Introduction to Probabilistic Methods in PDE
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Lecture 64
Mild-generalized solution to Semi-Linear Evolution Problem

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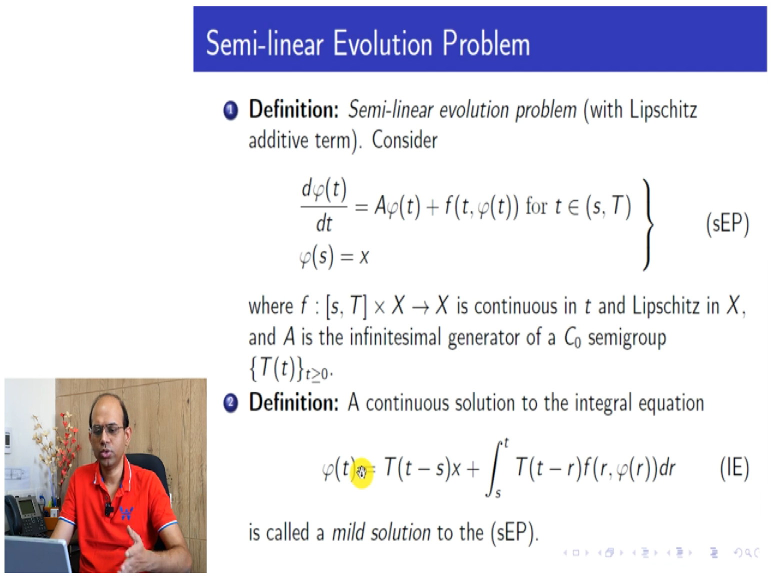

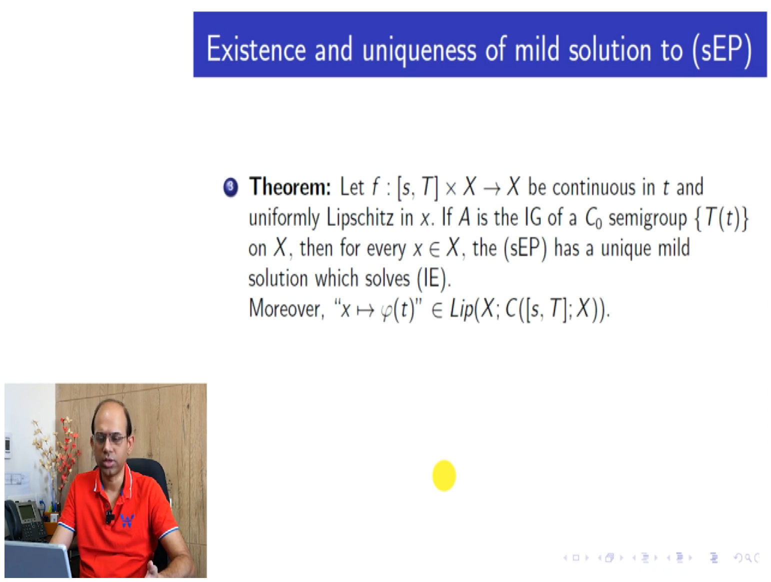

Welcome, today we are going to see a new class of equations. So we call this Semi-linear Evolution Problem, okay. So what is the difference between this problem with problems which have discussed earlier. In all earlier problems we had this phi this unknown appears only d phi dt and here and either we did not have any additive term or we have additive term but that is independent of the unknown phi.

But here for now we are assuming that this additive term also involves phi and this could be a non-linear manner. One can think that okay if it is just a linear function of phi then one can put that function inside this you know operator definition of the operator. So that is not a new problem.

So a truly new problem really appears when this dependency is not linear, however if it is not linear then things become completed also. So we are going to assume some sort of we are going to assume Lipchitz continuity of f on this second variable. So this kind of problem, so is new to which we have not yet discussed and since we are going to discuss this so we are going to put our main focus for this.

So we are assuming simpler sittings here about the operator A, we are here (allowing) not allowing it to depend on time t. We are just assuming that A is constant so it is autonomous operator here. So this is extension of iivp inhomogeneous initial value, inhomogeneous initial values problem where f of t is replaced by f of t comma phi t.

So here we consider this equation, so starting point is assume to be s, so phi of s is equal to x okay, for and then time t is between small s to capital T. So this equation we henceforth going to refer to as sEP, s for semi-linear, e for evolution, p for problem, semi-linear evolution problem and here f is assumed to be a continuous map so f is a function of function of two variables the time variable and the space variable here.

So the time variable this is coming from there close interval is small s to capital T and here this x variable this is enough because phi is a solution of every time t it is taking value in the Banach space. So phi of t is a point in a Banach space so that is here, so this is the member from capital X and the result of A for or the range of f is also in the capital X.

Okay so this map is assumed to be continuous in time variable t and Lipschitz continuous in this you know second variable and A is the infinitesimal generator of a C0 semigroup T t, okay that is also I mean included in our assumption. Then we define what we mean by a mild solution of this equation.

So here this notion of mild solution becomes quite different from earlier notions. Although equation looks similar why is it so, because the unknown appears here. So if we try to write down the formula for variation of constant. There that f appears in that integration but then this phi also appears, so this is not a formula because the unknown phi appears on the both sides.

This is basically an equation, got it, so here we define mild solution in the following manner, a continuous solution to the integral equation phi t is equal to capital T t minus t minus s of x plus integration of small s to t, capital T of t minus r f r phi r dr. So that is called a mild solution to the sEP, okay.

So here since A is time autonomous so I need there is a semigroup T t, so we are using that semigroup T t for writing down. So it appears exactly as same as we have obtained for formula for variation of constants only difference is that here this is not a formula but this is rather an equation because the phi is appearing on the both sides.

So what we are going to always you know refer to this integral equation. So if I say that okay let phi be a mild solution of sEP I mean phi be a solution of this integral equation, okay. So not arbitrary solution but a continuous solution I mean that is phi as a function of t is continuous, okay.

Now we should ask the question the existence and uniqueness of mild solution of to sEP. See this is a question which we never asked before, why? Because that mild solution was just a formula I mean formula is written here. So we did not ask need to ask the question whether that exists or not but here it is an important question that whether it exist.

So this you know equation is like you know Volterra equation type of things if it is (continuous) if it is linear, if correct, because here this integral phi appears here and here. And then you have a variable r running from s to t. So it is not very difficult to establish the existence theory of this equation. So we are going to recollect that thing, we are going to prove this thing rigorously, this theorem.

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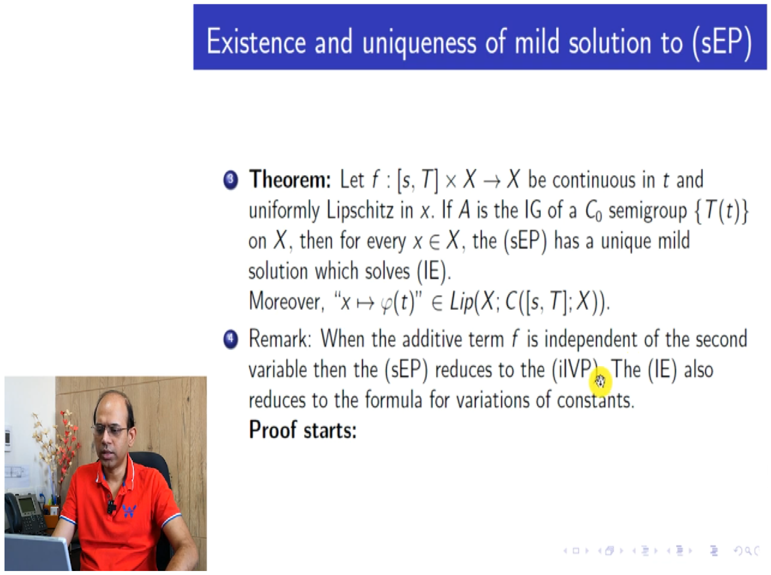

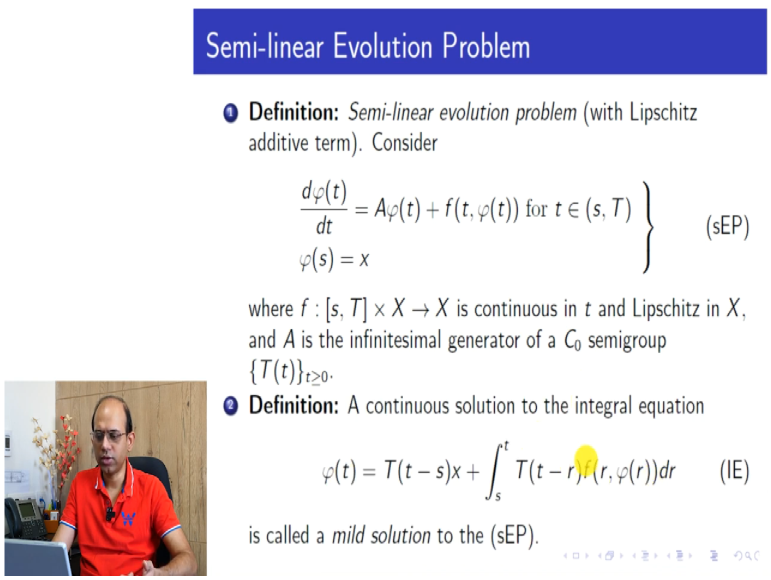

So let me state the theorem first. Let f be a function from close interval s, small s capital T cross X to X, okay so this be continuous map in time t in this T variable. And uniformly Lipschitz in x what does it mean that I mean so f of small t comma x okay that you know as a function of x is a Lipschitz function.

But the constant, Lipschitz constant that is independent of this time, okay that can be chosen independent of this time variable. If A is the infinitesimal generator of a C0 semigroup T t a on X then for every small x in capital X the sEP has a unique mild solution which solves the integral equation.

So this is the theorem see here we are choosing the initial point small x is not from inside the domain of the definition of operator A but from anywhere from the Banach space capital X. So because we are not going to, we are not asking here for existence or uniquness of classical solution we are just demanding just existence of mild solution.

So for existence of mild solution is concern we can still restrict ourselves although we can allow the small x to be anywhere from the Banach space capital X. And then conclusion there is a another conclusion is that as a function of small x where small x is the initial point, okay.

So for every small x we get the whole path phi t, correct, so this path now for if you vary small x you get another path because phi t you know for every time it moves in the Banach space. So as a function of time it is path, phi is continuous also so we get you know a connected path there. So that map, small x to the path okay, if you vary x you get various different paths and then this map x to the power so that this map is also Lipschitz.

So this is also another nice property we are going to get here. Okay before starting the proof, let us have some remarks here, remarks says that when the additive term f is independent of the second variable here in this case. So then this is just iIVP so this reduces to iIVP. And then the integral equation what we have written in the earlier slide so this integral equation so if f is independent of this same variable, so then it is not an equation itself.

So this is just a formula so this coincides with formula for variation of constants. So this is the remark I mean we are making here and that then the integral equation also reduces to the formula for variation of constants.

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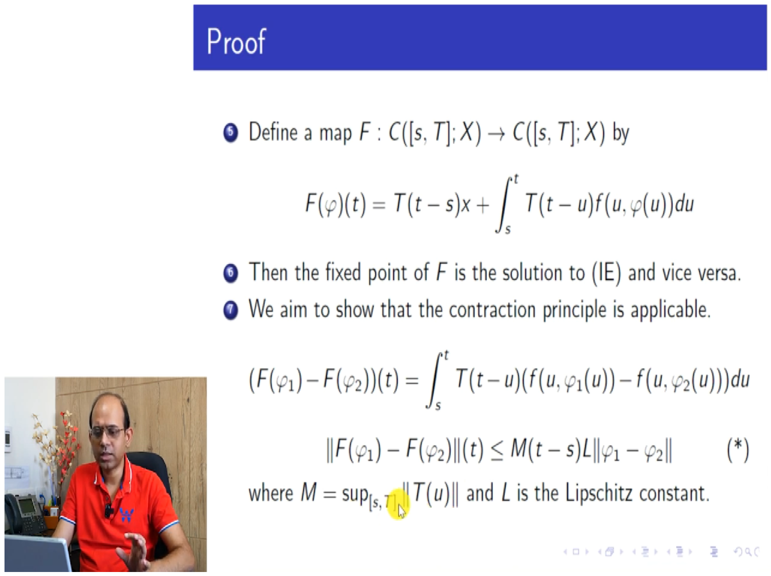

So proof starts, so this proof is little lengthy okay, but it is not tricky it is elementary. So the main idea is basically for you know existence of solution of integral equation. Most of the time relies on coming up with a map which should be established as contraction to argue that would have a unique fixed point and that fix point would solve the purpose of the solution of the integral equation. So that is the line of the proof, so we start here.

So define a map capital F, the map is a function, okay this capital F what does it do, it takes a path so path, I say path because you know it sounds you know more closer to this thing because the function from time to x. So F takes a continuous path to a continuous path so this by this following formula.

So this is basically the right hand side of the integral equation, correct, capital F of phi t is defined as capital T of t minus s x plus integration from small s to t capital T of t minus u f of u phi u du. So this is basically the right hand side of the integral equation. So if right hand side on the integral equation is written as capital F of phi then the integral equation becomes phi is equal to f of phi.

So the solution of the integral equation is nothing but the fixed point of the map capital F, so that is the idea. Okay so then the fixed point of F is the solution to the I mean integral equation and vice-versa. So we aim to show that the contraction principle is application for this map capital F. So that is the idea.

So what we do is that this is a very standerd approach so for contraction we just need to evaluate capital F at two different arguments. Here arguments are the paths phi 1 and phi 2 so capital F of phi 1 minus capital F of phi 2, okay so that evaluated at t, so that is obtained by integration I mean this difference of this right hand side.

So here I have capital T of t minus s x but this does not depend on phi 1 or phi 2 so this term will be the same and they would cancel each other for phi 1 phi 2 case. So this would not happen the difference only this part would appear. Here in this part I have integration small s to t capital T of t minus u since it is a linear operator.

So I would get this difference here small f of u comma phi 1 u minus small f of u comma phi 2 u du. So now here we are going to take the norm of this term because these are the terms which is in the Banach space capital X we are going to find a norm in that space X, X norm basically.

So for this what we do is that capital F of phi 1 minus capital F of phi 2 at t is less than or equal to here for this integration we know that this F is Lipschitz in second variable. Since it is Lipchitz is in second variable and capital if capital L denotes that Lipschitz constant and actually it is uniform Lipchitz, correct?

So capital L that is Lipchitz constant could we regarded you know independent of the first variable. So write down capital L here, L here so this would be you know norm of this can be upper bounded by L times norm of phi 1 minus phi 2, okay evaluated at u but that can also be (upper bound by) phi 1 minus phi 2 and then for capital T what we do is that we take supremum a supremum from over this interval small s to capital T, the maximum of this that is capital M.

So this is finite, why is it finite? Because capital T is a C0 semigroup and then since it is C0 semigroup of bounded linear operators, so on a compact set here on s to capital T, so this is a bounded thing. So its supremum I would get a finite number capital M, so that we use here capital M, okay.

And then t minus s is coming because of this integration, because if all these constant comes out of this then we get t minus s du. So I mean here this thing is written very briefly, this is actually you know obtain in 2 steps, one is norm of phi1 minus phi2 of u okay for each and every u but they but we would like to get rid of that, we take even supremum over all possible u.

So that it would give a norm in this sense, and that is phi1 minus norm of phi1 minus phi2, okay. So, you obtained this inequality and we write down that is star. We are going to use this later also.

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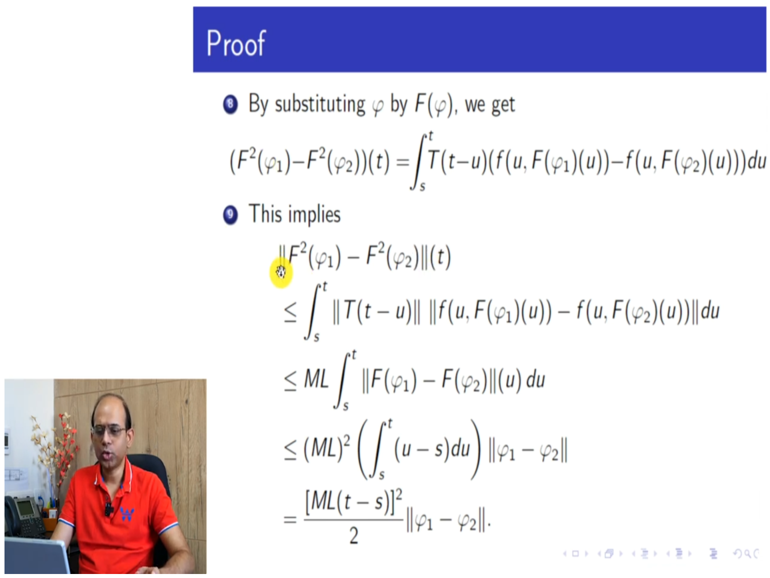

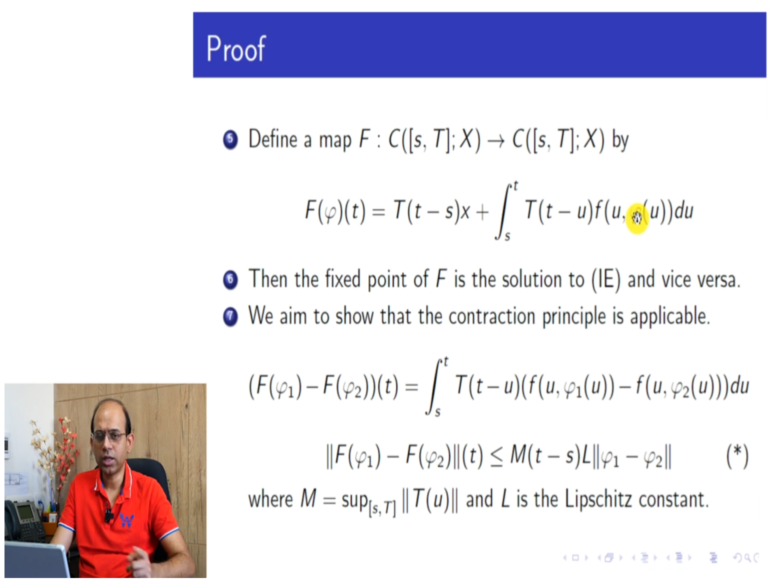

So now we substitute phi by capital F of phi in the earlier thing here, here if I substitute phi by F of phi then we get F square of phi, F of F of phi. If composition on F composition of phi. So here also instead of phi I should get F of phi here.

Okay so we do that by substituting phi by F of phi we get F square, okay so I call this F square is F composition F so this is my I mean like you know double composition, F square of phi 1 minus F square of phi 2 of t is equal to is for the same reason here as we have obtained this difference we are going to get same thing only except here instead phi 1, we are going to F of phi 1, correct?

So, here we get capital integration of small s to t with respect to u variable du and here your capital T of t minus u f of u comma capital F of phi 1 u minus small f of again u comma capital F of phi 2 of u du, okay so now we are going to get the estimates and norms etc. So here we take norm here both sides.

So norm of F2 of phi 1 minus F2 of phi 2 t is less than or equal to integration small s to t here this norm here and then this difference norm appears here norm of f of u, F of phi 1 u minus small f of u, capital F of phi 2 of u norm du here. And then already you have defined capital M in the earlier slide exactly using that capital M I can upper bound this by capital M.

And for the same reason as earlier since this is the second variable and small f is Lipschitz continuous in second variable, uniformly Lipschitz continuous second variable, so we can upper bound these by capital L times norm of difference of F of phi 1 minus F of phi 2, so you do that, F of phi 1 minus F of phi 2 u and du here. And then we get this integration sign there.

So next what do we do for this again we apply this thing, norm of F of phi 1 minus F of phi 2 is less than or equals to M L phi 1 minus phi 2 t minus s, correct, so here also we have that F of phi 1 minus F of phi 2, so there I am using the inequality star okay, to get again M L and then u minus s and then phi 1 minus phi 2 norm of phi 1 minus phi 2.

And then here this integration u is running from small s to t. So what do we get is that M L, so here if we do this integration I would get there is a result of these t minus s whole square by 2.

So that is did not together M L, these are all constant t minus s whole square by 2 norm of phi 1 minus phi 2, okay

So now this thing we are I mean if we now take supremum overall possible small t, small s to capital T we get a further this thing, so here we get I mean first instead of F2 by induction we can get F3, F4, etc. And only instead of two I mean it will be 3 and 3 factorial etc would come, due to this reason we always would have this. And then, so there you would get u minus s to the power of some orders.

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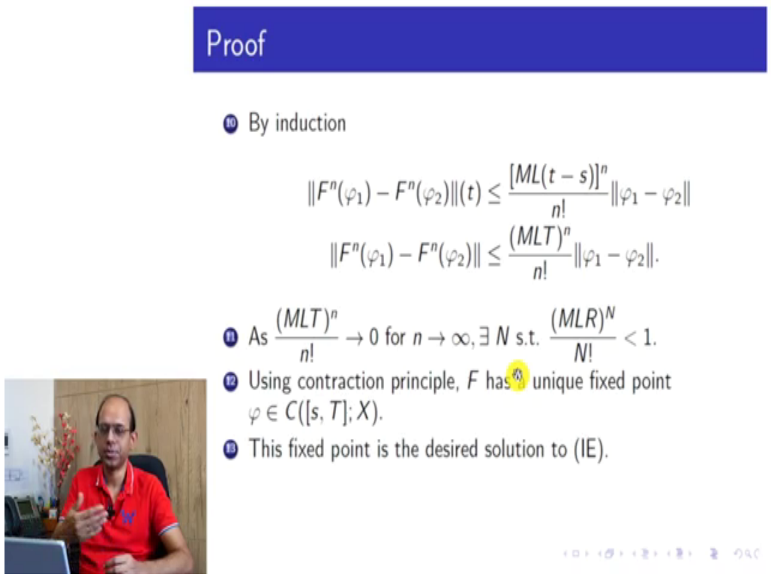

So we would get Fn, norm of Fn phi 1 minus Fn phi 2 t is less than or equal to ML t minus s to the power n by n factorial times norm of phi 1 minus phi 2. Here after this we take supremum over all possibile t between small s to capital T. So when you do that left hand side is this and right hand side that would be t minus s is replaced by capital T, so that it dominates all other terms.

So this to the power of n by n factorial norm of phi 1 minus phi 2, so this is for a particular n whatever n you can there is no upper limit, you can make n as large as possible. But you would still get this inequality and then as n tends to infinity we know that this term converges to 0, because this term appears in the expansion of exponential function.

So that constitutes the term of a convergent series. So this converges to 0, so given I mean does not matter how large M L or T are this whole thing may go, would be as small as possible for a large M. So one can choose sufficiently large n so that this is less than 1, correct? So as MLT to the power n by n factorial converges to 0 for n tends to infinity, there exists a capital N such that ML so there is typo it is capital T.

T to the power N by N factorial is less than 1. What about that N is? We fix that N, then we have this relation that capital F is a map such that capital F to the power of n phi 1 minus F to the power of capital N phi 2, this now is less than equals to some constant which is less than 1 times norm phi 1 minus phi 2, if that holds then we can actually apply the contraction principle.

So we apply now contraction principle for the function F using contraction principal F has a unique fixed point. F has a unique fixed point and let us denote that unique fixed point by phi. So phi is function from time to X so is a path continuous path. So these denotes the fix point. So by this we have I mean although this is unique fixed point, but by this we have not really proved uniqueness. But we have surely prove existence.

Because since F has a fixed point and since fixed point is the same as the solution of the equation of the integral equation. So we have surely assured existence of a solution of the integral equation. However, why does it not imply uniqueness because this is just a one particular way to obtain a solution of the integral equation, okay?

I mean ofcourse if the integral equation has another solution then one can ask that both should be fixed points of the same function. And if both are but then it is unique fixed point. So in this sense it is saying that but we cannot conclude from here that Lipschitz property from X to this thing.

So what we do is that here now next few arguments is going to establish that uniqueness part again and also the additional property that Lipschitz property of the Lipschitz dependence on initial data.

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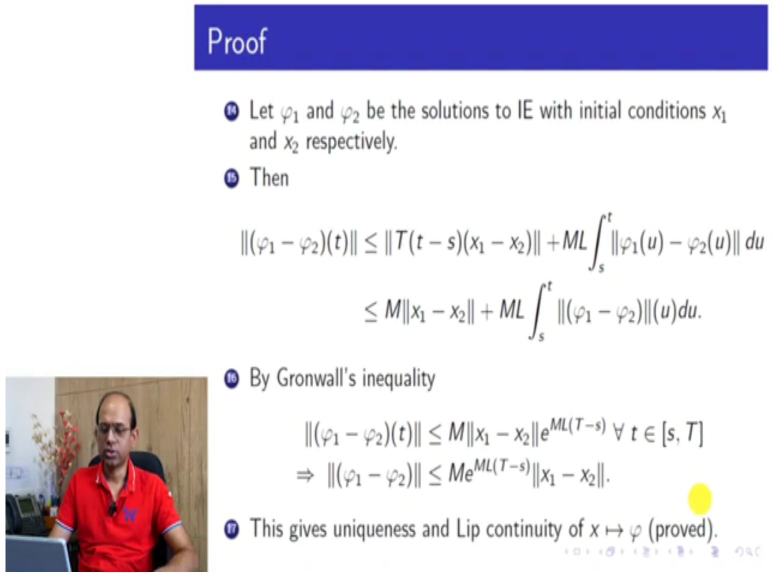

So next let phi 1 and phi 2 be the solution of integral equation with initial condition x1 and x2 respectively. So assume that with two different initial conditions x1, x2 we have phi 1 and phi2. So then phi 1 minus phi 2 of T norm this is less than or equals to capital T of t minus s x1 minus x2 norm plus ML times integration s to t phi 1 u minus phi 2 u du, okay.

So this we obtained directly from the equation because integral equation if I write down phi 1 and phi 2 and if we take difference we are going to get T of t minus s x1 minus x2 because both starts with different thing. Since both starts from two different points x1, x2 so this term survives here earlier since we talked about both start from same x so they disappeared.

But here they are here, so norm of phi 1 minus phi 2 of t but this whole thing is a real valued, correct so real value function of time t plus ML. So here we had this capital T t minus s capital T of t minus r f of you know r phi r dr. But there you know when we apply the Lipschitz property and then the boundedness property or the semi-group and we get M and L.

So we get this estimates of our that line and then again here also we apply that upper bound capital M, so we get capital M times norm of x1 minus x2 here. So here I have ML times integration of small s to t norm of phi 1 minus phi 2 u du. Okay so by Gronwall's inequality what we because why, how can I use the Growall's inequality because now one can consider phi 1 minus phi 2 as t as a function. So it is real value function, correct, because this norm is there.

So real this value function so here also this norm comes, so this real value function appears in both sides I have this inequality. So whenever one has these inequality we have seen earlier Gronwall's inequality that for this we can write down estimate for the solution I mean the function which satisfies this inequality.

What is that estimate that norm of phi 1 minus phi 2 t is less than or equal to M times this constant would appear. M times norm x 1 minus x 2 and then this would appear in the power of e. So e to the power ML and then here this difference is t minus s so small t minus s but then we are taking you know all possible t so if t I mean at most it could be capital T, it could be at most capital T.

So we upper bound by capital T minus s, so the right hand side becomes independent of small t it does not depend on small t so that, okay e to the power capital ML times capital T minus s, left hand side of course depends on small t right hand side does not. So if we take supremum over all possible small t from the left hand side it will still obey this same upper bound.

So it implies that norm of phi 1 minus phi 2 is less than or equals to capital M times e to the power of ML capital T minus s norm of x 1 minus x 2. This is same thing I just rearranged, so this norm of difference appears here, M appears from here and then e to this part appears here. So by this we can we would be able to conclude what you wanted with the desired Lipschitz property of the solution I mean Lipschitz dependence of the solution to the initial point.

Because the x 1 minus x 2 if you change you know that thing is changing this way. So this gives uniqueness and I mean uniqueness also we have obtained earlier, say for example if we choose x 1 is equal to x 2 both are equal than it is 0. So phi 1 and phi 2 be identical, okay

So the norm of that will be 0, correct? So that would give again uniqueness and also it would give Lipschitz continuity because here this is Lipschitz constant of this norm. So this is the end of the proof. Thank you very much.