Introduction to Probabilistic Methods in PDE
Professor Dr. Anindya Goswami
Department of Mathematics
Indian Institute of Science Education and Research, Pune
Lecture 66
Existence of Classical Solution Part 2

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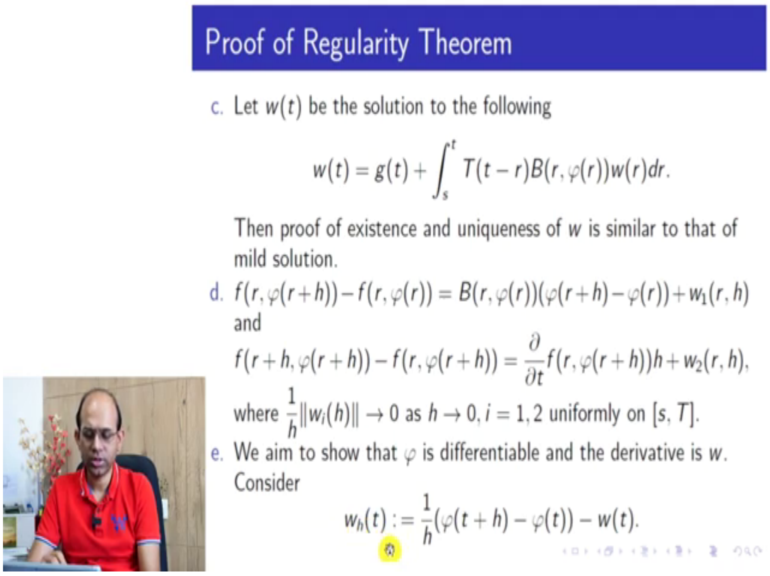

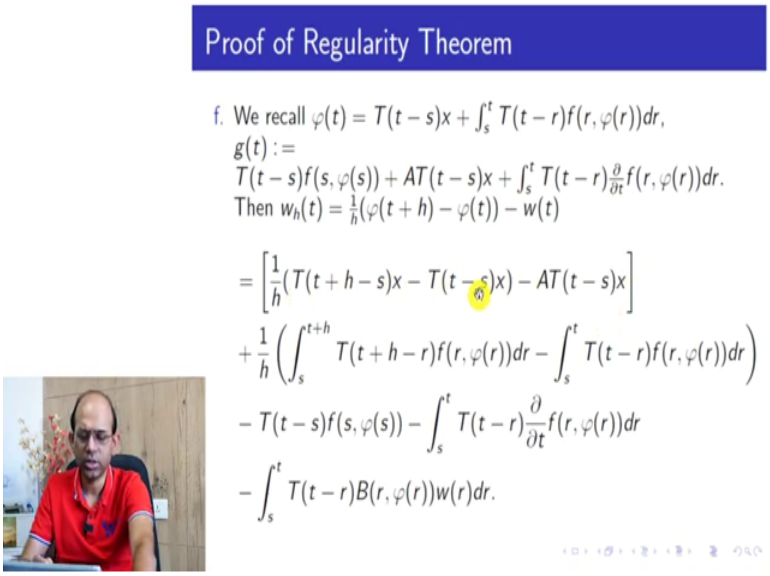

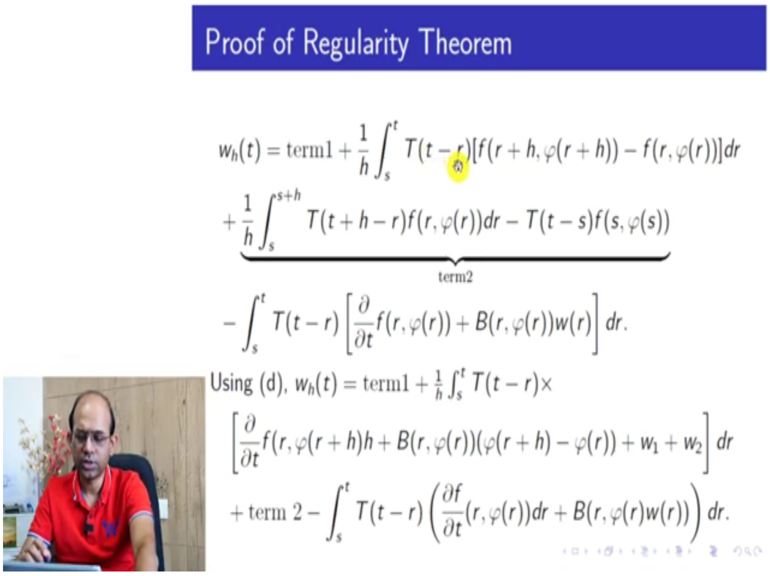

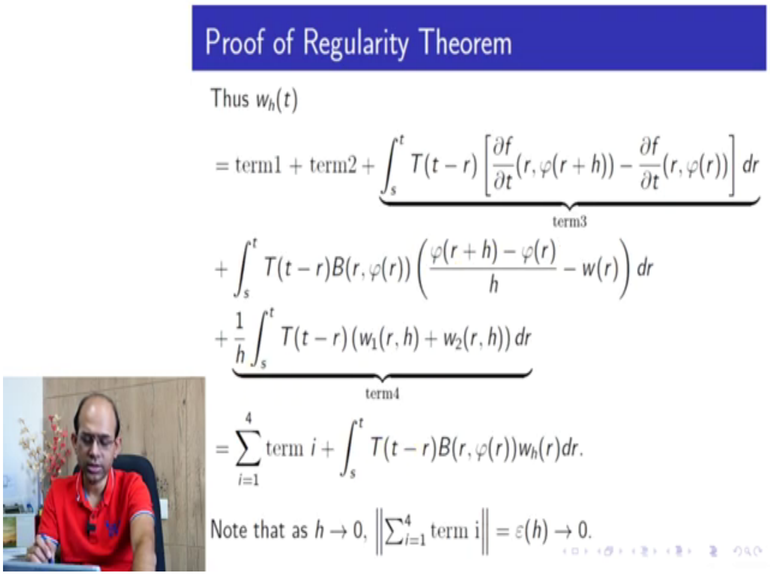

Welcome. What we have seen till now is that, Wh t, which is defined as 1 over h phi of t plus h minus phi of t minus w of t. So, this term, okay, what we would like to show that this goes to 0, okay. Why, because if we show that this goes to 0, then that asserts that phi is differentiable, because this is the finite difference. And then we would also confirm that the derivative is nothing but w, okay so this is the thing what we are going to prove now in few steps.

So, we recall all the relations here in the same slide, so that we can use those to manipulate some terms here. So, here, phi is satisfying this integral equation. Phi of t is equal is equal to capital T of t minus s x plus integration small s to t, capital T of t minus r f r phi r dr, okay.

And g of t is defined as capital T of t minus s f s f of s comma phi s plus A with capital T of t minus s x, plus integration of small s to t, capital T of t minus r del, del t of f of r phi r dr.

And then we also remember that the definition Wh t, this is this fraction minus w t. And w t satisfies this thing. Integration is equal to g t plus integration s to t, capital T of t minus r, B r phi r w r dr. So this integral equation, okay so, which is satisfied by w.

Okay, so this w appears in Wh expression. So, we would like to show that Wh goes to 0, basically, correct? So, what we are going to show here actually is that after a manipulation of terms, we are going to show that this satisfies the sufficient conditions for Gronwall's inequality.

And then that would assert that wh is going to less than or equals to some term and that right-hand side term, we would be able to show that, that goes to 0. So, that is the main idea of the proof. So, here, phi of t plus h, I can compute from this expression, phi of t is written this way.

So, phi of t plus h is capital T of t plus h minus s x, plus integration small s to t plus h capital T of t plus h minus r f r phi r dr. So here phi t also has this expression. So, when we take the difference, term by term difference we can discuss. First the difference of this first term and then we can talk about the difference coming from the integral term and then of course the full expression of w also.

Okay, w also solves the integral equation. So, there also we have 2 terms. Okay, so those are the things we are going to and w involves g and g also involves, you know I mean g constitutes 3 additive terms, 1, 2, 3. So, there will be many terms now. So, wh t is equal to this, okay so, phi t plus h minus phi t, this difference I am writing down now first 1 over h is there so we should not forget about it.

So 1 over h, so here for this, capital T of t plus h minus s t plus h minus s is coming. So phi t okay, so here capital T of t minus s, capital T of t minus s difference. And then this term is coming from g actually, because here w has g term. But here, w is coming with a negative sign. So, minus of this term would be present here. So, minus A T t minus s x is coming from that thing.

Okay, and now for the difference of integral term coming from this phi t plus h minus phi h, we are writing down now. So integration here, s to t plus h first, because it is coming from t plus h, s to t plus h. All the places, where t appears, I should write it t plus h. Capital T of t plus h minus r, f r phi r dr.

Now for, from this term phi t, I would have just this integration. So, I am writing just this integration with a negative sign here. So, negative sign here, in minus integration s to t capital T of t minus r, f r phi r dr, okay. So for this term, this difference is concerned all the terms coming from this I have written.

And for w term only I have written this term. I mean, we still need to write down these 2 terms and also the integral term which appears in the equation of w. So, we do this. So, these 3 terms we write down now. So, these first we write the first term. So minus capital T of t minus s f s phi s.

And then this term, minus integration s to t, capital T of t minus r, del, del t of f r phi r dr. And then w, since w involves the integral equation, this whole thing, so, this whole thing with a negative sign should also come there, the T t minus r, B, w dr. So, that we write down here minus capital T t minus r B r, w r dr. Okay, so wh t is written in this way. So, there are so many terms. So, but, some terms I have combined to make it parenthesis, because those terms are easy to manage.

For example, I have taken parenthesis in the, okay, for these 3 different terms. So, here, this difference divided by h. We know that since x is in the domain of A and then this is differentiable and the derivative of Tt x is AT t x that appears here. Okay so t plus t minus s plus h and t minus s. So, this difference is going to give me capital T of t minus s x, okay, A T t minus s x, okay or t minus s, A x, okay.

So, that is exactly this term. Okay, so as h tends to 0, this first term goes to 0, okay. Okay so that is the reason that I have just dumbled around 2 terms I mean I have written this term first here, okay. Okay for remaining terms also we need to do some sort of you know rearrangement, so that we can group them together as to some small term so which we can show easily that those goes to 0, okay, so that we will do next.

So next onward I am not going to write down this in details. I would just replace this by term 1. We call this term 1. And you know that this goes to 0, as h goes to 0, okay we are going to use that. So wh t is equal to term 1 plus okay, so, this, whole is term 1 plus this thing, 1 over h, this thing.

So now for this term, we can do it a little bit more (simpler) you know, these 2 things are quite looking similar. Okay only difference is that it is s to t plus h and here, t plus h appears. Okay but, one can of course change variable, okay, so substitute r minus h as r prime. Okay so then r is equal to r prime plus h. So, I would get f of r plus h, phi r plus h here, okay.

And here I would get capital T of t minus r, okay. And then the limit would also change. So, when r is s, r prime would be s minus h, correct? Okay so and then when r is t plus h, r prime would be t, okay. So this s minus h to s is additional term. If we separate that, then I would get, I mean it decomposes the domain of integration.

Then I would get 1 s to t here, s to t here that we can manage, because then this operator T t and this look similar, because here I would not get h here. Okay so, that is the main idea. So, what we do is, 1 over h, s to t, capital T of t minus r that is common in those 2 terms we take common.

And then, the difference of the remaining terms, so here I would get f of r plus h, phi r plus h here f of r phi r. So, this difference would appear there. And then so f of r plus h phi of r plus h minus f of r phi r dr and as I told that while doing this substitution, you know this (interval) this we do need to do decomposition of the domain of the integration.

So, here, I need to separate integration s minus h to s there. And there, if I mean back substitute it again to the old thing, so basically here it is nothing, but you know s to s plus h this integration. So, that is the thing. So, I write s to s plus h, capital T of t plus h minus r f of r phi r dr.

And then 1 over h is not forgotten so it should be there, 1 over h is there. Okay, so from here we are going to get this you know this term. And then, we are writing minus the next term. So, next term is here. So, after this you know difference, we have this term, minus capital T of t minus s f of s phi s.

So, we write here capital T of t minus s f of s phi s, this term. Okay. So, these 2 terms, we are now clubbing together okay and we call this term 2. Why do we call this? Because you know, this is very easy to manage this term. Here, if we see very carefully, that here, capital T of t minus s f s, f s phi s appears.

And here, this 1 over h, this is you know average, okay, s to s plus h, 1 over h this thing. And then, we know already that continuity of this integrant. Since the integrant is continuous, so this s to s plus h 1 over h (you know) as h tends to 0 that is converged to that value where of the integrant, where this r would be replaced by s and h would be replaced by 0, okay.

So that means capital T of small t, plus 0 minus s that means capital T of t minus s, f of s phi s. So, that is this. So, in other words, this integration converges to this thing, as h goes to 0. So, all this convergence, we are talking about in the Banach space norm, okay, good. So, this is term 2, okay. And then, but we still have so many other terms. So, this term is there.

And then, the remaining term. So, I talked to you up to this thing I have not yet talked to, talked about these two terms, minus s to t. So, this you know in one place I have a partial derivative of f with respect to t and another plus B. So, that is these 2 terms, minus capital T minus capital T of t minus r, because this is common okay, t minus r is common.

All these are not common. So, we would have this in an additive sign. So, minus sign integration s to t capital T of t minus r del f I mean del del t of f plus B r phi r w r dr, okay. So wh t has now this reduced form, where this term 1 and term 2 are assured to go to 0. And we still have 2 more terms, okay so, to deal with.

Okay, so we now analyze the remaining things. So, w, so, now we are using the d. What is a d thing? d is about w and w 2. So, here you see that f of r plus h phi r plus h minus f r phi r plus h and then f r phi r plus h minus f r phi r. So, if you add these 2 terms, what we are going to get there, f of r plus h phi r plus h minus f r phi r, okay that is equal to the sum of these things, correct?

Okay, so because these 2 cancels, correct? So, this thing we are going to use now. Why do we need to use this? Because we already have this term f of r plus h phi r plus h minus f r phi r okay. So, here we are going to use the, you know bullet point d, to write down this, you know addition of the derivatives and the error terms.

Okay, so the term 1 as it is plus 1 over h, okay and integration s to t, capital T of t minus r okay, as it is. And then, now this difference is del, del t f okay. So, del, del t f times h. So, this h will be actually cancelled with this 1 over h later. So, h plus, now B r phi r, okay so this is the partial derivative with respect to time and then this is derivative with respect to space.

So B r phi r times phi of r plus h minus phi r. So, this is exactly obtained from the verbatim from the point d, plus w 1 and w 2. I am not writing here it is dependence on h and r. But that is understood and just to you know fit everything in the same slide phi w1 plus w2. So this integration is d r, okay.

So this term is elaborated here and then we have this whole thing as term 2, okay I have written simply just term 2. And then minus this then this integration s to t capital T of t minus r, okay. So, exactly, what is written here I am writing exactly the same thing here, I mean no change here.

So, notice that here, already I have a partial derivative with f and then B. So, like these things are appearing here. So, I mean these 2 terms are coming when you know looks are, are looking very similar. Here, only thing is that we have w here I have phi r plus h minus phi r. So that is the thing you know, we need to really check.

So, this is the main thing, main term, which would give me again wh. So, I would get an equation in wh. Okay, so let us go to the next slide. So, there we are going to write down term 1 plus term 2 and so, term 1 plus term 2, plus this additional term here. So s to t integration capital T of t minus r del f del t, okay, minus. So, as I told that okay, this term and this term should be compared with.

So del del t f of r comma phi r plus h whereas we have r comma phi r here. So, here, that is why we get del f del t r phi r plus h minus del f del t r phi r and dr, okay. So, we call this whole thing as term 3. But we have, we still have more terms, because we still have B terms here, B terms here. That we are going to delete now.

But we can take the count, take B r term common, because here we have B. So, this B term, capital T of t minus r B r phi r. Here also, we have capital T of t minus r B r phi r, okay. There is a typo here bracket should parenthesis, this parenthesis should come here actually, okay.

Okay so here we had phi r plus h and r phi, okay see here we both the cases, we have phi r, okay. So, you can take common, okay good. So this term we call term 3. And we take common those terms. So integration s to t capital T of t minus r B r phi r. And then phi t plus h minus phi, phi r plus h minus phi r divided by h that comes from here phi r plus h minus phi r and then 1 over h is there. So, divided by h comes from there.

And here, we just have w r. We do not have 1 over h, correct, we just have w r here. So, the minus w r, okay so we got this term. But we know this term, this is precisely wh t okay, wh t which we started with. You know the definition of wh t is 1 over h phi t plus h minus phi t minus w t. So that is the wh t. So, this term is known, this is wh t, okay.

So, what we have obtained that wh t is equal to term 1 plus term 2, where both the terms goes to 0. And the term 3, so, here term 3 is also interesting. What happens here that as h tends to 0, since you know f is you know continuously differentiable, that means derivative is continuous.

So and phi is also a continuous function of r, okay so here this thing converges to this, as h tends to 0, okay. And this is bound-linear map on this you know a compact set, a finite interval. So, this whole thing converges to 0, as h tends to 0 using the Bounded Convergence Theorem.

Okay so term 3 also goes to 0. Okay and now the term 4, this is additional term what appears here is that in w1 and w2 also are present here, in these terms. So, that thing, you know would be capital T of t minus r, composed on w 1 w 2, okay. So that should also be kept. So, I have written plus 1 over h, integration of small s to t, capital T of t minus r then of this addition of w1 and w2, here, okay.

So, this is the expression of w h. And her term 1, term 2 and term 3. For these 3, I have already argued that all these 3 converges to 0, as h goes to 0 in the Banach space norm. And then, for these, this is, I am replacing wh t. So, we are going to replace this by the wh t notation. And in term 4, term 4 already we have I mean by virtue of the differentiability, we know that this errors.

Okay error terms, 1 over h w1 goes to 0, 1 over h w2 also goes to 0. So, 1 over h is appearing here. So, 1 over h, w1 or 1 over h w2 goes to 0. So, then again use the Bounded Convergence Theorem of integrals to ensure that the term 4 also goes to 0, okay. So, what we have obtained is that w h is equal to term 1 plus term 2, 3 plus term 4, plus some term, where w h converges.

So I write down this whole thing in a more abbreviated form, the wh t is equal to the sum of all these 4 terms, term 1 to 4, 1, 2, 3 and 4. And then this term that is integration s to t, capital T of t minus r, B of r phi r, as it is and then this whole thing is by definition is wh r dr. So, wh satisfies this equation, okay, this integral equation.

Okay so, from here, actually our main goal was to show that w h converges to 0. So, from this, I need to find out the norm of w h. I need to show that norm of w h goes to 0, as h goes to 0. So, we take the norm both sides then, correct? And then we do not need to worry about this, this is something which goes to 0.

So, now as h tends to 0, this you know this norm of this thing, you know this whole thing, I write down as epsilon. Okay, so, norm of all these things term 1, term 2, term 3, term 4. Okay so, these are actually the summation of these 4 terms, actually, a point in the Banach space. It is function of h also, as h goes to 0 that converges to 0.

And the norm of that function, we call this epsilon h, epsilon h is a positive real number, non-negative real number. So, what we know about these terms that okay. This epsilon h goes to 0, okay. So, this property, I mean this expression, we are going to use now, after when we take the norm of both sides.

Okay, if we take norm of both sides okay, so, here norm of addition is less than equal to, you know addition of the norms, the triangle inequality.

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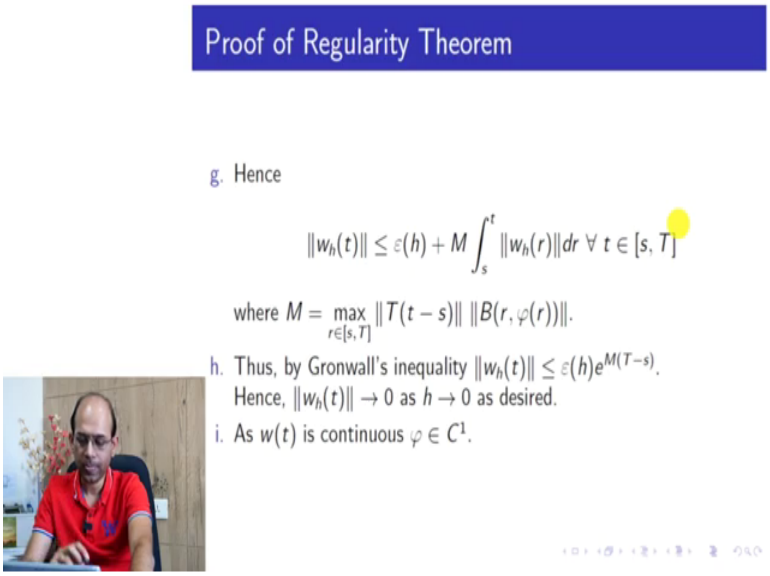

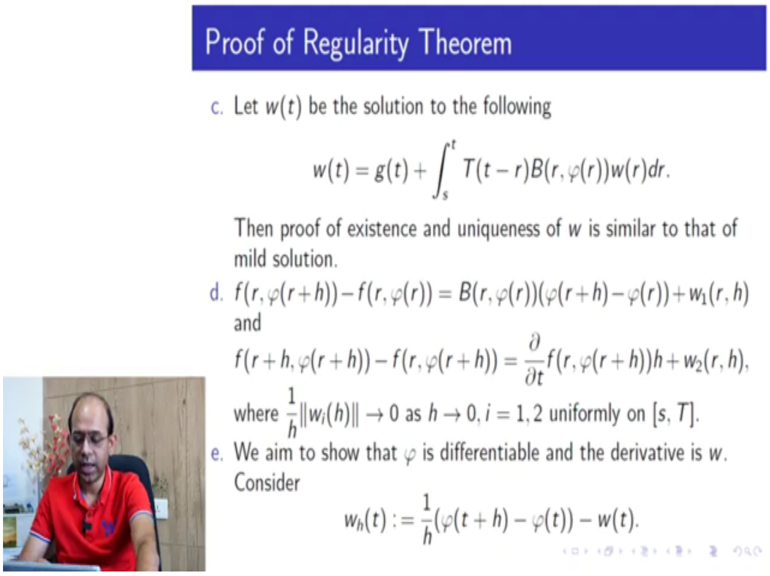

So, we are going to get norm of wh, is less than or equals to epsilon h plus capital M times, M is something, I am define it later, integration of s to t norm of wh r dr, okay so this wh r dr is here. So this thing, okay.

So if we now upper bound this thing, this operator okay, the norm of this operator. Yes, of course we can, because this is bounded-linear operator and these are also with respect to this time variable, this is also continuous. So, we can actually take supremum over all r on this closed interval s to capital T.

Okay so, this capital is some future term capital T. And then this capital T is of course the semi-group. So, norm of capital T of t minus s and norm of B r phi r, if we take this, we know for sure that this is finite. We do not know exactly what value, but we know this is finite capital M, M is finite.

So, that M, I replace here, because I can do that, because it is less than equal to sign that we can replace by a large quantity. Capital M times integration s to t, wh r norm of the dr. So, now, this has a real valued function, correct, a function of time, because the norm I have taken a real value function norm this whole thing.

So, this real valued function is satisfying this equation, the norm of, I mean this thing is less than or equal to the same thing appears here, right, r is from s to t. So, then we can use the Gronwall's inequality. Thus, by Gronwall's inequality wh t okay, it can be written as you know less than or equals to this quantity epsilon h times e to the power of this constant M times capital T minus s, okay.

Okay if I put t is equal to s, then it is 0 here so wh. So, basically, I should have written wh t minus wh s minus epsilon h is replaced. But that epsilon h anyhow goes to 0. So, it is not much a matter here. So, w, norm of wh is less than or equals to this term. Here, we see that this you know, this is a finite quantity.

And as h tends to 0, this epsilon h goes to 0. So here we conclude that norm of wh goes to 0 as h goes to 0, okay. For every small t in the interval, small s to capital T, okay. So by proving this what we have done that, that we proved that w t is I mean w t is differentiable, correct? I mean not w t I mean phi is differentiable, because that was the main goal here.

So, we had to show that yes, wh t was this. So, since we have shown that wh goes to 0 and incur the phi is continuously differentiable and that derivative is w, okay. Next.

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Proof of Regularity Theorem

g. Hence

$$\|w_h(t)\| \leq \varepsilon(h) + M \int_s^t \|w_h(r)\| dr \;\forall\; t \in [s, T]$$

where $M = \max_{r \in [s,T]} \|T(t-s)\| \, \|B(r, \varphi(r))\|$.

h. Thus, by Gronwall's inequality $\|w_h(t)\| \leq \varepsilon(h) e^{M(T-s)}$.
Hence, $\|w_h(t)\| \to 0$ as $h \to 0$ as desired.

i. As $w(t)$ is continuous $\varphi \in C^1$.

Proof of Regularity Theorem

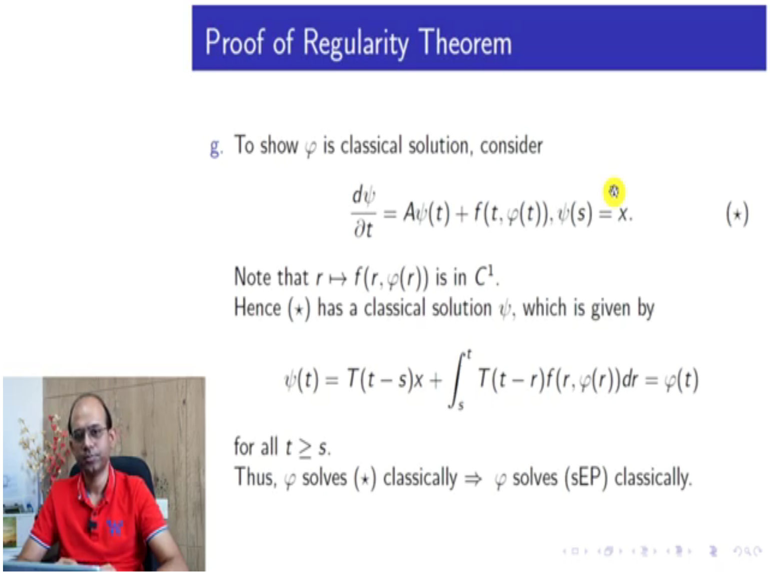
f. We recall $\varphi(t) = T(t-s)x + \int_s^t T(t-r) f(r, \varphi(r)) dr$,
$g(t) :=$
$T(t-s) f(s, \varphi(s)) + AT(t-s)x + \int_s^t T(t-r) \frac{\partial}{\partial t} f(r, \varphi(r)) dr$.
Then $w_h(t) = \frac{1}{h}(\varphi(t+h) - \varphi(t)) - w(t)$

$$= \left[\frac{1}{h}(T(t+h-s)x - T(t-s)x) - AT(t-s)x\right]$$

$$+ \frac{1}{h}\left(\int_s^{t+h} T(t+h-r) f(r, \varphi(r)) dr - \int_s^t T(t-r) f(r, \varphi(r)) dr\right)$$

$$- T(t-s) f(s, \varphi(s)) - \int_s^t T(t-r) \frac{\partial}{\partial t} f(r, \varphi(r)) dr$$

$$- \int_s^t T(t-r) B(r, \varphi(r)) w(r) dr.$$



Next we know that w t is continuous. Since phi is continuously differentiable and w is a derivative of phi. Also, we know that w satisfies that equation. From that also, we get that this is continuous.

Okay, so continuous. Since, so, now we need to show that phi is a classical solution, okay. So, here, till now what we have proved is that phi is differentiable. Okay, but we would like to show that this is the right candidate, where phi was just a mild solution I have taken.

So and we have shown that the mild solution is sufficiently smooth. That means it is differentiable. But then, what is the guarantee that this mild solution, which is actually a solution of the integral equation indeed, solves the evolution problem, okay, classically. So, we need to do that.

So, to show that phi is classical solution, we consider another equation so, d psi dt okay, exactly the same operator A, d psi dt is equal to A psi t plus f of t comma phi t psi s is equal to x. See here I have kept the same phi, this phi is the mild solution of the semi-linear evolution problem, SEP, okay.

So, I kept phi. So, phi is considered here as known. Okay and we have already proved that the mild solution is sufficiently smooth. So, we know a lot about the function phi. So, that f of t comma phi I am considering is a known quantity and unknown psi appears here. So, it is a new equation actually. It is not the same equation there.

So, we call this equation as the star, okay, it is denoted by star. And at starting, psi s is equal to x, the same initial condition. Now for this, this is nothing but IIVP, I mean it is an inhomogeneous initial value problem, where this f is very nice, because phi is differentiable, f is also differentiable.

So f of t comma phi t is differentiable. So, as a function of time t, it is differentiable. Okay so from that, so I write down here this thing, in that r to f of r phi r is in C1. So, from that, we can apply now the regularity theorem for IIVP that that asserts that this equation has a classical solution, okay. Let us call that classical solution as psi, okay so, hence this star has a classical solution psi.

And then this psi also I mean we know that, okay, that it has a classical solution and the classical solution can be written using the formula for, formula for variation of constants. So, the variation of constant the formula we write down here. So, psi is given as psi t is equal to the semigroup generated by the operator AT t minus s x plus integration s to t, capital T of t minus r and then this additive term, additive term f of r comma phi r dr.

So, remember this phi is intact, kept intact there okay, so psi is equal to this. However, this is something which we have seen earlier. So, what is this? We have already seen that the SEP has a mild solution. We have already seen. And that mild solution satisfies an integral equation. And that integral equation exactly looks this, okay. And here, we have exactly that, that you know, that mild solution exactly here.

So, we know that for the mild solution, phi satisfies this integral equation for each and every time t between s to capital T. So this whole thing as a function of t is nothing, but the same as phi t. After studying this, what we have obtained is psi t can be written this way using the formula for variation of constants.

And then, we recall what is the mild solution of SEP. The mild solution satisfies some integral equation. But here, in that, so, this appears at the right-hand side, this appears at the left-hand side there in that mild solution. So, let me write down, let me show you that again.

Yes, phi of t is equal to T t minus s x integration s to t capital T minus r f r phi r dr. So, this phi appears. So, now, even if I mean hide it so, if I take the solution of this equation, which is the mild solution and plug in here, I would get something which would exactly match with this phi t, for all time t, correct?

So, that appears here exactly. So, this thing you know T t minus s x integration s to t capital T of t minus r f r phi r dr. So, this appears here. So, that mild solution of the integration that appears here so that is equals to phi t. So, what we obtain is that psi is same as phi, okay. So, for all t greater or equals to s and in between s to capital T, okay, so this psi and phi are equal.

So, what does it mean? That means that the mild solution of the SEP, okay, is indeed the classical solution of this. So, that means this phi satisfies this. That means I would be able to write down d phi d t. So, it satisfies classically. So, I can write down this, I can replace phi, psi by phi here, because psi and phi are same. So, we get d phi d t is equal to A phi t f t phi t, okay.

Earlier I could not, because you know here by fixing phi here, I know that it has a classical solution and this you know relation is satisfied by psi classically. And when we have that, then we show that this psi is same as phi. And then we can conclude that, okay phi solves this classically, okay. Here, the argument has some, you know steps and some, you know steps, okay. So we are done.

So, thus phi solves this star classically. But what is this? This is exactly the SEP. You know; if I, so, if I solve this classically, if I replace phi here so, it is d phi d t is equal to A phi t plus f of t comma phi t d t. So, that means phi solves this classically, okay, good. I mean of course the way we have proved, it does not prove uniqueness, right?

Because here, I just show that okay, that if I replace phi, then it solves. But it does not, but I am not saying that whatever solution it may have, classical solution, you know of the SEP that would have you know this particular formula or anything.

So, here, I mean this round-about way, we have obtained just the existence of the solution of SEP, okay. Thank you very much. So, by this, I am finishing this portion of the semi-linear evolution problem, okay. Thank you.