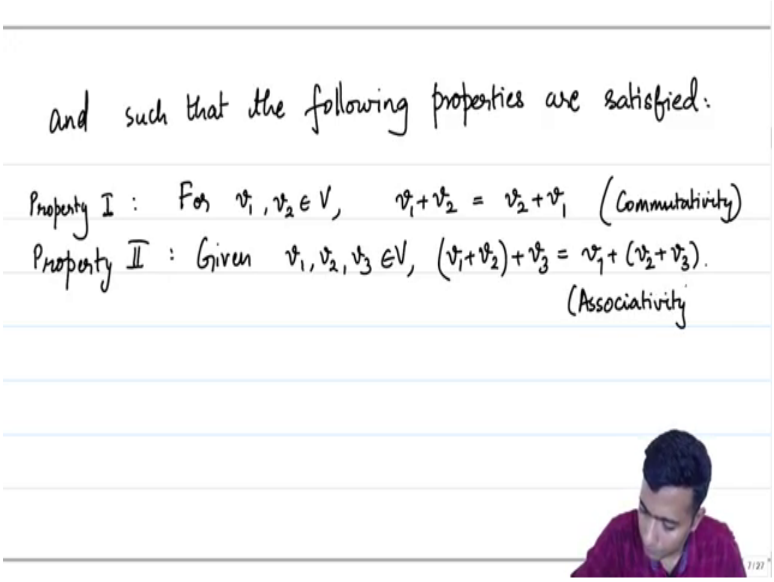

Property one: let me just write it as one here or maybe let me write property one. For v1 and v2 in V, v1 + v2 is equal to v2 + v1. That is one of the properties which we would like to generalize. The second property is again associativity. This is called Commutativity. Next is what is called as Associativity. If you take three elements, v1, v2, v3 in V and you look at v1 + v2 + v3, which one we add first should not matter. Given v1, v2, v3 in V, v1 + v2 if you add them first and then add it to say v3, this should not affect what the answer is when compared to say adding v2 and v3 first and then adding it to v1. This is what is called as associativity.

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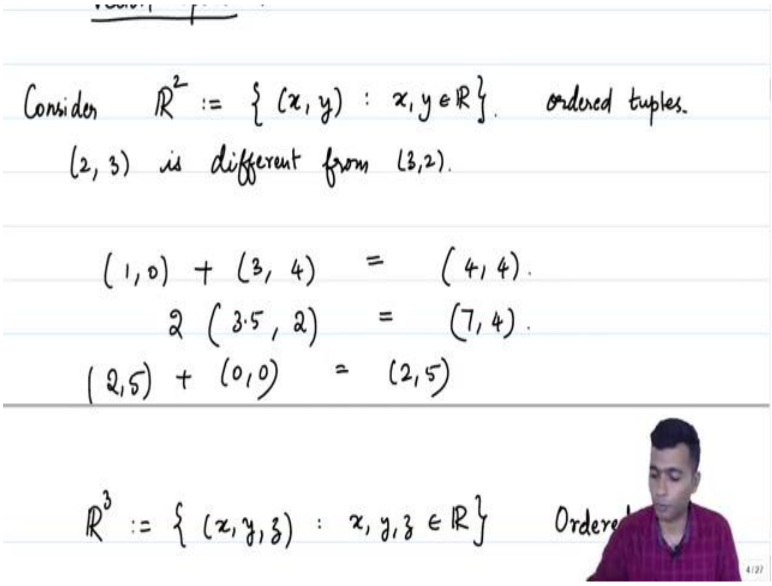

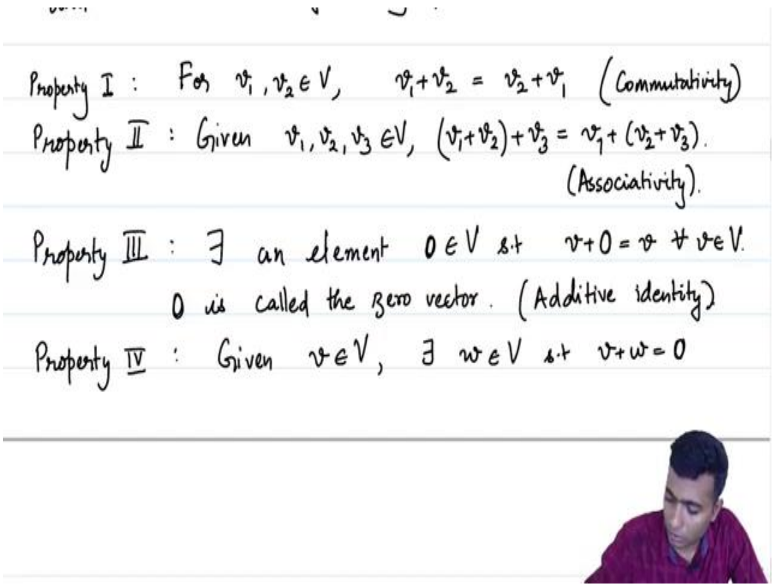

Property three is the existence of a vector like (0, 0) in R2. If you look at our example, any vector, say (2, 5) if you add it to say (0, 0), then you are going to get back (2, 5) itself. Right? And this is satisfied for any element in R2. We would like our vector space to always have one such element. So there exists an element 0 in V such that v + 0 is equal to v for all v in V. 0 is called the zero-vector. This is called, the property is called additive identity, existence of additive identity.

Not just (0,0), R2 has some very nice property that given a vector say (5, 6) if you look at (-5,-6), and if you add it to (5, 6) we get back our zero element or (0,0). This is satisfied for every element in R2. Property four demands that... This is a desirable property which we would like to generalize and property four captures exactly that. Given v in capital W, there exists some w... sorry, V, there exists a w in capital V such that v + w is the zero-vector. This is what is known as the existence of the additive inverse. Alright.

Those are the properties which we would like to generalize when it comes to the addition operation. We have not talked about the scalar (multiplication) operation at all. Right? Scalar multiplication operation. The next few properties will capture what are the desirable properties of scalar multiplications which we would like an arbitrary vector space to satisfy.

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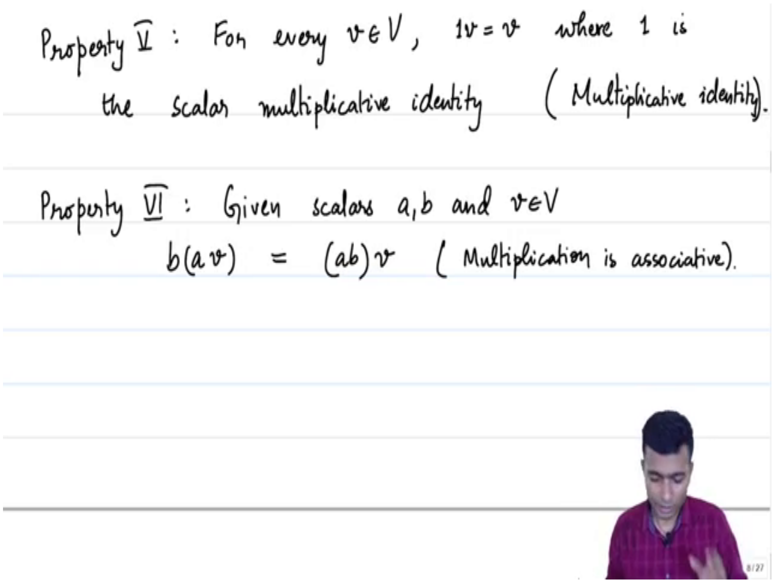

The first one is the existence of a multiplicative identity. For every vector v, every v in V, every element v in V, 1 times v is equal to v where 1 is the scalar multiplicative identity. So, if you take the number 1, if you take the scalar 1 and if you multiply to the vector v, it should always give you back the vector. This is the existence of multiplicative identity.... this property.

Next is the property which captures that the multiplication scalar multiplication is also an associative property, the order in which we look at the scalar multiplication does not matter. So for example, if you take say (2, 3) and you look at 2 times (2, 3) that is (4, 6), and if you then look at 3 times (4, 6), it is (12, 18). Right? But if you started off with (2 3), and if you multiplied 6 to it, 6 times (2, 3) is just (12, 18) directly. The order... whether we did 2 times the (2, 3) and then 3 times the resulting vector, or we directly multiplied the scalars here and then multiplied it to the scalar multiplications the vector it did not matter. The result remained the same. That is precisely what this property tells. Given scalars a, b and an element v in V, let us look at av and then b on av. That means scalar multiplication of a to v and then this scalar multiplication of b to the element av in V. Remember that av is an element in V. Right? This is nothing but... this has to be necessarily, first look at the products of the scalar ab and then do this scalar multiplication of that to our given vector v, this will be the same. Multiplication is associative... scalar multiplication is associative. Sometimes, the word scalar is dropped but it is important to keep in mind that we have two operations and only two operations; one is vector addition and another is multiplication of a scalar to an element or a vector.

Why is it called a vector addition? We will come to that. The elements of a vector space will be called vectors. We have however not finished (defining) what a vector space is. So, what other properties we would like to have in a vector space?

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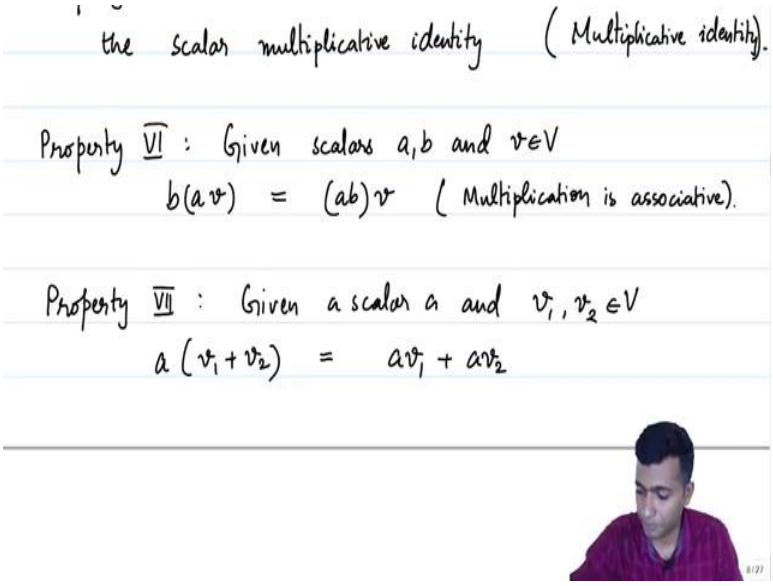

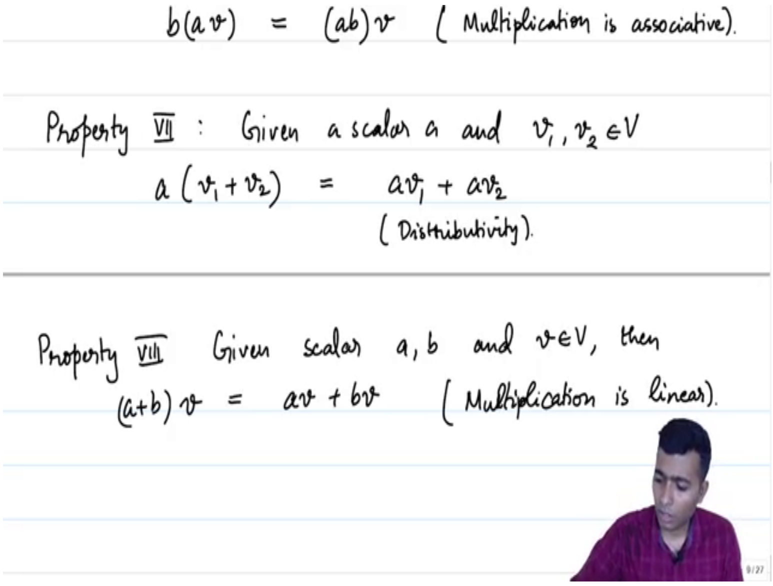

The next property is that the scalar multiplication interacts with the vector addition. This is called the distributivity property, so given scalar a and v1, v2 in V, the scalar multiplication of v1 + v2 this is the same as the scalar multiplication of a to v1 and then we add it to the scalar multiplication of a to v2. Say, for example, take (1, 0) and (0,1) and look at (1, 0) + (0,1) and it is (1,1). And if you look at 2 times (1, 1) we get (2, 2). But rather you look at 2 times (1, 0) and 2 times (0, 1) we get (2, 0) and (0, 2) and if you add them we again get (2, 2). This property says that..... this is the exact property which we just talked about, should generalize to any arbitrary vector space which we would like to define. Okay, so this is a property which is called distributivity.

And finally, the final property states that our multiplication is linear. So, given scalars a and b and let v be in V, then a + b times v is equal to av+bv this multiplication is linear. So, what is this property telling us? So, observe that the one I am circling in green is addition of scalars and the one I am circling in blue is the addition of 2 elements in V. This is a vector addition. Even though we are using the same notation +, the abuse of notation is not going to create any confusion as can be observed from the context. What this says is that suppose you do the addition operation first in the scalars and then multiply it to the vector, or you first multiply it to our vector and then do the vector addition, the answer should be the same. Notice that this is a property which is satisfied in both R2 and R3.

All right, so these are all the properties that our vector addition and scalar multiplication should satisfy. Let us just quickly go over what we have just written down. What is a vector space?

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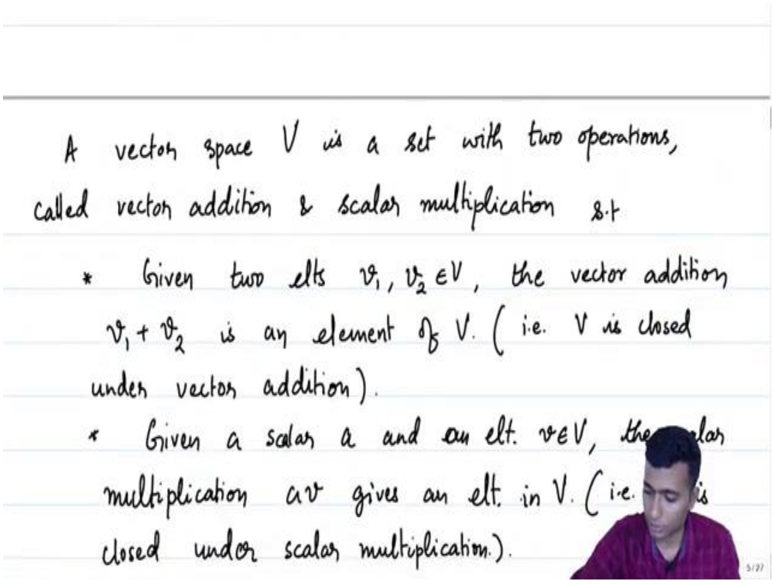

As you can see, it is quite a long definition. So what is a vector space? A vector space is a set V which has two operations a vector addition and a scalar multiplication. The vector addition is such that if you take two elements in the set V, if you add it, you get back an element in V. The scalar multiplication is such that if you get a scalar and if you get a vector, if you get an element v in the set V, then the scalar times the element should give you back an element in the set V itself. So that is how.... that is what is described as saying that V is closed under the vector addition and scalar multiplication.

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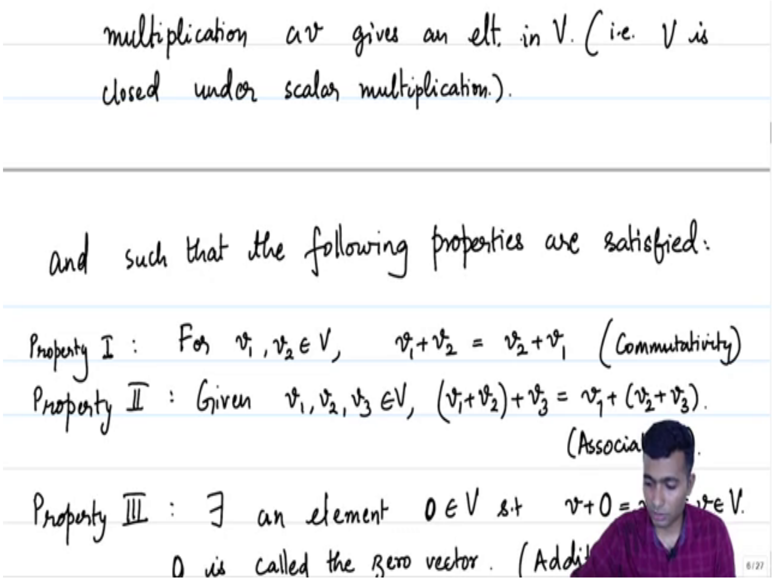

That is not enough, the operations vector addition and scalar multiplication satisfy a list of many properties; commutativity, associativity, existence of additive identity, existence of an additive inverse, existence of multiplicative identity, that multiplication is associative, scalar multiplication and the vector addition distributes over scalar multiplication and that the scalar multiplication is linear. All the eight properties are to be satisfied by these two operations. If we have two operations such that all this is satisfied, then we say that V is a vector space.