Introduction to Probabilistic Method in PDE
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Lecture 63
Y-value solution

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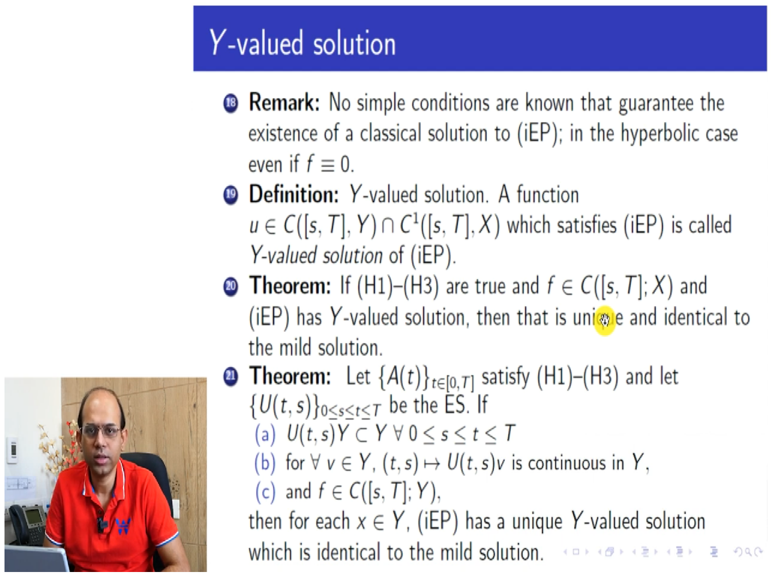

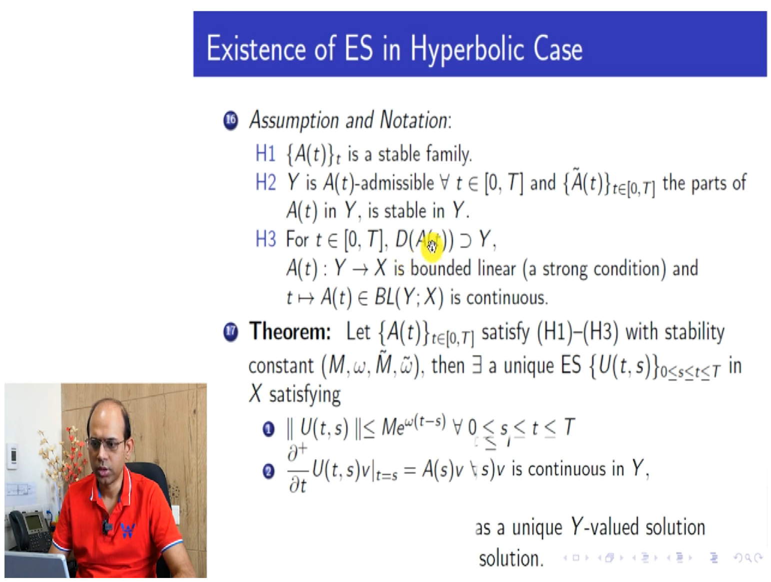

Welcome, let us see, what we have done in the last lecture towards the end, towards the end in the last lecture, we have introduced Y valued solution. Y valued solution is basically, is little stronger in notion than the classical solution. So, there we have seen this definition and that if Y is you know here is considered to be subset of the domain of the generator of the operator.

So, let us look at quickly what is Y, so here see H3, so this H1, H2, H3 are these three conditions for hyperbolic case. Where it is assume that Y is a subset of the domain of the operator for each and every time t, whereas this is a time, non-autonomous system for each and every t you have different different operators. However the common domain of definition of this operators do contain in Y.

So, that was the assumption that was Y here and then using that Y we have defined what we mean by Y valued solution that this is a function u which is continuous with respect to time and I mean from t to, for each time it is going to Y. So, it is a function from time domain to the space Y. So, it is a continuous map and also it is once continuously differentiable as a map from time to X.

Although Y is the subset of X, but Y has a different norm than X. So, I mean thing is not same as C1 s T, Y. So, here put that differentiability in this sense C1 s T, comma X. So, it is a map from this interval to the Banach space X. So, a map, a function which satisfies the inhomogeneous evolution problem is called the Y valued solution and then we have just stated theorem, where two theorems.

So, here if this hyperbolic assumptions are true H1, H2, H3 are true and if that additive term is assumed to be continuous then and also in addition to that, if you assume that the inhomogeneous initial value problem has Y value solution then that is unique and that is identical to the mild solution. So, here basically it is saying that the mild solution becomes the Y valued solution.

The next theorem say, includes one sufficient condition under which the existence of Y valued solution can be ensured. So, here we assume that A t satisfy H1, H2, H3 and let U be the evolution system we can talk about evolution system, because the family of operators satisfy H1, H2, H3 and the earlier theorem, which ensures existence of evolution system associated with this family operators.

So, using that if we have this additional conditions that the image of Y under U is contained in Y, so this a part. Then the b point is for every v in Y t is to U t s v is continuous in Y. So, that means U t s you know gives a continuous mapping and then, third thing is that f is a continuous map, so that was also assumed here earlier also.

But these are a and b are the two new conditions which appear here. So, under this one can conclude that the inhomogeneous evolution problem has a unique Y valued solution. In earlier theorem we did not put any condition. So, we had to assume that it has Y valued solution but here under these condition a b c we can conclude that it has unique Y valued solution and then we also assert that, this the same as the mild solution, this coincides with the mild solution.

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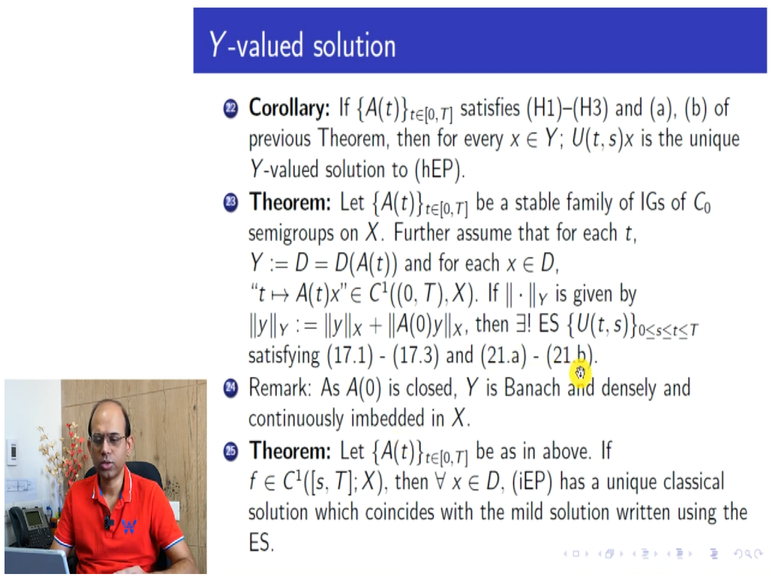

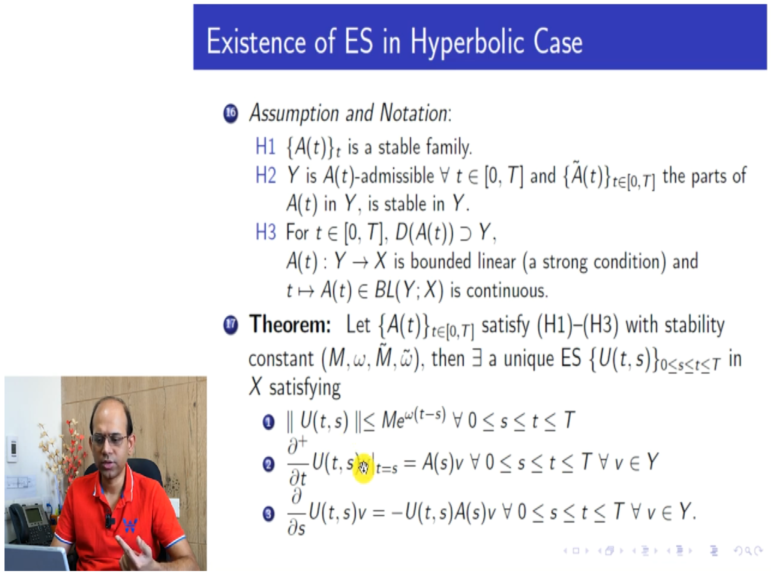

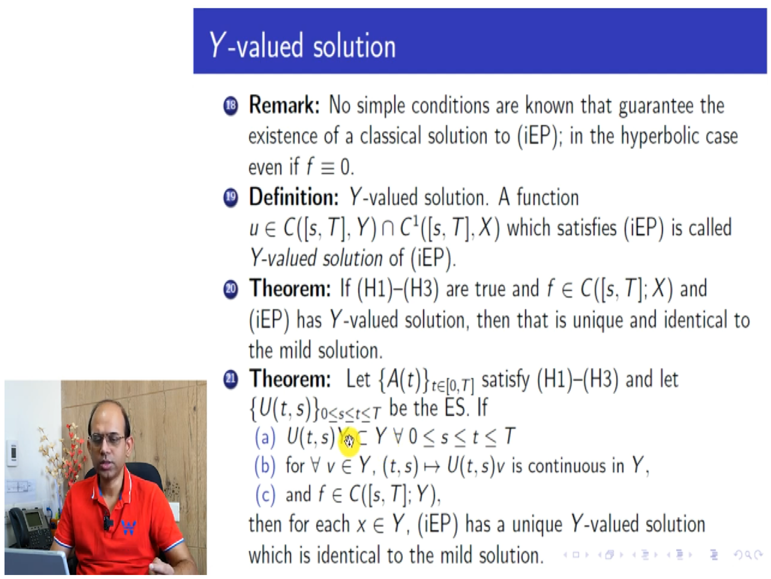

So, next we see few more discussions about this Y values solution which you have not done yesterday in the last lecture. So, this Corollary, so here if do not have the additive term f. So, that is the homogenous part. So, but capital A is still non autonomous it depends on time. So, if At is satisfies H1, H2, H3 and a b of the previous theorem that also holds then, for every x in Y, every x is an initial point.

So, initial point is being chosen not from the any part of the Banach space capital X. But in the inside the common part of the domain of every operators and that we call Y, Y is contained in that, so it is in Y, so x is in Y and you know that for these cases I mean when you choose initial point inside the domain of definition of the operator. Then you can get, then you can get a classical solution.

So, here also that anticipation holds true here U t s x. So, this you know the evolution operator x is a unique Y valued solution to the homogeneous initial value problem. So, where f is equal to 0 case. Now in the next this theorem this says that, let. So, let us read this. Let A t this collection of operators be a stable family of infinitesimal generators of C0 semigroup, what does mean? That if you fix t, then A t is a operator and that particular operator is generator of this C0 semigroup.

So by these we mean, so for each and every t this is infinitesimal generator of this C0 semigroup. So, the semigroup what we obtain will also remain on t small t. So, but this t will be differ from the semi groups own time. So, than that semi group should be written at capital T say subscript small t and the parenthesis s, so Tts.

So, s is the time parameter for the semi groups you know for family parameter and small t just corresponds to the parameter coming from the operator. So, A t be I mean we assume that it is stable family of infinitesimal generators of C0 semigroup on capital X. Further assume that for each small t, real time I mean this time t. Y which is defined as D that is domain of A t. So, basically you are assuming that domain of definition of each of every operator A t coincide with each other and we call that capital D and we take that as our Y.

For and for each x in D then or in Y, we consider this map t to A t x, t to A t x is this map is assumed to be continuously differentiable, continuously once differentiable. So, what is this map doing it is taking a time point t and then it is giving a point in the Banach space, in Banach space capital X because A t x goes to somewhere. But this x, I mean this is meaningful why because I am choosing x from D, D is in the domain of our definition. So, A t x is meaningful. So, t to A t x this map. So, this map is assumed to be continuously differentiable.

Now if this I mean as I mentioned earlier, that the norm on Y may be different from norm of X. So, basically it should be different, so this norm is written here. So, this norm of Y, so that is given by this is graph norm correct norm of Y small y is a member in capital Y. So, norm of Y is defined as norm of Y in X norm plus the image of y.

So, under A0 because there are plenty of A1, I mean A t, t ranging from 0 to capital T, we just pickup A0. So, A0 y and then norm of that. So, here these thing is giving me a norm. So, basically this is the graph norm of the operator A0. So, if we assume this norm, then under this norm etc, then we can state the following.

Then there exist a unique evolution system U t s satisfying the 17.1 to 17.3 those are actually this things that U t you know satisfies this conditions this is differentiability basically that these U is associated with A and then this is the growth property. So, and in addition to that we would be able to conclude also 21.a to 21.b these are just two properties that U t s Y is a subset of Y.

That means image of Y under U t s is contained in Y and also this property the strong continuity of U variable. So, these conditions are true, so let me go back and check what is the conclusion of this theorem, this theorem just assumes the family of operators A t is stable family of infinitesimal generators of C0 semigroups and then this is the first condition we are assuming.

Second thing what we are assuming is that, the domain of definitions of A t coincides and that is Y and then third thing what we are assuming that for each x in D, the t to A t x, so this map is continuously differentiable. So, under this three conditions and also I mean this is clarification of the norm what we are choosing on the space Y. So, we can conclude that there exist an evolution, an evolution system U t s associated with A and satisfying these nice property, this properties.

So these properties implies that, this equation would have Y-valued solution. For a homogenous case it would imply if I mean this homogeneous case you do not need to assume c part because f is 0 it is trivially true, so this is a nice result. So next, this remark says that, as A 0, so this theorem is important because, here we assume this property but we do not give a condition on A under which this is true, but this theorem is interesting because it gives that.

It says that if A has this a very nice property. So, for this special case, those two assumptions are true. So now, this is a remark that as A 0 is closed, so about this norm basically if since A 0, so closed, so that means you know its graph of A0 is closed and then this norm what we are going to get would give me capital Y as a complite metric space under this norms, so Y is Banach space and this is, you know as we have already seen the A is, if your A generates a semi group then the determine of definition is dense the Banach space, so it is dense and continuously imbedded in X.

Next theorem, this is also theorem what I am stating with our, your proof here. So, let A t be as in above and so like this theorem, if f is in addition to this is in C1 or that means it is continuously differentiable, a map from time to X Banach space. Then for every x in D, so the initial point we choose inside the domain of definition. The inhomogeneous evolution problem has unique classical solution, has unique classical solution.

So, here this is the reason as I have just explained that because this is very strong condition, this is a very strong condition and under this condition, the a and b part of point 21 is true. So, we can apply the theorem 21, which assures the existence of Y-valued solutions also. So, it is not, so actually this is like a corollary of the earlier theorems. So, has a unique classical solution which coincides with the mild solution written using the evolution system.

So, by these we conclude the discussion about inhomogeneous evolution problem. Here we have as a sum for summary, let me recollect that what we have done is that we have stated, what do you mean by inhomogeneous evolution problem and then what we have done, we have written down the mild solutions formula. So, that formula is motivated from the formula for variation of constants and then from after that, we asked that mild solution is okay, but when can you assure a classical solution.

Then these last few theorems they are talking about that. So, here we need to assume some further strong conditions on the operators A. So, under which one can, so these are sufficient conditions, under which one can assure existence of classical solution, when initial point is chosen from the domain of definition. So, none of these actually talk about, then when initial point is not in the domain of definition, then how to assure existence of the classical solution. If one wants to do that for some particular special case, then one has to do something more.