Introduction to Probabilistic Methods in PDE
Professor Dr. Anindya Goswami
Department of Mathematics
Indian Institute of Science Education and Research, Pune
Lecture - 07
Preliminary for Stochastic Integration
Part 01

Okay, so today in this lecture we are going to see many definitions so that we can use those to define what is stochastic integration. Stochastic integration is a different treatment than the classical integration, the main reason is that when we see integration with respect to a bounded variation process typically in the situation when we talk about Riemann–Stieltjes integration.

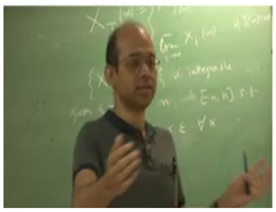
So in typical Riemann–Stieltjes integration situation what you do, you take one integrant and that is a for example is a function of time T and you take an integrator that is also a function of T and you integrate this integrant with respect to integrator, but to define such integration in the sense of Riemann–Stieltjes what you fundamentally need is the integrator should be of bounded variation. Why is it so?

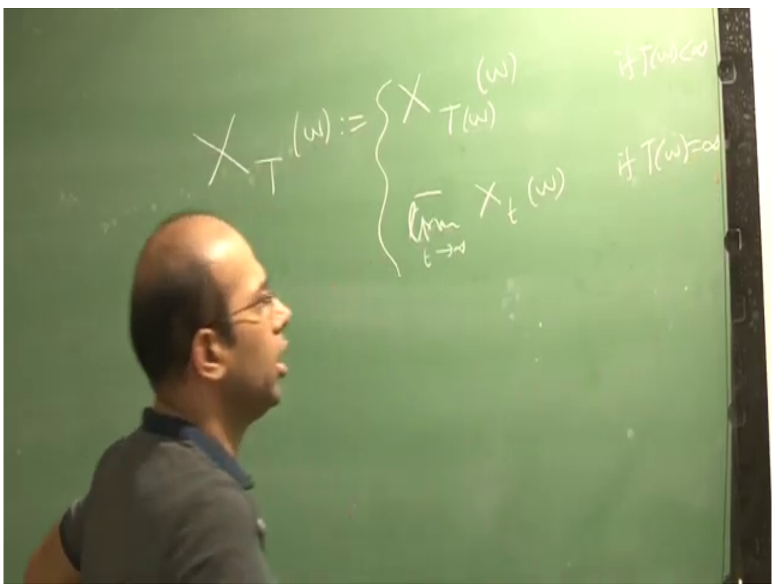
Because we know there a bounded variation function can be written as difference of two monotonic increasing function. And if you have a monotonic increasing function then you can take that as integrator and you can define integration with respect to that integrator. So, that is the limitation of Riemann–Stieltjes Integration.

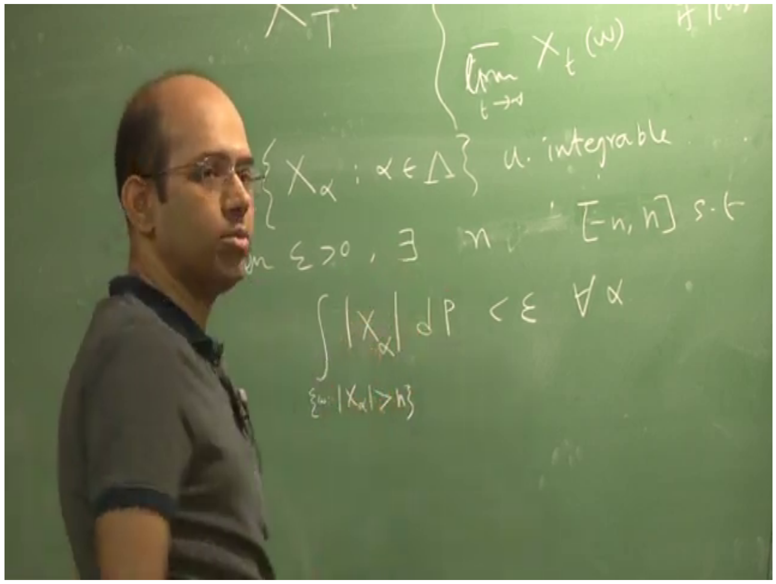
However, when we think about say finding out integration of some stochastic process with respect to another that another integrator is typically say for example, Brownian motion which has unbounded total variation so for those cases the Riemann–Stieltjes sense is not appropriate, one cannot define such integration using the sense of Riemann–Stieltjes. So, that is the reason that one should have a proper notion of integration with respect to some integrator which is not of bounded variation.

(Refer Slide Time: 2:46)

- **Definitions:** $T : \Omega \to [0, \infty]$ is $\{\mathcal{F}_t\}_t$ stopping time if $\{\omega | T(\omega) \leq t\} \in \mathcal{F}_t \forall t \geq 0$.
  $\mathcal{S}_\infty := \{\{\mathcal{F}_t\}_t$ stopping time $T | P(T < \infty) = 1\}$.
  The right continuous process X is in class D if $\{X_T | T \in \mathcal{S}_\infty\}$ is uniformly integrable;
  in DL if $\{X_T | T \in \mathcal{S}_a\}$ is uniformly integrable $\forall 0 < a < \infty$ where $\mathcal{S}_a := \{\{\mathcal{F}_t\}_t$ stopping time $T | P(T < a) = 1\}$.







Here, for stochastic calculus we encounter this type of processes which are not of bounded variation, so we deal with this integration in this topic. So, first I start defining few things which may appear as not so relevant but over few more slides it would be clear that how are these things building blocks and they are going to help us to define what we want to do at the end.

So, first we define what is stopping time? A stopping time is a random time, what does that mean? That its value is non-negative real number or could be plus infinity. So, this capital T is further assumed to be Ft stop measurable. What do I mean by that? I mean that the set of events, the event the T is less than or equal to small t so this event is Ft measurable.

So, given a filtration we can if we find one capital T if map from omega to extended positive real number such that the set of omega such that T of omega is less than or equal to small t is Ft measurable for each and every t positive then we call capital T as stopping time with respect to the filtration Ft. Or in short sometime we say Ft time.

The rationale behind this type of consideration of time is that not every random time is important or relevant. Why is it so? For example, if we ask, like you know, a gambler is gambling, playing gamble and that gambler would continue playing until some particular time and that is possibly a random time and after sometime the gambler stops. And if you ask what is this random time I mean how one can determine this random time? What does it depend on?

Of course that depends on the past or present performance of the gambling on the gambler. It does not depend on the future anticipation on the gambler so it just depends on past and present. It does not depend on the future realization because that has never happened. So, that is the thing I mean that is the also historical background why this type of time is called stopping time.

So, here it is saying that the random time capital T is less than or equals to small t that means its value is small t or below that, that event is Ft measurable that means that event would be determined if you have information throughout from 0 to t, you do not need to, I mean that does not depend on the information of the future. So, that is the definition of stopping time.

So, what are the stopping time examples? One can think that see gambling this is one now just you know realistic example, another is that you can think that you are considering a simple symmetric random work in one dimension that means you are always going to right with half probability, left with half probability and you are continuing.

And then your continuing, you started, after starting from origin you are continuing till you hit the boundary of the interval minus n to n, so n is equal to say 100. So, minus 100 to 100 and then you stop. So, how much time do you need to wait to hit the boundary, so that time is also a stopping time. And that time does not depend on what would happen after time you know the future it depends what is the path till time t. So, this is stopping time.

Now, we consider a class of stopping time, so this is very important class to define I mean for Doob–Meyer decomposition theorem. So, what we denote this by S subscript infinity. So, Ft stopping time capital T this is class of a Ft stopping times such that, that capital T is finite valued almost surely. Capital T is finite with probability 1 that means it is real valued with almost surely, it is not, it does not take infinity value with a positive probability, now we proceed.

The right continuous process X is called that it is in class D, if this X subscript capital T, what is X subscript capital T? Because I have a process X and X subscript T that is you know the value of X at time T. Now, if I replace small t by capital T. What we mean by X subscript T is that X subscript T of omega is X subscript T of omega of omega. So, T of omega if you fix omega then

T of omega is a particular, positive number, if it is infinity then you take lim sup of Xn, Xt and if it is finite number then you just take X subscript T omega of omega.

So, now whatever I was telling so let me write down it here XT of omega is defined to be X of T omega so T omega if it is finite, if T omega is finite then this is the definition, if lim sup of t so this Xt omega is the definition if T omega happens to be infinite. So, that is the meaning of FX subscript T.

Now, the right continuous process X is called to be in class D if Xt this family where capital T is from S infinity is this family this family has possibly uncountably infinite of members. This family is uniformly integrable. So, what do you mean by uniformly integrable? Basically if you have a family X alpha, alpha is coming from some you know family, you say this is uniformly integrable.

That means that given epsilon positive there exist a compact set K so compact K such that integration of X alpha I mean dP beyond that compact set so we write down this as set of all omega such that mod X alpha is I mean this compact set actually is real valued, so I do not bother I can actually take this to be for this special case say just this there exists n such that. This greater or equals to n is less than epsilon for all alpha.

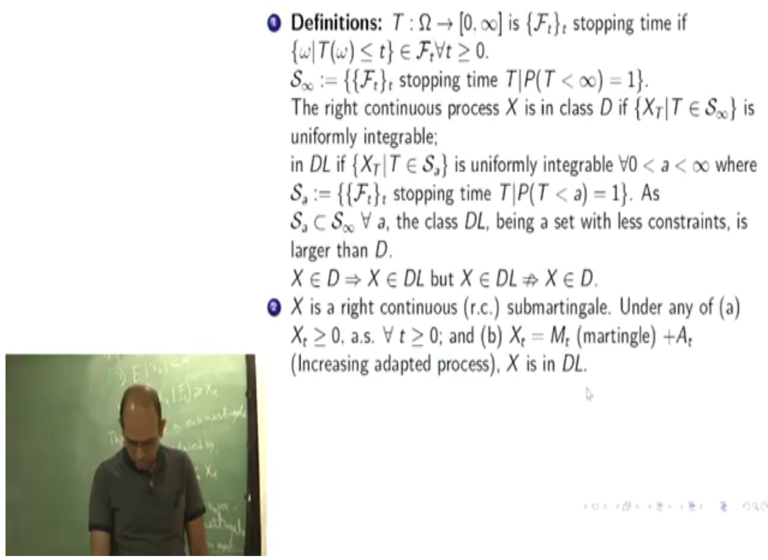
So, that is what we mean by uniform integrability, basically saying that if we have I mean the, the other way of saying that we can find out a delta given epsilon we can find out delta such that when the measure of the set is less than delta on that the integration is less than, less than epsilon. So, I should have written modulus here.

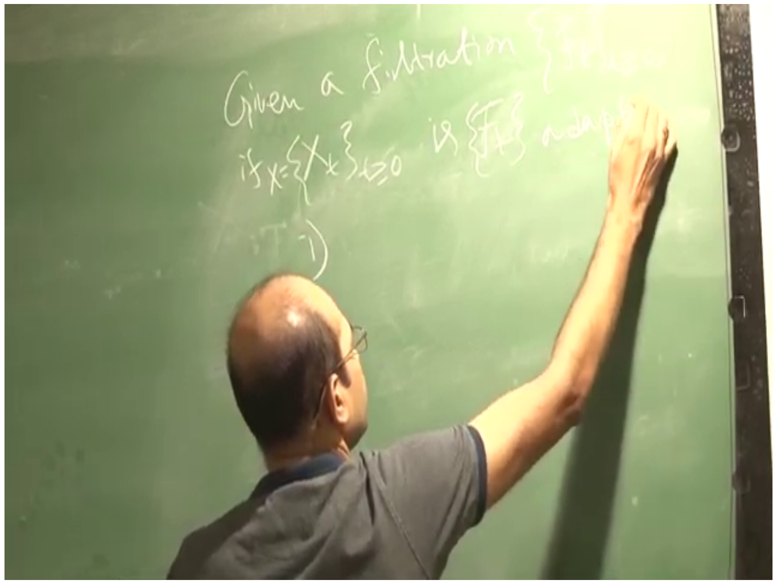
Here this index is capital T and that capital T is running over S infinity. So, so that we call in D and we would call the process X is in DL, is in DL if we consider different collection X subscript capital T where capital T is coming from S subscript a. So, what is that S subscript a? This is actually a condition I mean stronger condition it is saying that all Ft stopping time T such that the probability that capital T is less than a is equal to 1.

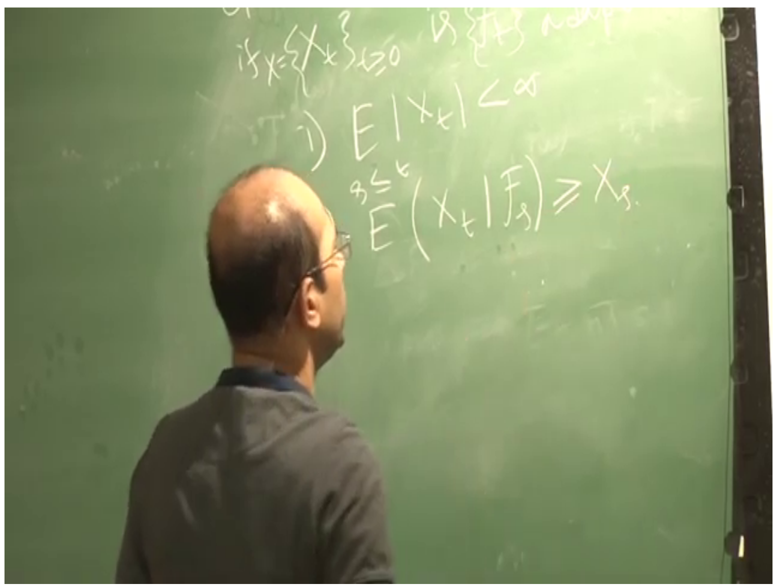
That means it is bounded by a, essentially bounded actually, essentially bounded by a, so that is a very strong condition. There are many stopping times capital T which would be in S infinity but would fail to be in Sa, so Sa is a small class. Smaller class. Now, if we say that we are looking for the stochastic process X such that this S subscript T where T belongs to Sa is uniform integrable that means we are putting less number of constraint because the class I am looking at is smaller than earlier class.

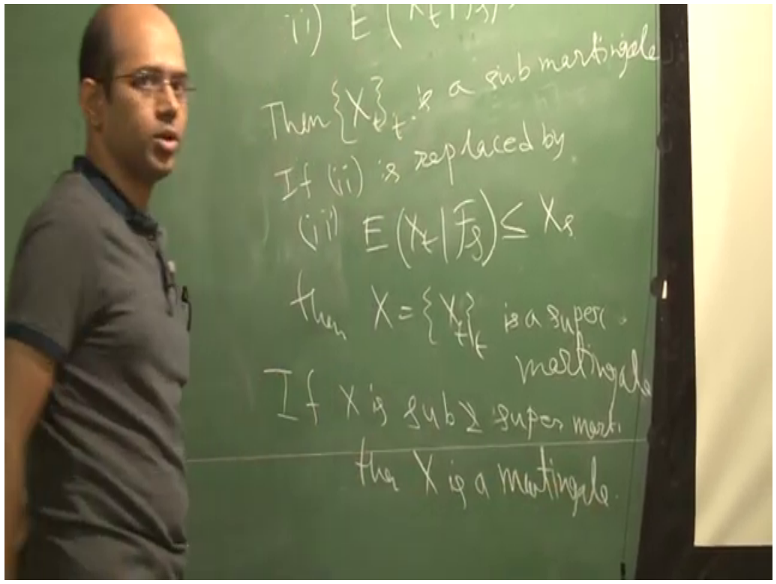
Say, for example, if my class is just singleton so then that is of course I mean in that contains 1 integrable random variable that is of course uniformly integrable. So, here we are putting less constraint since we put less constraint so as a result DL is larger set.

(Refer Slide Time: 14:14)




Given a filtration
if $X=\{X_t\}_{t\geq 0}$ is $\{F_t\}$ adapt
i)


if $X=\{X_t\}_{t\geq 0}$ is $\{F_t\}$
i) $E|X_t| < \infty$
$s \leq t$ $E(X_t | F_s) \geq X_s$



So, here I write down it even clearly as a Sa is subset of S infinity for all a positive the class DL being a set with less constraint is larger than D. So, X belongs to D implies X belongs to DL but of course opposite need not be true. Next, we see this result, this result is saying that X is right continuous so hence I mean I may use r dot c dot to denote right continuous this word.

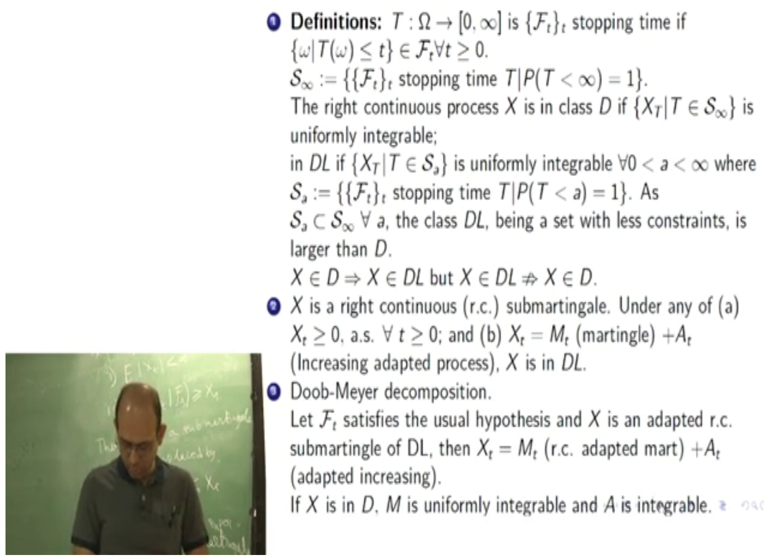
X is right continuous sub martingale and under any of these two criteria (a) and (b) we can say that X is in DL. So, we are I mean here quoting sufficient condition for a right continuous sub martingale to be in DL class. What are these two conditions? (a) is that it is simply non negative and (b) is saying that X has a particular decomposition that X can be written as sum of a martingale and an increasing adapted process.

So, if X can be written as sum of martingale and increasing adapted process or otherwise you just have a X is non negative either of the cases that is sufficient to ensure that right continuous sub martingale X is in class DL. So, that means this DL is really large enough, it includes lots of such processes.

So, let me define quickly what do I mean by sub martingale? In the earlier class possibly I have not mentioned that so given a filtration Ft if you have a process you call X such that, is Ft adapted and expectation of mod Xt is finite and conditional expectation of Xt given Fs where s is less than or equals to t is greater or equal to Xs.

Then X is, Xt is a sub martingale. On other hand, if second I mean if 2 is replaced by 2 prime which say I mean all others are kept intake, all are prevailing only 2 is replaced by 2 prime namely Xt given Fs is less than or equals to Xs then X is a, is called a super martingale, super martingale. And if X is sub and super both super martingale then X is a martingale, then X is a martingale. So, that is the definition of martingale, sub martingale and super martingale.

(Refer Slide Time: 18:39)



So, now we see Doob–Meyer decomposition, it actually says the I mean in some sense the reverse direction within two points what we have written is if the X has a decomposition Mt plus At where At is increasing and Mt is the fluctuation, then this is in class DL. And Doob–Meyer decomposition is saying that on the other hand if we have say here filtration satisfies usual hypothesis and X is an adapted right continuous sub martingale in DL. Then X has this decomposition X can be written as sum of a martingale and then increasing adapted process, adapted martingale and increasing adapted process.

So, this decomposition is called Doob–Meyer decomposition, we often write just D, M decomposition if X is in D so D is a smaller class then perhaps you would be able to say something more stronger is that is the case if X is in D then M is uniformly integrable I mean this martingale what you get as a result of Doob–Meyer decomposition that is uniformly integrable and A is integrable, we get that as output.

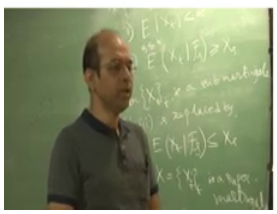
(Refer Slide Time: 20:01)

- **Definition:** (Natural) An increasing process A is called natural if for every bounded, r.c. martingale $M := \{M_t\}_t$

$$E \int_{(0,t]} M_s \, dA_s = E \int_{(0,t]} M_{s-} dA_s \ \forall \ t < \infty.$$

Every continuous increasing process is natural

- If A in Doob-Meyer decomposition is chosen natural, then the D-M decomposition is unique.



Next, we see some more few definitions, here we define what do we mean by natural process. An increasing process A is called natural if for every bounded right continuous martingale M, expectation of this integration is same as expectation of this integration. So, in the both the integration you have the same integrant that is A and we are actually talking about the property of A.

But integrate, sorry it is an integrator A but integrands are little different. In the first place you have M subscript s, second place you have M s minus. Does the leftly meet of the process M. And if you have this equality for all possible bounded right continuous martingales then this is a property of A if your process A is such that, that this equality holds for all possible right continuous bounded martingales then you call that A to be a natural process.

So, an increasing process which has this property would be called a natural process. I mean this is just one example, every continuous increasing process is natural. If you have continuous increasing process is natural, why is it so? I mean this is very easy to see because when we consider right continuous martingale then of course given a finite time interval for every omega it has all countable many discontinuities.

Every omega in the sense that almost every omega, we have finite many discontinuities and then for that omega if you consider this integration, this integration is in the sense of Riemann–Stieltjes so we can talk about path by path integration. And then of course both the integrations would exactly match because there the integrant is differing only at you know countably places.

If A is on the other hand is continuous increasing process so then you would get that they would be equal. And therefore, expectation would also be the same. If A in Doob–Meyer decomposition is chosen natural then the Doob–Meyer decomposition is unique. So, this is a nice I mean result why is it so because in the Doob–Meyer decomposition when we stated it just talks about existence subset decomposition, it does not talk anything about uniqueness.

One can of course take other type of decompositions. Say, for example, if you consider Poisson process say so in Poisson process Mt is the Poisson process that means it takes only integer values for every time t and then it is non decreasing and therefore, given any finite time interval it has only finitely many discontinuities actually with probability 1 and then if you look at how I going to write down this as you know martingale plus increasing process.

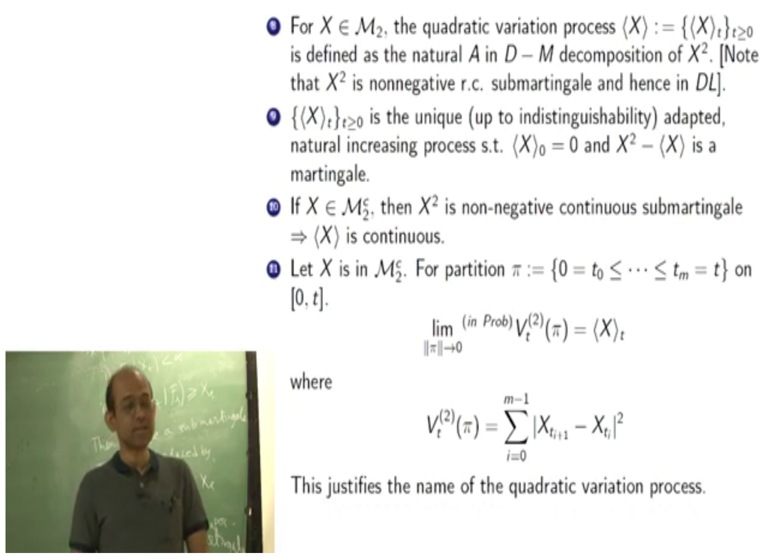
You can write down Mt is equal to 0 plus Mt, 0 is a martingale and Mt is also increasing otherwise you can also write down Mt is equal to Mt minus lambda t plus lambda t while lambda is the intensity of the Poisson process. Then we know the property of Poisson process the Mt minus lambda t would have you know martingale property that also martingale and lambda t is also increasing.

So, of course there are occasions then one has the multiple decompositions. However, if we choose only natural increasing process then we get unique decomposition. So, we are going to use that unique decomposition in the next slide. So, before that let us see this for X right continuous martingale we say X is square integrable. If we get that it is in L2 for a fix t it has finite variance. Expectation of Xt square is finite for a every time t.

If in addition that X0 the starting point is 0 then we write X is in script M subscript 2. So, this script M subscript 2 is now for hence forward, forward would denote a space of martingales which are square integrable and mean 0. Starting point 0 means mean 0, why is it so? Because here I am explaining that if X belongs to M2 then expectation of Xt can be written as expectation of X, I mean I have just you know using tower property of conditional expectation, I condition with first F0 and then again take expectation so then Xt given F0 is X0 itself but X0 is 0 so expectation X0 is 0.

So, for every time t Xt would have the mean 0 property if X is continuous as well as X belongs to M2 if X is continuous and as well as belongs to M2 we say that X belongs to M2 superscript c. So, M superscript c 2, Mc2 is also a notation which I would use hence forward this is of course a subspace of M2, this is also a linear space because if you add or subtract or you know multiply with scalers any member in Mc2 you know you would get here only you stay there only.

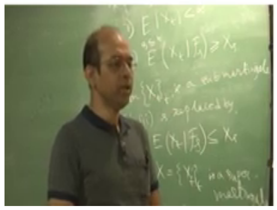
(Refer Slide Time: 26:28)



- **Definition:** (Natural) An increasing process A is called natural if for every bounded, r.c. martingale $M := \{M_t\}_t$

$$E\int_{(0,t]} M_s\, dA_s = E\int_{(0,t]} M_{s-}\, dA_s \;\forall\, t < \infty.$$

  Every continuous increasing process is natural
- If A in Doob-Meyer decomposition is chosen natural, then the D-M decomposition is unique.



Next we define, what is quadratic variation. So, this is the important thing since we wanted to talk about what is quadratic variation that is why I had to talk about what is Doob–Meyer decomposition and the uniqueness of Doob–Meyer decomposition is via only natural selection of the natural process. So, whatever we were discussing earlier for this two slides that is essential for this point 8.

So, let X is in M2 the square integrable martingale, the quadratic variation process X is defined to be the natural A in Doob–Meyer decomposition of X square so you have X square martingale you take X square. So, of course whenever you apply some convex function on a martingale you get a sub martingale. So, X square is a sub martingale no doubt and X is right continuous so X square remains right continuous.

So, X square is non negative right continuous sub martingale. So, the Doob–Meyer decompositions I mean there we have seen that if we it is so, so then I mean they are in Doob–Meyer decomposition theorem that if a member is DL class then it has decomposition but before that we have mention that if you have a non-negative right continuous sub martingale that also in the class DL. So, that we have mentioned as a result. So, bottom line is that X square has a Doob–Meyer decomposition and if you select the natural increasing process then that natural increasing process is called quadratic variation.

Next, we consider the stochastic process of quadratic variation. So, one thing is clear that by definition this quadratic variation is increasing process or non-decreasing. Because it is coming from Doob–Meyer decomposition. So, since it is increasing so of course it is a bounded variation with probability 1.

So, whatever is written in the point 8 is actually rewritten, it contains no extra information, it is just saying that this is the unique up to indistinguishability adapted natural increasing process such that the starting point is 0 and X square minus quadratic variation of X is a martingale. So, this is just rewriting of the above thing.

Next, we consider the subclass the continuous square integrable martingales. So, X belongs to Mc2, then we get the X square is also continuous X continuous implies X square continuous and then there is additional property of such continuous square integrable martingale that it has some regularity property which I am not discussing here.

So, due to that one can prove that after the decomposition you would see the both the parts like martingale and the increasing processes both would be continuous. So, this result also we would use in future. So, for Brownian motion actually we can compute this object. However, till now whatever you have discussed about the quadratic variation that does not give indication that why should such process be call quadratic variation. Because no such variation appear in this definition.

So, this point 11 actually justifies it quotes a result which justifies this name of the quadratic variation. So, let us read this thing. Let X is in Mc2 that means square integrable continuous martingale for a partition pi which contains say for example n plus 1 number of points so it is actually partitioning the interval 0 to T close interval 0 to T.

So, T I have fixed so for fixed T I come up with the partition pi and then we consider sequence of finer and finer partitions such that the mesh size goes to 0 and we take limit in probability of the functional Vt so what is Vt2? Vt2 is nothing but the sum of the quadratic variation. So, this ti plus 1 minus ti so these are the time increment and Xti plus 1 minus Xti is the increment of X in time subinterval ti and ti plus 1 and we take square of that.

So, it is giving me square of the increments. And then when you sum over all possible i is going 0 to m minus 1 so this is actually quadratic variation of X with the granularity of the partition so given the partition, along the partition this is the quadratic variation of this part. But if you increase you know your partition points to make it finer and finer then I mean you can discuss that where does this limit go.

So, here it says that this family if I tends my mesh size pi goes to 0 then in probability it converges to the process quadratic variation process as described above. So, this justifies the name that why should such process be called quadratic variation.