Introduction to Probabilistic Methods in PDE
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Lecture 65
Existence of Classical Solution Part 1

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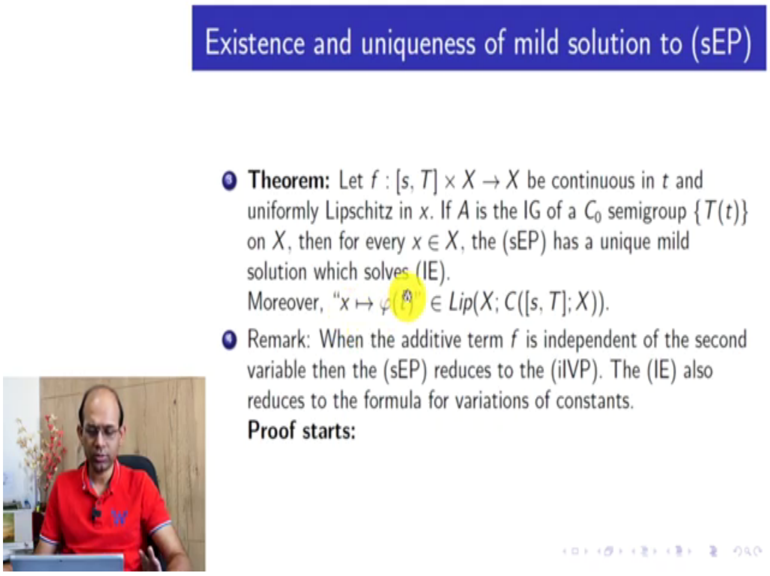

Welcome. In the last lecture, we have studied semi-linear equation, semi-linear evolution problem. And there, we have stated one theorem, saying that if the additive function is continuous in time and uniformly Lipschitz in the second variable or space variable, okay then, and also the operator A is infinitesimal generator of C0 semigroup. Then, for any initial point from the Banach space, one can get one mild solution, okay.

It has a unique mild solution and moreover that, the dependence on the mild solution, on the initial data, the dependence is Lipschitz, so it has Lipschitz dependence. Okay so this theorem, we have seen in detail. So, we have seen the proof in details.

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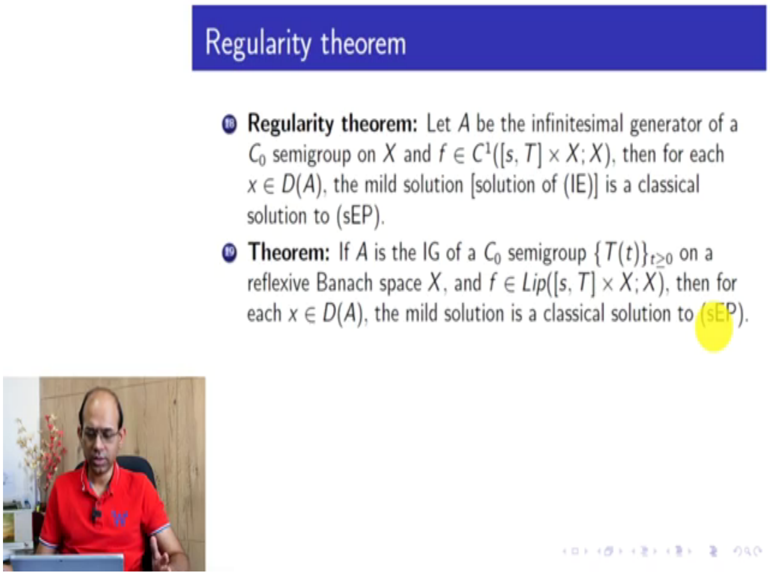

Now when this is done, we ask the next level question, that what about the classical solution? Okay, for all other earlier problems like (you know) initial value problem, homogeneous initial value problem, inhomogeneous initial value problem or even the evolution problem, homogeneous and inhomogeneous evolution problems.

So, for those, we did not need to ask the question that whether the mild solution exists, because we used to write down, we actually have written down the mild solution formula, provided the operator A has sufficient stability. I mean in the sense that okay that generates a either a semi-group, C0 semigroup or it generates a evolution system, okay.

So, only, I mean immediately after obtaining an evolution system or a semi-group, we could write down the mild solution for those problems. But here, when we are discussing semilinear evolution problem, then even after obtaining the semi-group, generated by the operator A, since the data, your data is non-linear, one needs more property on the data, how it depends on the space variable.

So, and then this, I mean if that is Lipschitz, then only we can ensure existence of a mild solution. Here, the mild solution does not have any expression. Rather, we write down mild solution of the equation, of the original problem as solution of an integral equation, okay.

So, the question arises, that when the integral equation has a solution. So, that was the reason of discussing this. Okay, but however the question what we generally ask for every Cauchy problem whether it has a classical solution or in other words, what are the sufficient conditions on the operator initial data and additive terms, so that one can assert that the problem has a classical solution?

So, here, for this problem, this regularity theorem asserts that. So, let A be the infinitesimal generator of C0 semigroup on X and additive term f is continuously differentiable. So, this f is a function of 2 variables time variable and space variable.

So, here so, we have assumed f is continuously differentiable. So, it is in line of the assumptions what we have made for IIVP. So, there, we have stated couple of sufficient conditions for existence of classical solution. There also we have taken the additive term to be C1, okay, in one of those theorems.

And also we must take the initial data coming from the domain or definition of A. So, x belongs to DA. Okay, so we have taken these assumptions in the line of earlier theorems, for some special cases. So here, one must be careful, that okay this is C1 in with respect to this, it is the function of 2 variables. Correct? Not only only time to X, but time and space together.

Okay so differentiability in that sense. Okay, so the mild solution is a classical solution. Okay? Second assertion is that, first assertion is that okay. So, it has a classical solution and then the classical solution whatever we write, coincides with the mild solution, okay, fine. Or in other words, basically one can say that the mild solution becomes the classical solution.

Or in other words, the mild solution gains sufficient regularity that it belongs to the domain of definition of the operators, I mean whichever which appear in the equation. Okay, so, that is all about the Regularity theorem. We have stated similar theorems for earlier problems. So, for this semi-linear problem, this is the statement.

There are some other alternative, you know theorems, with some other type of sufficient conditions. And then, after stating those, we would prove these, you know regularity theorem, this point, this statement we are going to prove in details. Okay so, let us see some other, I mean this type of statements.

If A is the infinitesimal generator of a C0 semigroup Tt on a reflexive Banach space, capital X. So, here we are assuming lot to the space X, okay. So, then we may need to assume less on the additive term f. So, here we are not requiring that to be differentiable, but only Lipschitz. So, Lipschitz function on the, on these 2 variables time and space variables, okay.

So, then for each x in DA the domain of operator, the mild solution is a classical solution to the semilinear evolution problem, SEP, okay. So, let us recall what is the SEP, the full expression which appears here.

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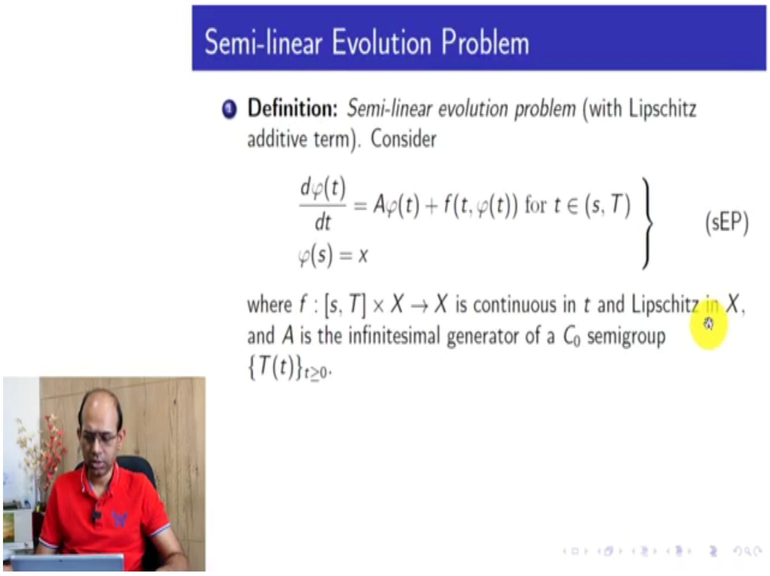

So, this is SEP, d phi dt is equal to A phi t plus f of t phi t. So, here, the dependence of f on phi need not be linear. So, that is the reason we call this semilinear evolution problem, okay.

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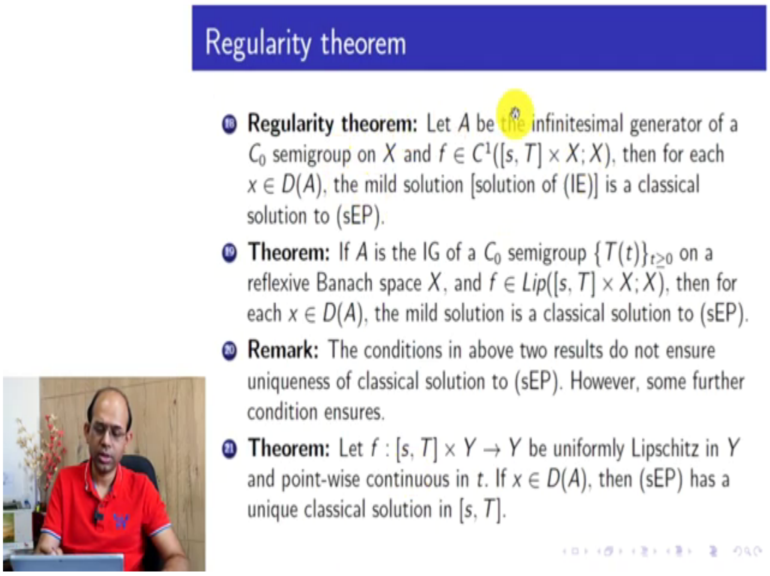

Next, before going to the proof of this Regularity theorem, we make some comments. The conditions in our 2 results do not ensure uniqueness, do not ensure uniqueness of classical solution to SEP.

However, some further stronger condition is required. So, that ensures the uniqueness. So, this is one of the sufficient conditions. So, let f is a map from, you know, so, this is an additive term, for time and space, but here in space instead of x, we put capital Y. So, assume that if you restrict f to the time domain and then then instead of whole Banach space just restrict to the subspace Y, where Y is a domain of differential of the operator A.

So, then the range is, the range is also in Y okay. And then this map is uniformly Lipschitz in Y. So, here this is uniformly Lipschitz in Y. When we say that, that means that to get the Lipschitz property, we would consider the norm, which is endowed I mean which is given in Y, okay. So, Y is endowed with a graph norm. Correct? So, with that, we need to have this f to be uniformly Lipschitz, in Y, okay.

Uniformly Lipschitz, that means that, I mean the Lipschitz constant does not depend on time, okay. One can get a Lipschitz constant which does not depend on time. And pointwise continuous in time variable t so with respect to time variable, it is continuous, so second variable is Lipschitz.

But here, we are putting much smaller subsets and some other different norm. So in under this condition, if x is in a domain of A, okay and A ofcourse generates C0 all those things are as before, okay. So, then this semilinear evolution problem has a unique classical solution in the interval, smallest to capital T in that total duration. Okay. So, next we go to the proof of the Regularity theorem.

So, we are going to prove this theorem, okay. So, here, the main idea of the proof is that since f is given this way, okay, so, we would use the earlier theorem which ensures the existence of a mild solution. So, given these conditions, we would first ensure that it has a mild solution. And then we are going to show that the mild solution has sufficient regularity, okay. So, that is the main idea. So, it has many steps. We would go step by step.

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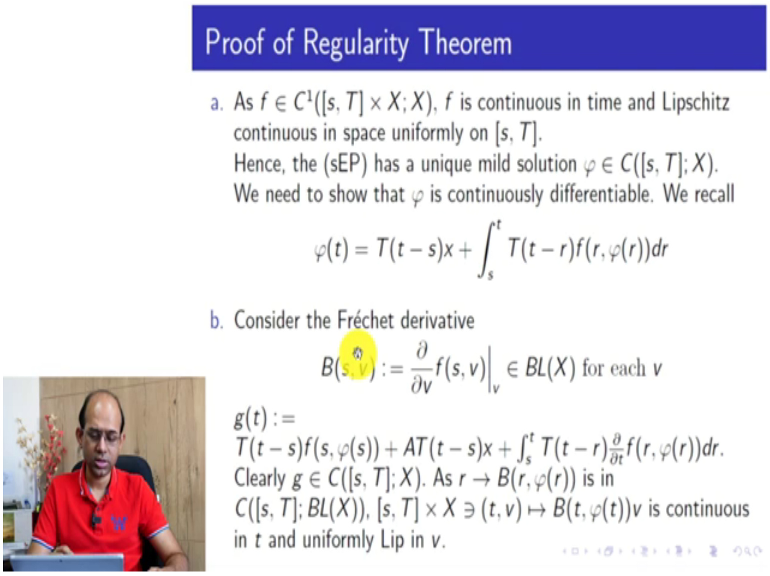

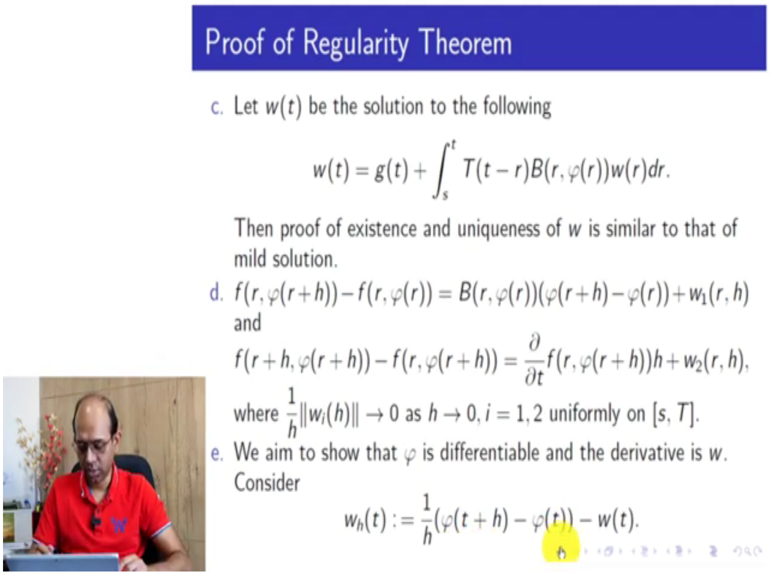

So, as f is in C1, okay, f is continuous. Correct? So, here we are assuming it is continuously differentiable. So, it is indeed continuous. And since it is C1 here, okay, so that means it is differentiable. And the derivative if we consider the derivative, okay so, the derivative, okay. So, that is on the time domain, okay.

And then we are going to get the Lipschitz continuous in space uniformly on s to capital T. Since this differentiable so derivative we are going to get, but that derivative may depend on time. But that time varies on a compact set. So and because the continuous derivative, with the continuity of derivative, we would get the derivative would be bound there. So, we would the derivative, okay.

So, that (function) I mean that the Lipschitz constant, whatever I am going to get would be upper bounded by a fixed constant. So, you get Lipschitz uniformly on the interval s to capital T. So, then we can apply the theorem, which asserts that the SEP has a unique mild solution. So, SEP has a unique mild solution and we call that mild solution as phi, okay so, it is actually varphi. So phi which is itself a continuous function of time.

So, it is actually path, a continuous path okay, on the Banach space capital X. Okay so next, we need to show that the phi is continuously differentiable okay that means, you know in the equation, phi appears on both sides, left-hand side d phi dt appears. So, that should be meaningful, classically, correct, so d phi dt should exist.

So, phi we are going to show that it is continuously differentiable. So, we recall the definition of phi. So, phi of t is, this is the integral equation, which the mild solution must satisfy. So, phi of t is equal to capital T of t minus s x. x is the initial point, capital T is a semi-group generated by the operator A. So, phi t is equal to T of t minus s x, plus integration small s to t, capital T of t minus r f of r comma phi r dr, okay.

And then, we consider the Frechet derivative okay so, this Frechet derivative is the derivative of f with respect the second variable. So, B s comma v is defined as del, del v of f s, v. Okay. So, this thing belongs to BL X, okay. So, that is the meaning of derivative here, because a derivative of these functions with respect to the Banach space value function if I am in the Banach space.

So, here the derivative with respect to v would be a bounded linear map from the space to space itself. So, that is a derivative. And we can write down this, because it is already given that f is in C1. So, that means this derivative exists. So, we can do that. We also define g or another function g of t, which is defined as capital T of t minus s f of s phi s plus A composition of capital T of t minus s. This is semi-group, this is a generator. x plus integration small s to t, capital T of t minus r, del, del t f r phi r dr.

So, what it means that means del, del t f of something is basically the partial derivative of f with respect to the first variable, okay. Whenever I want to get the derivative with respect to second variable, we write down del, del v. Whenever we would like to write down the derivative of f with respect to the first variable, we write down del, del t.

So, this is the form of gt. It is not now yet clear why do I need to consider such complicated, long expression, function. Yes, so actually this appears. This was, this appears in the proof. And intuitively I can say that okay, when we are going to take derivatives you know some terms would arise. You understand that if you take derivatives of this, you know the A terms appear.

So, this term basically appears you know as you know terms which arises after taking the derivative. You see the derivative also appears here. Okay, this is gt. So, we at least know from the expression, that g is continuous path okay so it is a continuous function of time t. Why is it so? Because this this as a function of t, this is T t of something, okay some constant. So, T t is is C0 semi-group. So, this is continuous in t.

Here, for the same reason, that since x is coming from the domain of definition of f. So, T t, so this is in the domain of definition of s. So, this makes sense. And since, T t is C0 semi-group, so this is continuous. So, A T t minus s x is same as T t s, T t minus s A x. So from that we are going to get this continuous in t. So, here, what we need just the integrability of the inside term to get the continuity of t.

And that is true, because here since f is in C1 and then on the close interval it is continuous also. So, the derivative is also continuous on closed interval. So this is bounded during that interval. So this would be, you know more locally integrable or basically it is integrable also. So, we are going to get this thing, okay. So that asserts that g would be continuous.

Okay, so now what we have, we have this B notation, okay. So, that we have which this is nothing, but a member or the member in the BL X and then g t. And now, we are studying the property here. So, r goes to B r phi r. So, phi is this okay, phi of r is a point in the Banach space. So, as the function of r and the first variable is r here.

So, now we are studying this map r to B r phi r. So, this map is also continuous map, okay. Why is it so? Because of this, you know C1 function. So, this derivative is continuous. So, due to that and since phi is also continuous, so it is composition of these 2 maps. So, this is continuous, okay? So, r to B r phi r, so this this map is continuous, okay.

So, this is continuous and then we consider this map t comma v to B t phi t v. Because B, B is okay this whole thing is a member in the BL X. So, this if you apply to a member in the Banach space X then you get a point in the Banach space X. Okay, so, here if you apply that to v, you get some point there.

So, now if you choose an arbitrary a small t in this interval and an arbitrary of small v in the Banach space. And then you consider B t phi t v okay so this as a function of time t and v okay, so this is also continuous in t and uniformly Lipschitz in v. uniformly Lipschitz in v is trivial because this is actually linear in v. Correct?

And then, this changes with respect to time, but this changes continuously so as stated here. And on the compact set, we are considering here. So, the norm of this linear operator it can be upper bounded by a fixed constant. So, that would give me uniformly bounded. So, this is family of linear operators, which is uniformly bounded. So, that would give me that uniform Lipschitz property also. So, this is uniformly Lipschitz in v.

Now next, so what we do is that we write, introduce another function. So, here at this stage, I understand that I am not giving much motivation, why am I using these (you know) introducing these new-new operators and functions, say B, g and now we are introducing w also. But that would be clear, when we go to the details of the proof.

So, let w t be the solution to the following, you know integral equation, w t is equal to g t, g as earlier defined. So, g is defined here. So, it has 1, 2, 3 additive terms, 3 additive terms. And then, we have w t is equal to g t plus integration small s to t, capital T of t minus r, B r phi w r dr. So, w appears both sides. So, this unknown appears both sides. So, this is basically an equation.

So, consider this equation, this another integral equation, okay. And assume that w be the solution to the following integral equation. Okay so, then then the proof of existence of uniqueness of w okay, so basically existence and uniqueness of the solution of this equation is very much same as the existence and uniqueness theorem of the mild solution, okay.

So there we had the different term here T t minus s x, something like that. And here, we did not have B, but some f terms were there. But what I can write down this whole thing as you know we can treat this as, this part as f, okay and you can check that whether the conditions in the theorem is satisfied by the condition here.

Here, that is so, okay so then we can assert that; the existence and uniqueness. Okay, so this is true, because the conditions which just we have expressed here, that the Lipschitz continuity in the v variable, okay. So, as a function here, so for mild solution, this is the form. And then in the equation, new equation w, there instead of this, we get g t. And instead of f, we get this B these things. Correct?

So, here, B r phi r w r. Okay, so, here it is not semi-linear operator; this is a linear operator. Okay, so here we are using proof of earlier theorem the sufficient conditions for the applicability of the theorem is true for this case also. So, we can apply that to obtain that this has a unique continuous solution.

Okay, so w t exists. So, why am I talking all this? Because, just to ensure that w t exists okay. Such, you know there is a continuous function w t okay, which satisfies this. And now, we consider f of r comma phi r plus h minus f r comma phi r. Okay, so this increment.

So, this increment, if we would like to apply the differentiability property of f here, we would be able to get that this difference of you know this this second variable increment is equal to B of r comma phi r, the derivative, okay and the point phi r. So, this left hand point here. And then, multiplication with this difference phi r plus h minus phi r plus some other, I mean error term w 1 r h.

As h tends to 0, this 1 over is w 1 must go to 0. Correct? I mean that is, I mean that is implied by the differentiability of f with respect to the second variable. Okay, so this is the notation basically I would like to introduce the w 1 here, error term here. Okay so, next we consider the increment to the first variable here, the difference here.

Second variable looks as it is. So, first variable is f of r plus h phi r plus h minus f of r comma phi r plus h is equal to del, del t. So, now since it is increment with respect to first variable and f is you know differentiable, continuously differentiable. So, I would be able to write down this difference, using the derivative with respect to the first variable. So, we write down del, del t of f of r comma phi r plus h, okay times h, h is the increment here.

This is r, r plus h, so h is the increment plus say another error term w 2 r h, so only one thing we know about this term is that, 1 over h of w i this goes to 0. Correct? So, 1 over h, norm w i h, okay so that goes to 0 as h goes to 0, okay, uniformly on this. Okay so next, what we do is that, we aim to show that phi is differentiable. Okay?

So, that is our main goal that phi is differentiable. And the derivative is w, okay. So, now it is clear, that why am I introducing these notations like g, B, w, etc. See I mean, phi I would like to show that it is differentiable, that then the derivative is also very relevant and important thing.

And there, that the derivative of phi okay, we are going to show that is w. And to write down w, I will need to write down this integral equation. And this integral equation involves g and the operator B, okay. So, that was the main reason to introduce this function g and B just to get the w, okay.

Now consider this w h t, this is also another error term, so, 1 over h phi of t plus h minus phi of t minus w t, okay. So, we need to show that this error term goes to 0, okay. If I can show that this error term goes to 0, as h tends to 0, then we would be able to justify that phi is differentiable and the derivative is w, right? We would be able to assert both the things, okay. If we just prove that w h is equal to or goes to 0 okay, in the subsequent slides.