Introduction to Probabilistic Methods in PDE
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Lecture - 09
Definition and Properties of Stochastic Integration
Part 01

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- **Definition:**

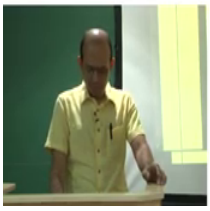
$$\mathcal{L}_0 := \left\{ X \middle| \begin{array}{l} \text{(i)}\ X_t(\omega) = \xi_0(\omega)1_{\{0\}}(t) + \sum_{i=0}^{\infty} \xi_i(\omega)1_{(t_i,t_{i+1}]}(t) \\ \text{(ii)}\ \xi_i \in \mathcal{F}_{t_i} \\ \text{(iii)}\ \sup_{\omega}\sup_{i} |\xi_i| < \infty \end{array} \right\}$$

$\mathcal{L}_0$ is the class of simple processes.

- **Result:** Let X be a bounded, measurable $\{\mathcal{F}_t\}$ adapted process. Then there exists $\{X^{(m)}\}_m \in \mathcal{L}_0$ s.t.

$$\sup_{T>0} \lim_{m\to\infty} E\int_0^T |X_t^m - X_t|^2\, dt = 0 \text{ (see not } d\langle M\rangle_t)$$

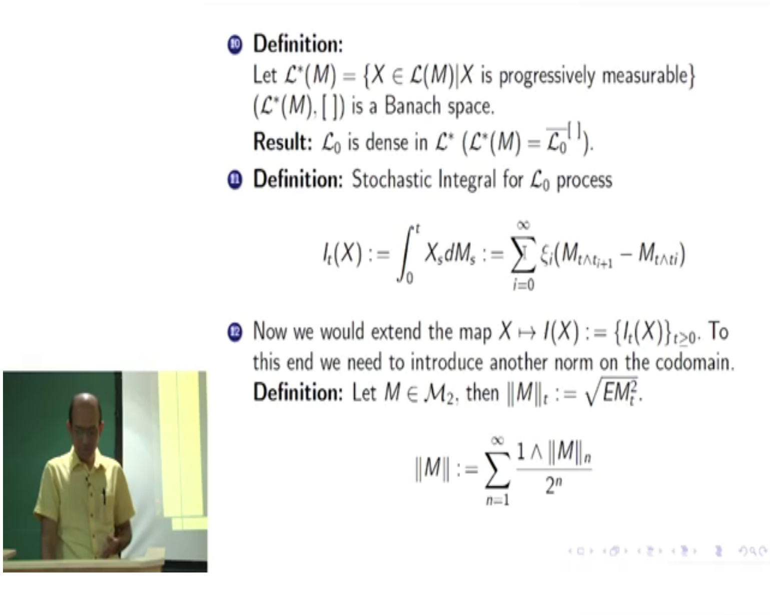
- **Result:** If $t \mapsto \langle M_t\rangle$ is absolutely continuous almost surely, then $\mathcal{L}_0$ is dense in $\mathcal{L}(M)$. ($\mathcal{L}(M) = \overline{\mathcal{L}_0}^{[\,]}$ closure in $[\,]$).

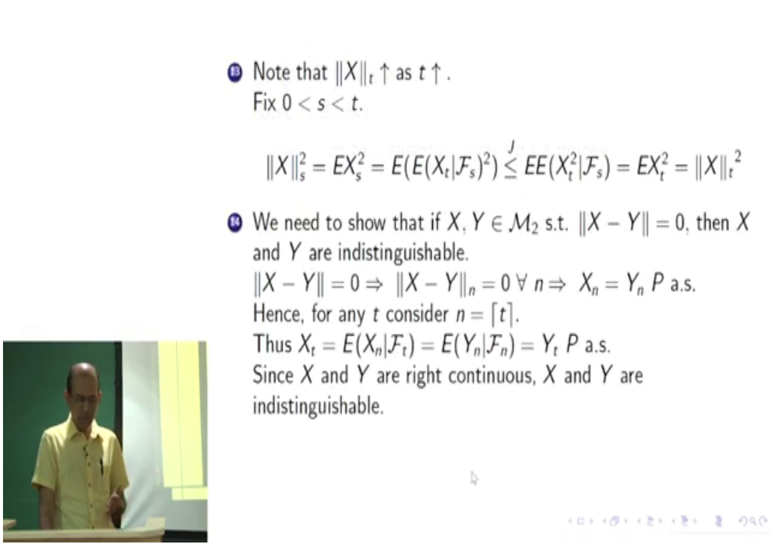


- $\mu_M(A) := E\int_0^\infty 1_A(t,\omega)d\langle M\rangle_t(\omega)\ \forall A \in \mathcal{B}_{[0,\infty)} \otimes \mathcal{F}$.
- Two measurable adapted processes X and Y are equivalent if $X_t(\omega) = Y_t(\omega)$ a.e.$[\mu_M]$.
- For a measurable $\{\mathcal{F}_t\}$ adapted X, define $[X]_T^2 := E\int_0^T X_t^2 d\langle M\rangle_t$, provided this is finite.
- $\mathcal{L}(M) :=$ equivalent classes of measurable and $\{\mathcal{F}_t\}$ adapted process X, s.t., $[X]_T < \infty\ \forall\, T > 0$.
- $(\mathcal{L}(M), [\,])$ is a normed linear space where

$$[X] := \sum_{n=1}^{\infty} \frac{[X]_n \wedge 1}{2^n}.$$

$(\mathcal{L}(M), [\,])$ would serve as the space of integrands for a stochastic integral w.r.t. M.





In the earlier lecture we have seen the space of integrands, okay we have first considered the space of simple processes because that is the easiest thing to integrate that we have defined as is L0 in earlier slides and then we have defined one particular norm this box norm and we have shown that under this box norm, okay, the closer of a L0 is L star M okay.

So, this result I mean we have not proved this we have just quoted this result L0 is dense in L star M. In general, when the quadratic variation process of the martingale although the square integral martingale is continuous, if one has more regularity property for example, the quadratic variation process of the martingale is absolutely continuous then one can actually have that the closer of L0 ease actually L of M. L of M is the space of in right continuous processes.

So, here this you know the definition is there is the equivalence class of measurable and Ft adapted processes, okay. So, we do not even have the right continuity property assumed here. We just have that equivalent classes of measurable and Ft adapted processes X okay such that this you know Ft norm is finite for all t so that one can make sense of this norm X okay.

So, that is the space of integrands and then what we did we considered the space of inte, the we have considered the integration of the of the process of the integrands. So, we have defined, we have started only for first simple process, for simple process this is the direct definition that I t of X is just the sum, okay, and this coincides like kind of you know, riemann stieltjes sense, okay. So, this is the most natural way to define integration of simple processes with respect to a square integral continuous martingale. Then we have proved that this I t x okay this is a martingale and then we have talked about a norm of martingales, okay.

So, this norm is defined here and then we showed that okay if two processes are indistinguishable their norm I mean the difference norm of the difference is 0, okay so this is done.

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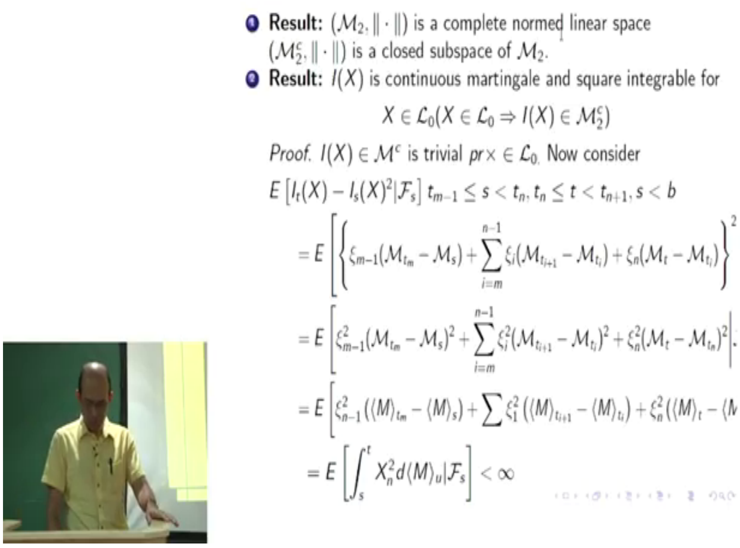

And then we have obtained the space of square integrable martingales with this norm and we have quoted the result that this space is a complete nonlinear space. So, it is the Banach space and a particular subclass of that is the space of square integral continuous martingales. So, that is a closed subspace of this space, okay.

Here we have seen that this integration of x where x is a simple process is a continuous martingale and square integrable and we need to prove that the stochastic integration of X with respect to a square integrable martingale is itself a square integrable martingale and continuous path. So, continuous square integrable martingale.

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- **Result:** $(\mathcal{M}_2, \|\cdot\|)$ is a complete normed linear space. $(\mathcal{M}_2^c, \|\cdot\|)$ is a closed subspace of $\mathcal{M}_2$.
- **Result:** $X \in \mathcal{L}_0 \Rightarrow I(X) \in \mathcal{M}_2^c$

Proof. For a fixed $s < t$, $t_m \le s < t_{m+1}, t_n \le t < t_{n+1}$

$$\begin{aligned} & E[I_t(X)|\mathcal{F}_s] \\ &= E[\sum_{i=0}^{n-1}[\xi_i(M_{t_{i+1}} - M_{t_i})] + \xi_n(M_t - M_{t_n})|\mathcal{F}_s] \\ &= \sum_{i=0}^{m-1} E[\xi_i(M_{t_{i+1}} - M_{t_i})|\mathcal{F}_s] + E[\xi_m(M_{t_{m+1}} - M_{t_m})|\mathcal{F}_s] \\ &\quad + \sum_{i=m+1}^{n-1} E[\xi_i(M_{t_{i+1}} - M_{t_i})|\mathcal{F}_s] + E[\xi_n(M_t - M_{t_n})|\mathcal{F}_s] \\ &= \sum_{i=0}^{m-1}[\xi_i(M_{t_{i+1}} - M_{t_i})] + \xi_m(M_s - M_{t_m}) \\ &\quad + \sum_{i=m+1}^{n-1} E[E[\xi_i(M_{t_{i+1}} - M_{t_i})|\mathcal{F}_{t_i}]|\mathcal{F}_s] + E[E[\xi_n(M_t - M_{t_n})|\mathcal{F}_{t_n}]|\mathcal{F}_s] \end{aligned}$$

So, for the proof we first consider s is a time which is less than t, why do we need to consider two different times? Because to prove martingales, we need to show that conditional expectation of the process at present t given the filtration value at s which is past, is actually the value of the process at past time s, that we need to do.

So, we consider s and t okay. And whenever we fix s and t we can always find out this t1, t2, tm okay so, which is actually the jump times of the process X, okay. The simple processes simple process has discontinuities at some points. In any finite interval it has only finitely many such discontinuities.

So, then we consider that tm is the time point where s is between tm and tm plus 1, tn is the time which is such that t is between tn tn and plus 1. So, m and n are these two integers. Now, we tried to prove the conditional expectation of I t of X given F s is actually I s of X, what is I t of X? I t of X is actually integration from 0 to t of X with respect to M.

So, here what we do is that we rewrite I t X in its simple form. So, I t X is I mean from the very definition it is like a summation I is equal to 0 to n minus 1, why n minus 1? Because t is within tn to tn plus 1. So, here we integrate till the interval t n minus 1 to tn. So, these intervals and then tn to tn plus 1 we do not integrate the full interval but only for tn into t.

So, 0 to n minus 1 for that we get Xi i times Mti plus 1 minus Mti plus Xi n times here we write Mt minus Mtn okay. So, this is the value of I t X. Okay I t X is rewritten here and condition is given with respect to the Sigma algebra Fs. Now, we rewrite the above expression from instead I 0 to n minus 1 I 0 to m minus 1 first. Because you know since s is less than t, so, m is not more than n. So, you can write down this way I is equal to 0 to m minus 1.

So, here this since is finite sum. So, expectation can be taken inside of the submission of Xi i times the increment of the martingale Mti plus one minus Mti given Fs, plus, so, I have here counted only till n minus 1. So, mth one is here. So, Xi m times capital N tn plus 1 minus M t n given Fs okay since expectation is a linear operators we can write down on the summations this way and then from m plus 1 to n minus 1.

So, this one single submission is broken in these three parts okay and the last term is as it is, only thing is that here expectation is taken here. So, this expectation mean of the sum is sum of the expectation that we have used. So, now for this part I is equal to 0 to m minus 1 what we have? We have this all know for all I which is less than equals to m minus 1, this is at most tm, but tm is less than Fs so, these are all Fs measurable. So, conditional expectation of this random variable given Fs is exactly this random variable itself.

So, these, since these random variables are Fs measurable. So, here we get exactly this Xi i Mti plus 1 minus Mti without any conditional expectation. And now for this part, when we take conditional expectation of this, we see that okay Xi m is Fs measurable so Xi m comes out of the expectation. When expectation of Mtm plus 1 given Fs is in s itself due to the property of martingale.

So, we get Xi m times Ms minus Mtm, so, this is Fs measurable as it is. Now, we talk about the last two terms. In this last but one term when I have I is equal to m plus 1 to n minus 1 expectation of Xi i times Mti plus 1 minus Mti given Fs. What we do is that we do another conditioning in between. And we are using the tower property of expectation. So, what we are doing is that, with this inside thing we are write now expectation Xi i Mti plus 1 and minus Mti given Fti.

Since, s is, so what is i here i is m plus 1 to n minus 1, so, i is at least m plus 1 so, that this ti is t m plus 1 or more than that but s is less than t n plus 1. So, this is a finer sigma algebra than Fs. So, we can use this tower property we are writing expectation of Xi i Mti plus 1 minus Mti given Fti, okay? And then condition again, given Fs.

So, this term is rewritten this way. And then we consider the last term, the last term is expectation of Xi n Mt minus Mtn. So, here also we do again conditioning in the same manner. So, here we condition with respect to Ftn. In this terms Xi i is Fti measurable, so I can Xi i outside of this expectation, and then what would be left inside is expectation of Mti plus 1 minus Mti given Fti, but then using the martingale property of M, conditional expectation of Mti plus 1 given Fti is Mti, and then the subtraction Mti is there, so I am going to get 0 here.

For the same reason, I would also get, you know, Xi n, Xi n outside of this expectation. And then here I would get 0 here.

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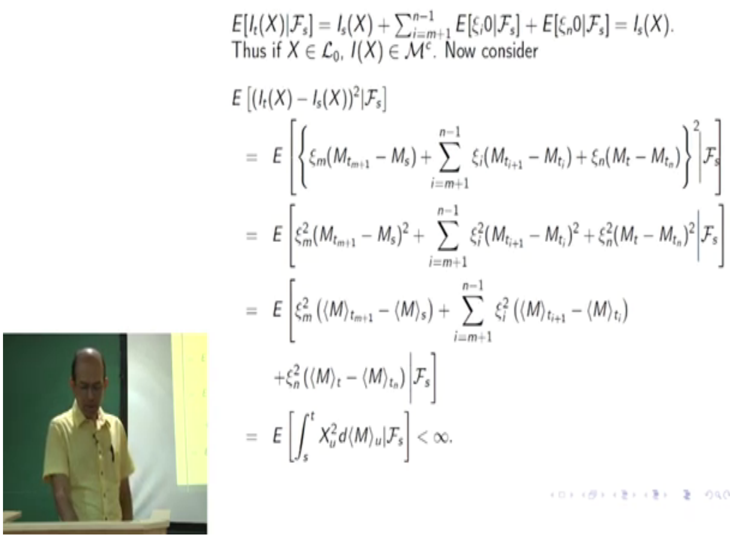

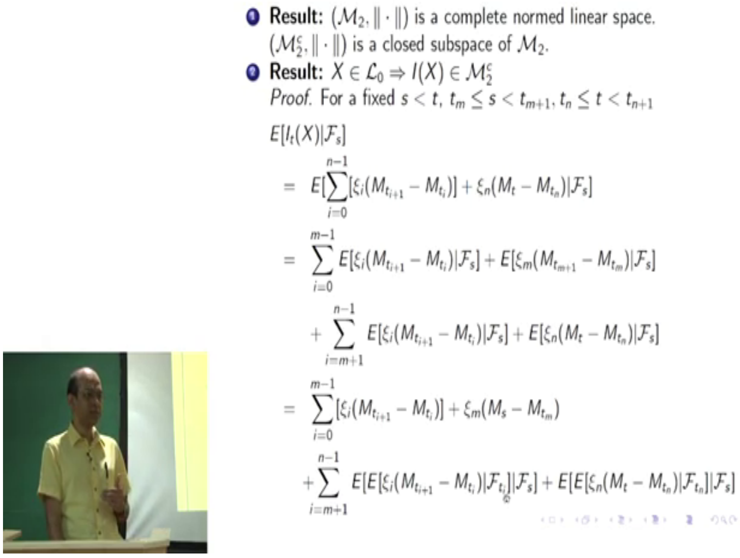

So, in the next slide I write down this thing, right, I am summarizing. So, left hand side, the left hand side is expectation conditional expectation I t s given Fs. So, conditional expectation I t s given Fs, right hand side I have already I s X, why is it so because this sum what we see is nothing but I sX, correct from the very definition. Okay. And then the other two terms is basically the expectation of Xi i and 0 given Fs and expression Xi n times 0 in given Fs, and these are zeros so I am going to get I s X.

So, this proves that the stochastic integration of simple process given a square integrable continuous martingale is itself a martingale. The square integrabolity part is not yet proved that you are going to do now and why is it continuous? The continuous because you know that when we integrate the simple process with respect to continuous integrator at the end we are going to get continuous function itself.

Whenever we integrate a stape function okay so, some you know measure, actually any measurable function you take which is integrable and you integrate with respect to a continuous integrator from 0 to t then the integral as a function of t becomes continuous. Actually it becomes absolutely continuous, fine. So, now we get this integration of the simple process is a continuous martingale.

We need to just show that it is also square integrable that means it is, that I t X minus I s X the whole square this square expectation of this thing is finite. So, conditional expectation of

I t X minus I s X the whole square given Fs that we are again writing in earlier way as you know, using the definition.

So, here, when we let sees this is a difference, so from 0 to that part would be 0, so I am just going to have s to t part so s to t part so there I am going to get Xi m times Mt m plus 1 minus Ms with s and onward okay. And then I is equal to m plus 1 to n minus 1. Okay, so if m and n coincide for some reason they it would be void, okay, nothing will be there.

So, Xi i times Mti plus 1 minus Mti plus Xi n times Mt minus Mtn, so that is coming from the direct definition of I t and I s so, so this is done and square is written here given Fs. Now what are we doing is that we are writing down we are finding out this square the square I mean square of sum is not sum of squares correct, because you are going to get to cross, cross terms also. But here for this case, what you are going to get is that expectations of these cross terms, interestingly are like differences of the martingales or the increment of the martingales.

Okay, and their increments are disjoined increments, okay. And then you can again further condition exactly in the I mean similarly, as in the last slide, you can again condition in between, to take one of the product term outside of that conditional expectation and then remaining part becomes just the increment of the martingales and that expectation is 0.

So, that would if you do that for each and every such cross, cross term then you are going to get exactly that, that this is square of this sum is sum of the squares. So, Xi m square times Mt m plus 1 minus Ms whole square plus summation i is equal to m plus 1 to n minus 1 Xi i square Mti plus 1 minus Mti whole square plus Xi n squared times Mt minus Mtn whole square.

So, after this now we are going to use the Doob–Meyer decomposition result and the very definition of quadratic variation. The definition of quadratic variation is that if we are martingale you take it square it, and then look at the Doob–Meyer decomposition where you have a martingale part and the remaining part is the quadratic variation.

So, now if we do that, then then what are you going to get? We are going to get that expectation of this thing this, this square of the martingale would be same as expectation of the martingale part in the Doob–Meyer decomposition that will be 0 again and then the expectation of the of the quadratic variation that you are going to get.

So, here we are not doing it directly because you know, you have to first you know, condition in tm time, okay. And then, then you have that martingale square, and then that expectation of square or conditional expectation of the square of this difference is same as expectation of the martingale, not this martingale, but the martingale which comes to the Doob–Meyer that be 0 plus this quadratic, difference of the quadratic variation so that appears here.

For the same reason for all of this term, we are going to get exact the difference of the quadratic variation. So, ti plus Mti plus 1 minus Mti whole square is, from there we are going to get the quadratic variation of Mti plus 1 minus quadratic variation Mti. This is happening because of, I mean, I am not saying that this is equal to this, but I am saying that okay, after applying conditional expectation, you would get that this is equal to this thing.

So, we can get for all these three terms, these three terms. However, now these three terms are showing that if I integrate the indicator for I mean you know, the simple process, which is Xi m on the interval tm, to tm plus 1, etc, as Xi i is and it is Xi i at ti to t i plus 1. If we integrate the square of this process with respect to the quadratic variation process, then these are the increment of the integrator okay these are integral, integral and these are the integrands okay on that intervals.

So, this is nothing but the integration from s to t of this simple process Xi i Xi square u d Mu given Fs okay. So, that is a proof that this conditional expectation of the square of the increment of the integral is expectation of integration of the square of the integrand with respect to the quadratic variation okay.

So, let me repeat what it is, it is expectation of the square of the increment of integral and it is a expectation of integral of the square of integrant with respect to here quadratic variation. Here you had the integration this is a martingale here it is with respect to the quadratic variation. However, this term is from our assumptions is finite.

Because what it is it is the just the box norm. So, we have considered x to be in the L0 so that means you know, it is assumed that this is finite. So, at the end what we have achieved, we had achieved that way this squares the expectation of this square is finite. So, that or in other words we have proved that okay this stochastic integration is square integrable, it's expectation is finite.

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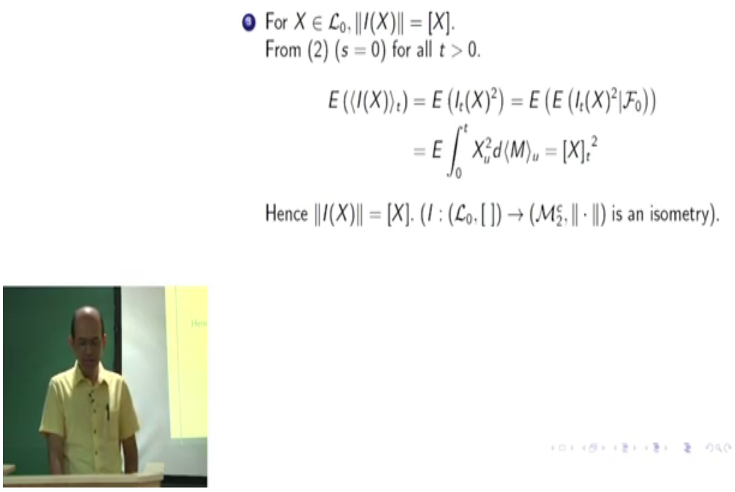

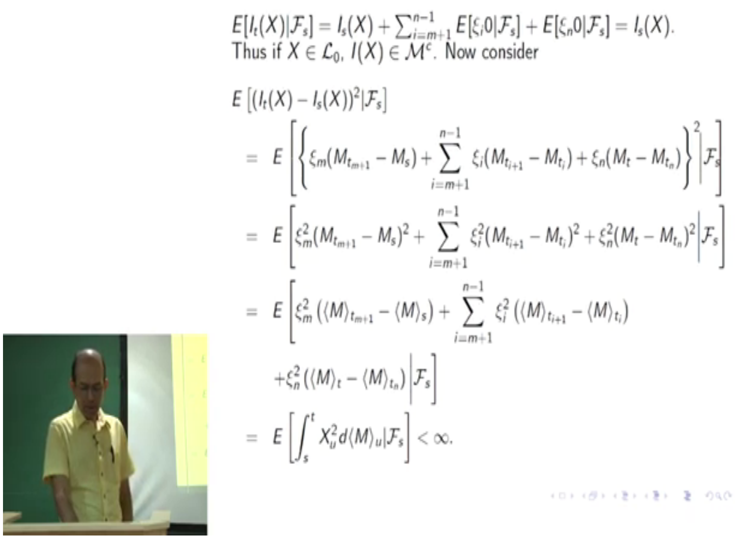

Now, for X in L 0 that means simple process, we consider norm of the integration norm of I X. So, that is we proved here that is nothing but the box norm of X. It is important to understand that this X is integrands correct and I X is integrals okay I X is a martingale but X is just an adapted process okay and the box norm so these are two different spaces and these two different norms are defining in separately okay.

However, of course, I mean the norm the box norm of X is defined using the martingale itself the same martingale which appears in the definition of IX okay so, that is the connection

Okay. Now, if we prove that these two norms are same, then what can we achieve we can say that okay these two are isometry.

So, that I is an isometry these two are isometrical. So, here we start in this manner. So, whatever we have obtained here we put s is equal to 0. So, then we just do have I t X on the right hand side and left hand side so left hand side I have just I t X whole square and I this is 0 and here you have also right hand side just the square norm of X.

So, here that that thing I am written it more lucidly, so, expectation of I X t this quadratic variation is equal to expectation of I t X whole square and that is equal to again we take condition with this respect to F 0 and then using the earlier result, this square is nothing but expectation of integration of I X u square dMu and that is by definition Xt square.

And this term is that I X t. So, for every n, every n this is true n is 1, 2 , 3 integers and then the definition of norm of I X is based on the values, these values for every integer 1, 2, up to n then we take the minimum that with 1 and divide it by n and sum over all possible n to get this norm. So, if these are matching for each and every t therefore, these two would be equal.

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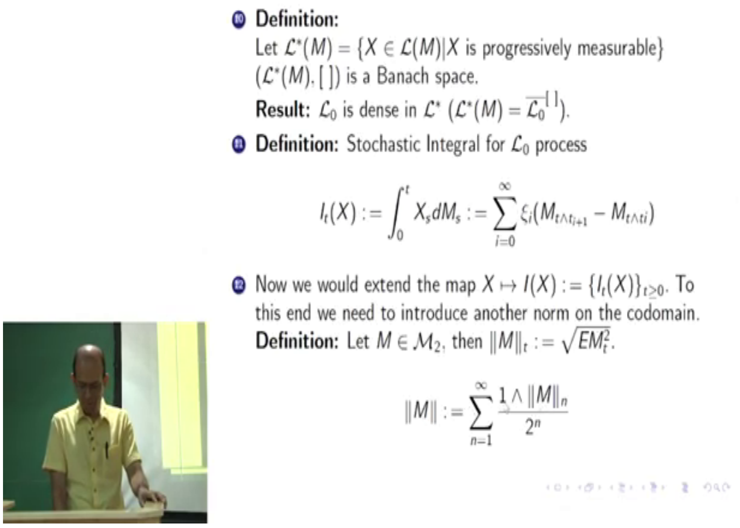

So, this box this norm is 1 minimum norm at n and this is if you put n is integer so expectation of Mt square okay so square root of that.

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- Two measurable adapted processes X and Y are equivalent if $X_t(\omega) = Y_t(\omega)$ a.e.$[\mu_M]$.
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- $(\mathcal{L}(M), [\,])$ is a normed linear space where

$$[X] := \sum_{n=1}^{\infty} \frac{[X]_n \wedge 1}{2^n}.$$

$(\mathcal{L}(M), [\,])$ would serve as the space of integrands for a stochastic integral w.r.t. M.

- For $X \in \mathcal{L}_0$, $\|I(X)\| = [X]$.
  From (2) $(s = 0)$ for all $t > 0$.

$$E(\langle I(X)\rangle_t) = E\left(I_t(X)^2\right) = E\left(E\left(I_t(X)^2|\mathcal{F}_0\right)\right)$$
$$= E\int_0^t X_u^2 d\langle M\rangle_u = [X]_t^2$$

  Hence $\|I(X)\| = [X]$. ($I : (\mathcal{L}_0, [\,]) \to (\mathcal{M}_2^c, \|\cdot\|)$ is an isometry).
- For $X \in \mathcal{L}^*(M)$, $\exists$ a sequence $\{X^n\}_n$ in $\mathcal{L}_0$ s.t.

$$X^n \to X \text{ in}[\,] \ (\text{as } \bar{\mathcal{L}}_0 = \mathcal{L}^*)$$

  Now we use such sequences to define stochastic integral of X w.r.t. M.

$E[I_t(X)|\mathcal{F}_s] = I_s(X) + \sum_{i=m+1}^{n-1} E[\xi_i 0|\mathcal{F}_s] + E[\xi_n 0|\mathcal{F}_s] = I_s(X)$.
Thus if $X \in \mathcal{L}_0$, $I(X) \in \mathcal{M}^c$. Now consider

$$
\begin{aligned}
& E\left[(I_t(X) - I_s(X))^2|\mathcal{F}_s\right] \\
&= E\left[\left\{\xi_m(M_{t_{m+1}} - M_s) + \sum_{i=m+1}^{n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n})\right\}^2 \middle| \mathcal{F}_s\right] \\
&= E\left[\xi_m^2(M_{t_{m+1}} - M_s)^2 + \sum_{i=m+1}^{n-1} \xi_i^2(M_{t_{i+1}} - M_{t_i})^2 + \xi_n^2(M_t - M_{t_n})^2 \middle| \mathcal{F}_s\right] \\
&= E\left[\xi_m^2\left(\langle M\rangle_{t_{m+1}} - \langle M\rangle_s\right) + \sum_{i=m+1}^{n-1} \xi_i^2\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}\right)\right. \\
&\qquad \left. + \xi_n^2\left(\langle M\rangle_t - \langle M\rangle_{t_n}\right) \middle| \mathcal{F}_s\right] \\
&= E\left[\int_s^t X_u^2 d\langle M\rangle_u \middle| \mathcal{F}_s\right] < \infty.
\end{aligned}
$$

So, and on the other hand box norm is defined here, here also, so after this you know defining nth one and then both have a similar formula. So, we can say that okay therefore both are equal. Now, for assume that you have one X in L star M the progressively measurable process okay that is X and there exists a sequence we want prove the sequence X n I mean you know we have already seen because L0 closer is L star M so of course we can find out one sequence X n in a L0 such that X n converts in X in box norm.

Because this L0 is dense in L star M. Okay before going to this part, let me again clarify that that this part okay this you know this statement is often referred to as Ito's isometry okay Ito's isometry. Mainly Ito's isometry is quoted for a special case when martingale is chosen to be a Brownian motion. However, this is what I mean grossly stated as Ito's isometry. So, now here X n, we choose from the space of simple processes, which converges to a given X in the box norm.