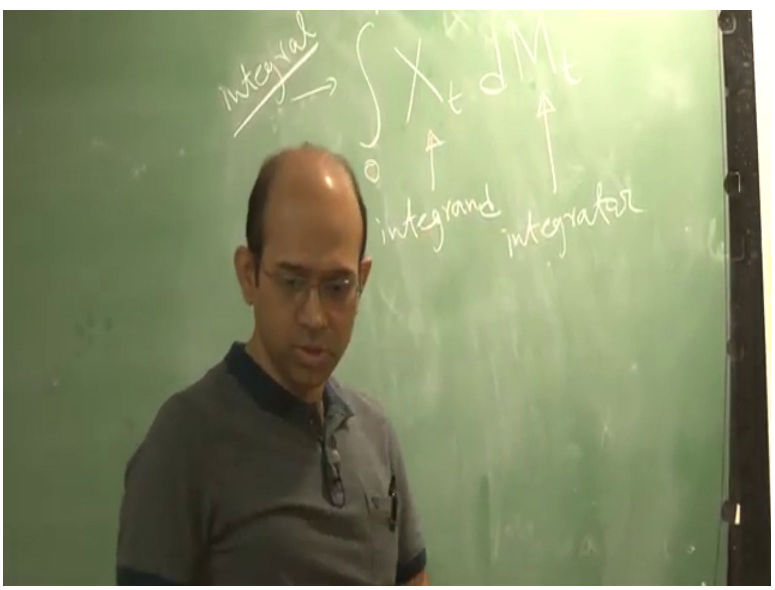
Introduction to Probabilistic Methods in PDE
Professor Dr. Anindya Goswami
Department of Mathematics
Indian Institute of Science Education and Research, Pune
Lecture - 08
Preliminary for Stochastic Integration
Part 02

So, till now we were talking about mostly the martingales, correct? Because martingales and its quadratic variations. Now, these martingales would constitute to be your integrator because that is the main purpose but we would like to integrate some integrand, some process but every such process, every you know arbitrary process would be integrable with respect to a given martingale is too ambitious plan, so that we might be not able to do.

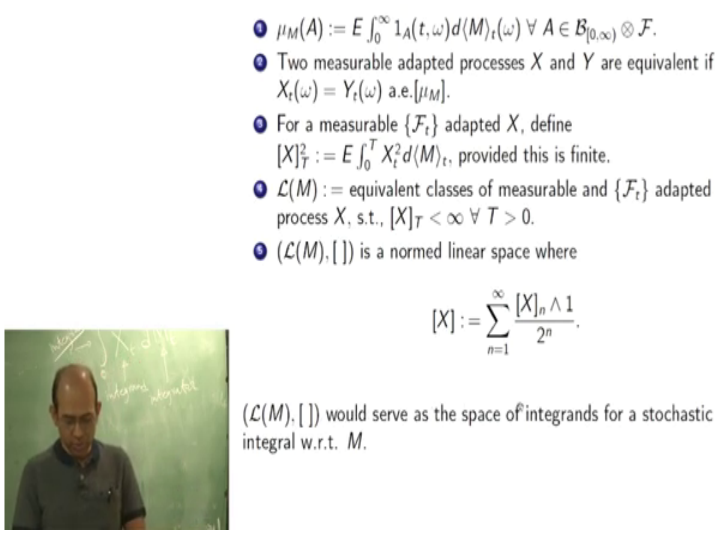
So, we should start thinking about what are the processes which can be integrated with respect to the martingales. So, we are now going to look at the space of integrands, what are the suitable space of integrands one should look for. So, let me all, since I am talking to much maybe I should keep this thing written somewhere.

(Refer Slide Time: 1:18)



So, we are talking about integration of Xt dMt 0 to T so this is integrand this is integrator and this whole thing is integral. This whole thing is integral, this whole thing is integral. So, now, I mean till now I was talking about this martingales, etcetera now I will focus on integrands. Basically you have to talk about space of integrands and appropriate norms, etcetera.

(Refer Slide Time: 2:14)



So, let us consider a measure that is on the set of a time and omega. So, we consider A to B a Borel set from the product sigma algebra of Borel sigma algebra on the time axis, so 0 to infinity and a F on omega. So, we take such A because for process I have two variables t and omega. So, A is actually you know set of t omega points but a measurable set with respect to this sigma algebra.

So, for such A mu M of A is defined in the following way, why mu M? Because given M, so after fixing M we consider this measure mu M A. So, that is defined as expectation of integration of integral function of A with respect to the quadratic variation of M. So, when I fix M, I have fixed quadratic variation of M, quadratic variation of M is an increasing process so I can define integration of something simple something easy things with respect to a increasing process.

And, so how do we understand this integration, I would think that, I would fix omega first. If I fix omega then dMt is an increasing process and then for that fix omega this 1 At is also a function, a function of taking only 0 and 1 value basically an indicator function. Since A is measurable so section, omega section of A is also measurable. So, this function is measurable and since bounded so and this M is need not be you know bounded.

However, so, but it is increasing and this is non negative so it can be infinity sometime does not matter, we can still give the notion of integration, so is measurable function non-negative measurable function we can define this the I mean it have the it has usual meaning of this integration 0 to infinity where time is moving.

So, now for every omega we are getting a number which is real number, positive real number, otherwise it is possibly plus infinity. But then we take expectation. So, without before taking expectation this is it behaves like a random variable. So, this is like a Fubini's theorem kind of thing you can say that if omega section so then this integration is also measurable with respect to omega. So, measurable function omega so this is a random variable.

Now, we take expectation basically a repeated integration. So, here that second t is doing the integration with respect to P measure. So, then whatever you are going to get is surely a non-negative value and it could be positive number or plus infinity so this would be a non-negative external real number valued. Valued map. So, mu A defined this way clearly satisfies all the conditions for this to be a measure on this sigma algebra.

So, this mu M we consider and then we use this mu M, actually this mu M is used only at this place to define an equivalence relation we call that X and Y are equivalent if X and Y are equal almost every mu M. If they are almost equal, almost everywhere with respect to the measure mu M then we call X and Y are equivalent.

So, now we agree that here this equivalence is depending on M after fixing the martingale M then we talk about that the space of integrands and their equivalence. So, this is very important thing to remember. So, for a measurable Ft adapted X, so assume that we have stochastic process X which is Ft adapted and measurable.

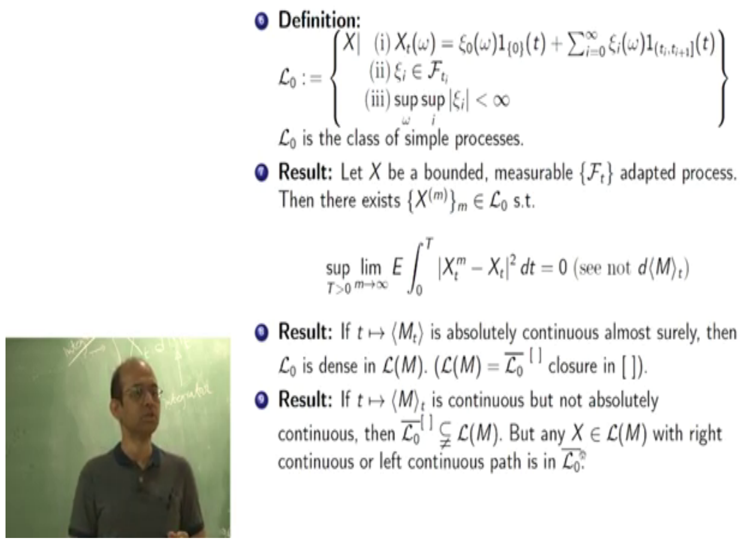
So, define this norm so we call that the square bracket X T square is equal to expectation of integration 0 to T Xt square D quadratic variation of M. So, we take integration of X square with respect to mu M and probability measure. This is basically with respect to mu M and probability measure, two measure, so repeated integration. So, that is Xt square so if you take square root both sides then you get square bracket X subscript t. So, definition is such that it is a norm.

Now, we define this class we denote it by the script L of M so given M. We define this class of integrands, so we say equivalent classes of measurable and Ft adaptable process X is such that we get this norm finite for every positive T. So, earlier remember that we found out this by fixing a capital T but when you talk about integration we do not need to just consider a finite horizon, finite time horizon. So, for every positive T we take if this is finite then we include that in this space L of M.

Now, we undo L of M with a particular norm so this, this you know square bracket t norm is insufficient, is not a appropriate because this is appropriate only when time only 0 to T is considered capital T is considered. So, here we define this particular manner that X box n so this you know square bracket n minimum 1. So, if that number, this value is more than 1, we replace that by 1 and divide by 2 to the power n and n is running from 1 to infinity this sum, this infinite sum, this infinite sum of course converges because the numerator is always less than or equals to 1 and denominator is 2 to the n. So, this is dominate by geometric series.

So, this value whatever you are going to get, we are going to call that is the norm of X. So, this L M box this norm would serve as the space of integrands of for a stochastic integral with respect to a given M.

(Refer Slide Time: 9:22)



Next, we define so we cannot actually straight away define this integration for X coming from L of M. We cannot do that we need to build it step by step so we starts from the very fundamental and simple thing. So, process which looks like this like step function, correct? So, let us read it what is it.

For every omega this is like you know that xi i of omega would be a real number and then you are considering open (t_i, t_{i+1}]. So, this is a step function and i is equal to 0 to infinity but it has possibly infinitely many. So, it is a step function basically has finite many things. So, it is not a step function in that sense, but it is, it has only countably many discontinuities and you have this you know process which you call simple process.

And then for omega dependency we must assume some sort of measurability condition what is that? We look at it that for every t between ti to ti plus 1 the process X picks up the value xi i. So, if I need to put some measurability condition I should put measurability condition on xi i, so we required xi i to be Fti measurable.

So, then as a result what I am going to get is the Xt is Ft measurable, why? Because Xt is always the t is within some ti to ti plus 1 and that is Fti measurable and Fti is sub sigma algebra of Ft so we get Xt is Ft measurable as a result of this. xi i is Fti measurable and then third thing is that we require xi to be a bounded random variable.

So, sup over all i n sup over all omega is finite. So, we just need xi to be bounded random variable. So, I mean nothing simpler than this can be considered as an integrand although this is a very small class but we are going to see it is not that small, it is actually dense in a very large class.

So, one can approximate a useful integrand using members from this L0 class. But in which sense and which way that we are going to describe in the following two slides. So, L0 is called the class of simple processes. So, we write down this result let X be a bounded measurable Ft adapted process and then there exist a sequence of Xm from L0 such that you would get this expectation goes to 0 as m tends to infinity.

So, what is this expectation? This is integration of Xm minus mod of Xm minus X whole square dt and time t is running from 0 to capital T but we are also taking supremum over all possible capital T. So, in this consideration the martingale did not playing any role, so we are not integrating with respect to the quadratic variation of M, we are only with T.

So, here for L0 also we did not put any condition, it is not depend on M, remember earlier we choose L of M so depending a martingale we define what is L of M. But L0 is like universal you do not care anything you just have a filtration, with the given filtration you come up with L0. And since this is a result of a L0 so we appreciate this, this result that here we get that mod of X so we can, we can approximate.

So, this basically says that so in this square this is actually L2 norm so because you know we are taking square and integrating with respect to time and then integrating with respect to probability measure. So, this is L2 norm in lambda cross P where lambda is the lebesgue measure on to time. So, it is saying that L2 norm is but here supremum capital T greater than 0 is also involved here but if you fixed capital T that means you are just considering only finite horizon case then it is just saying that okay, when we define horizon case every bounded measurable Ft adapted process can be approximated by a simple process from here. So, that is a nice result.

Now, we proceed further if we have M square integrable martingale such that it has a nice property. What is nice property? There is a typo this T should be outside, so if t to <M>t quadratic variation of Mt this map is absolutely continuous if it is absolutely continuous almost surely, why almost surely?

Because this map is also depends on omega for every omega you get different different maps. But you observe this map to be absolutely continuous with probability 1. That I mean by absolutely continuous almost surely. Then L0 is dense in L of M so what is the bottom line? The bottom line is that I cannot assure L0 to be dense in L M for any arbitrary M only if M satisfies some nice property.

Say M is such that its quadratic variation process which is increasing process is also absolutely continuous. One can just ask any anlysis student would know that there is a very pathological example which is increasing continuous function but not absolutely continuous function. You know it cantor ternary function so one can define that cantor ternary function which is increasing continuous but not absolutely continuous.

So, of course there are plenty such process of function which might not satisfy this condition but if such condition is true that it is absolutely continuous then we know that this result says that L0 is dense in LM. Or in other words, if you take closer of L0 in this norm then LM is this L0 closer.

Now, the other side if t to <M>t this map is need not to be absolute continuous but continuous, it is just continuous then what can we say at most or at the best? Then this closer of L0 in this norm, in this norm may not be equal but of course you know subspace of LM but any X if it is in LM with right continuous or left continuous path that is in L0 bar, so closer of L0. What do I say here?

I say here that these integrands which has nice continuity property like you know it is either right continuous or left continuous then that can also be approximated using L0 bar.

(Refer Slide Time: 17:50)

10. **Definition:**
    Let $\mathcal{L}^*(M) = \{X \in \mathcal{L}(M) | X \text{ is progressively measurable}\}$
    $(\mathcal{L}^*(M), [\,])$ is a Banach space.
    Result: $\mathcal{L}_0$ is dense in $\mathcal{L}^*$ ($\mathcal{L}^*(M) = \overline{\mathcal{L}_0}^{[\,]}$).
11. **Definition:** Stochastic Integral for $\mathcal{L}_0$ process

$$I_t(X) := \int_0^t X_s dM_s := \sum_{i=0}^{\infty} \xi_i (M_{t \wedge t_{i+1}} - M_{t \wedge ti})$$

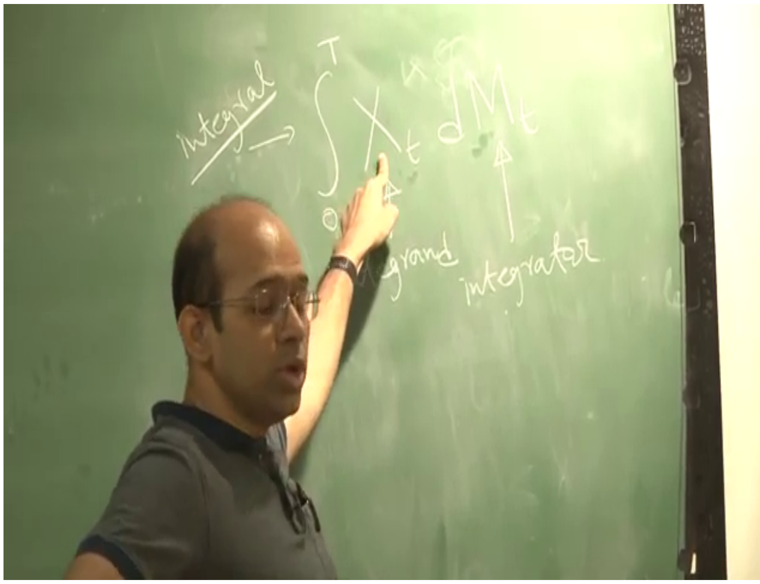
6. **Definition:**

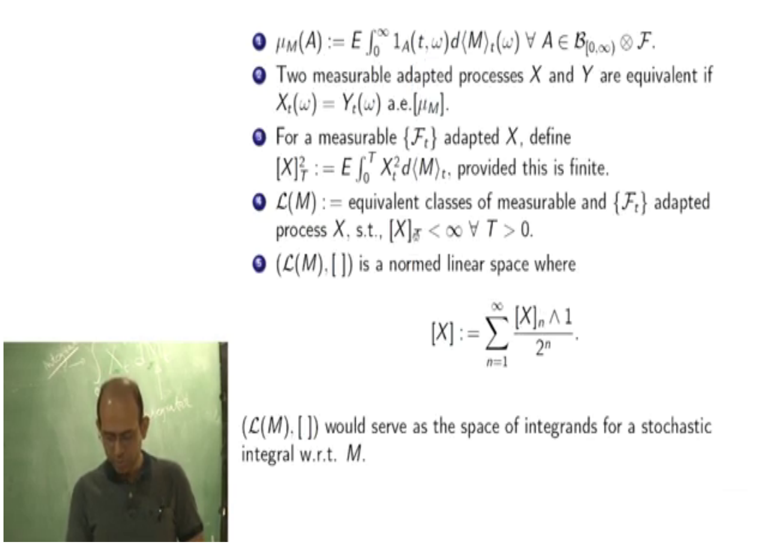
$$\mathcal{L}_0 := \left\{ \begin{array}{l} X| \quad \text{(i)}\ X_t(\omega) = \xi_0(\omega) 1_{\{0\}}(t) + \sum_{i=0}^{\infty} \xi_i(\omega) 1_{(t_i, t_{i+1}]}(t) \\ \quad\quad \text{(ii)}\ \xi_i \in \mathcal{F}_{t_i} \\ \quad\quad \text{(iii)}\ \sup_{\omega} \sup_{i} |\xi_i| < \infty \end{array} \right\}$$

   $\mathcal{L}_0$ is the class of simple processes.
7. **Result:** Let X be a bounded, measurable $\{\mathcal{F}_t\}$ adapted process. Then there exists $\{X^{(m)}\}_m \in \mathcal{L}_0$ s.t.

$$\sup_{T>0} \lim_{m \to \infty} E \int_0^T |X_t^m - X_t|^2 \, dt = 0 \text{ (see not } d\langle M \rangle_t)$$

8. **Result:** If $t \mapsto \langle M_t \rangle$ is absolutely continuous almost surely, then $\mathcal{L}_0$ is dense in $\mathcal{L}(M)$. ($\mathcal{L}(M) = \overline{\mathcal{L}_0}^{[\,]}$ closure in $[\,]$).
9. **Result:** If $t \mapsto \langle M \rangle_t$ is continuous but not absolutely continuous, then $\overline{\mathcal{L}_0}^{[\,]} \subsetneq \mathcal{L}(M)$. But any $X \in \mathcal{L}(M)$ with right continuous or left continuous path is in $\overline{\mathcal{L}_0}$.





Now, we see some more definitions, so L star M, what is L star M? L star M is nothing but you know set of all X in L of M such that X is progressively measurable so what is the use of L star M? Basically here we just could assure that it is just a subset but we could not categorize what it is?

So, next slide we are actually categorizing what it is. So, L star M we may consider as X is progressively measurable that means you put some more conditions on X. So, X is in L of M is one condition so it is in this large class but you put a additional measurability condition on X that is it is progressively measurable. Then L star of M with this squared bracket norm is a Banach space so and it is you know subspace of LM close subspace of LM and L0 is dense in L star.

So, this L0 closer is nothing but L star M. So, if we need to define integral for an M whose quadratic variation need not be absolutely continuous with respect to time then I should consider integrand not from LM but from L star of M, why? Because if I consider for L star M then I can approximate this using that L0 space and perhaps that would allow me to give a meaning to this integration but I have not come to that point yet. I would come to that soon.

So, now we define what do we mean by stochastic integral for L0 process. First we define that and then we are going to extend this notion for the larger class, we are going to extend this map. So, this is a functional, because I_t so we write down this I_t so given X and of course M is already fixed.

So, we define I_t of X for every positive t in the following manner. This is a integration 0 to t, this is a notation we write down this way integration 0 to t Xs, dMs this notation. What does this notation mean? It means that, means this, this summation this is xi i Mt minimum ti plus 1 minus Mt minimum ti. Why does it come?

It comes if we just consider this integration and we see that in an interval of time ti to ti plus 1 xi i is fixed and for that if I integrate such constant random variable with respect to a martingale Mt I do not know what should be the notion of the integration but whatever it is, if it does not give me that xi times M ti plus or minus Mt i then that notion I would not call as integration.

Because for integration of a constant with respect to integrator should be difference of the integrator multiply with the constant. So, from that point of view I should consider integration of such process with respect to Mt as Mti plus or minus Mti time xi i and sum of all xi i so that is only natural notion of integration so we are accepting that notion and we are fixing that notion.

So, integration of this is defined to be this is not a result, you are not proving it is a defining, you are defining this to be the sum of xi i Mt minimum ti plus 1 minus Mt minimum ti so where does it belong? What is it? So, let us see now we would extend this map so here first we must understand that if we take such kind of norm so here M, I mean this norm of M for a fixed t given as square root of expectation Mt square this is nothing but L2 norm.

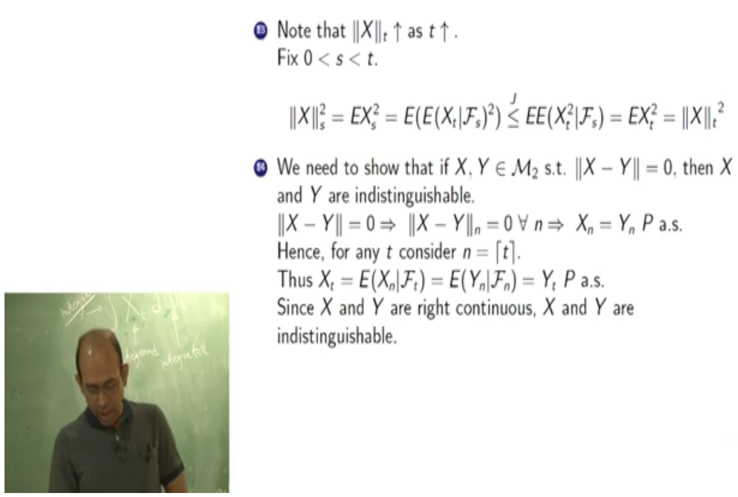
So, for a fixed t L2 of P L2 P norm in the probability measure so square of expectation of square and then square root. So, this norm we are going to talk about for fixed t and then for the whole time horizon we are going to take only t which are natural numbers why should it be sufficient that we are going to see later.

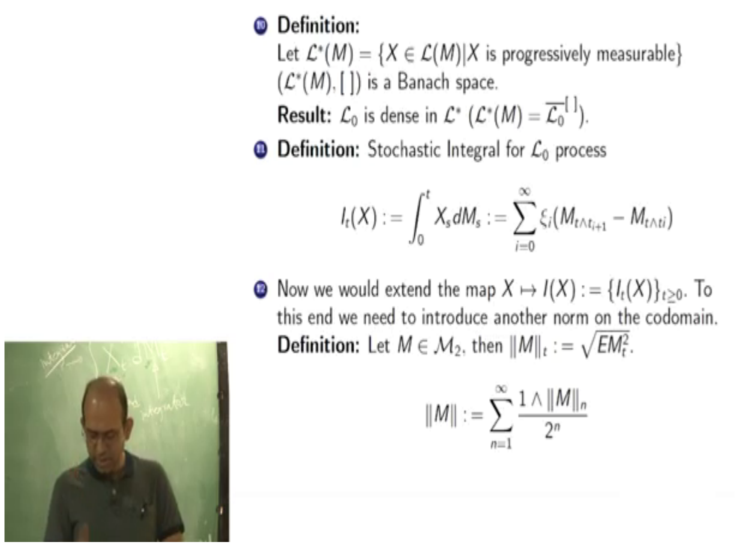
So, because you know here this norm when we consider we are not looking at what happens to the process between 0 to T, here this came only for the process at a time T so for each and every uncountable difference T there are many informations. So, when you consider this one can think that many other information are lost, we are not considering those but that is not the case because we know M is also martingale so from that we are going to get some inherent relationship between times.

So, this way I am emphasizing it because one may get confuse what is difference between this way, so here in this way when you write down capital T that took care of the process X from 0 to capital T. So, this norm actually captures the behavior of X for the throughout the time throughout this time.

But here it apparently it is not, apparently so supercially it is not. But actually it does so that I am going to show later. So, now we would extend the map X to Ix to, to decide what we need we need to introduce the norm. So, here M is like you know square integrable martingale and then we know that this is finite expectation Mt square is finite so this is a finite number and we can define this norm of F in this manner because these are all you know numerator is less than or equal to 1 so this it is converges.

(Refer Slide Time: 24:38)





Note that this Xt is increasing as t increases this is the first thing to note, why? Because the reason is that Mt is martingale, Mt square is, square of Mt square is a convex function so convex function applied to a martingale gives you sub martingale, a sub martingales expectations never decreases keeps on increases.

So, that shows that this M subscript t the norm M subscript t is never decreasing so ever increasing. So, that is the first thing to observe. Now, fix s and t such that s is less than t, s is positive. So, now norm of Xs square so that is by definition is expectation Xs square and that I can write down Xs I can replace by a conditional expectation Xt given Fs because you know X is a martingale.

So, this would be expectation outside but inside conditional expectation Xt given Fs but square is there then I apply Jensen's inequality to the conditional expectation then I get less than or equal to sign, why? Because convex function of average is less than or equal to average of the convex function.

So, here expectation of, expectation of Xt square given Fs so this double expectation is give me all expression Xt square and that is norm of Xt square so what we get this then norm of Xs square is less than or equal to norm of Xt square. So, this is also the same thing as noted before like we just have martingale it is proved here also.

We need to show that if X and Y are in M2 the square integral martingale such that X and Y are close like you know norm of difference that is 0, norm of X minus Y is 0. Then X and Y are indistinguishable so this is something we need to establish if you do not establish this then we do not know that what is the equivalence class to fix so that this I mean double norm whatever we have introduced is really a norm.

Because it is apparently it is not clear because it is just considering value of M at some integer points, not taking in between so in between what happens it is not showing anything but what if that way, that norm is showing 0 but the process is not 0. So, then it is not a norm. So, for that we need to understand this part.

So, assume that norm of I mean this I am saying norm because it would be proved but it is still not yet proved. Norm of X minus Y is equal to 0 implies that this nth norm of X minus Y is 0 for every n from the definition because if this is 0 that means each and every term is 0, so every Mn is 0.

So, every this thing is 0 for every n what does it mean? That Xn is equal to Yn almost surely so with probability 1 they are exactly equal. So, then what we do is that for a every for a given t we consider a next to it, the smallest greater integer, the ceiling function of t. So, thus Xt can be written as conditional expectation of Xn given Ft.

But Xn and Yn are equal almost surely so this is same as expectation of Yn given Ft so this is a typo, this should be Ft, and that is since Y is also a martingale so we are going to get this as Yt so we are establishing Xt is equal to Yt almost surely. So, for every fixed T Xt and Yt are matching with probability 1.

So, that means X and Y are modification or version to each other but we need to prove this is a these are indistinguishable. However, we have extra information that X and Y are right continuous since X and Y are right continuous then, then we get that they are indistinguishable also. Fine.