Introduction to Probabilistic Methods in PDE
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Sufficient condition for existence of evolution system
Lecture 62

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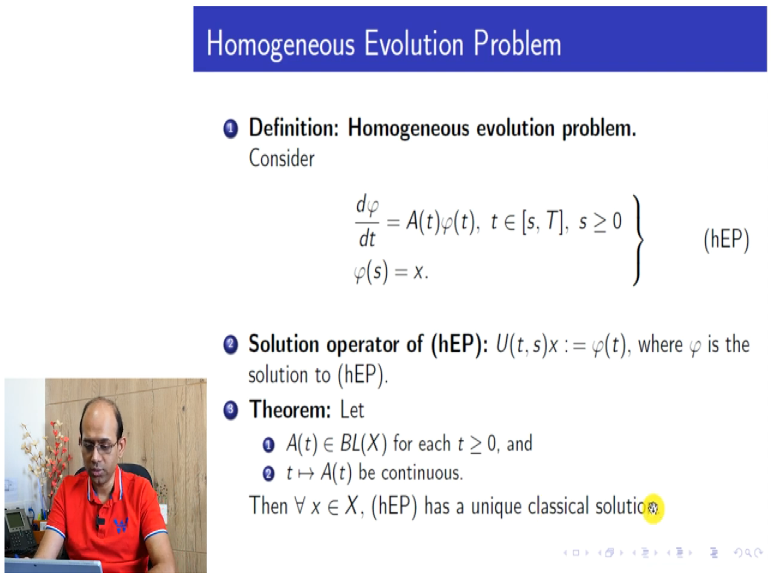

Welcome, let us revisit the homogenous evolution problem what we have done in the last lecture very quickly. So, this is the homogenous evolution problem, why do you call it evolution problem? Because in this Cauchy problem the operator it is non-autonomous operator. So it depends on time explicitly.

So, here this d phi dt is equal to At phi t is the equation and with initial condition phi s x where t is between s small s to say some capital T. So, we are here presenting all these finite horizon case and then what we have done? We have introduced the solution operator. So, homogenous evolution problem is this one, for this solution operator is written as capital U of t, s x, what is this, how is it define, it is defined as a solution of this equation.

See this equation gives me one solution which depends on time. So, it is you know time varying, so this is like a path on the Banach space on the Banach capital X. So, at time s it takes the position small x and then it varies. So, this is phi of t and this solution operator capital U is a two perimeter family operator is defined this way that, when you apply this operator at the initial position small x and then what you are going to get is a solution at time t. So, given if at s, if the position was x, small x, time t you are going to get the solution at time t.

So, this is the definition of the solution operator. Then we have saying the theorem, this is very special case where this operator is a bounded linear operator like matrices. So, it is bounded linear operator, so then one can explain it very nice result because our, the initial value problem, that is time autonomous case there one could get actually that for each and every initial point small x, in the Banach space one can get the solution and the solution was also written in a very simple manner just, just it was T t x where T t is the semigroup generated by the operator.

So, here also we are going to get similar kind of things if we assume certain things. So, here t to A t if this map is continuous map for autonomous case it is just constant map, constant map is continuous anyway. So, but here when we allow A to vary with t this t, A t is continuous and then for all small x in capital X. This homogenous evolution problem has a unique classical solution.

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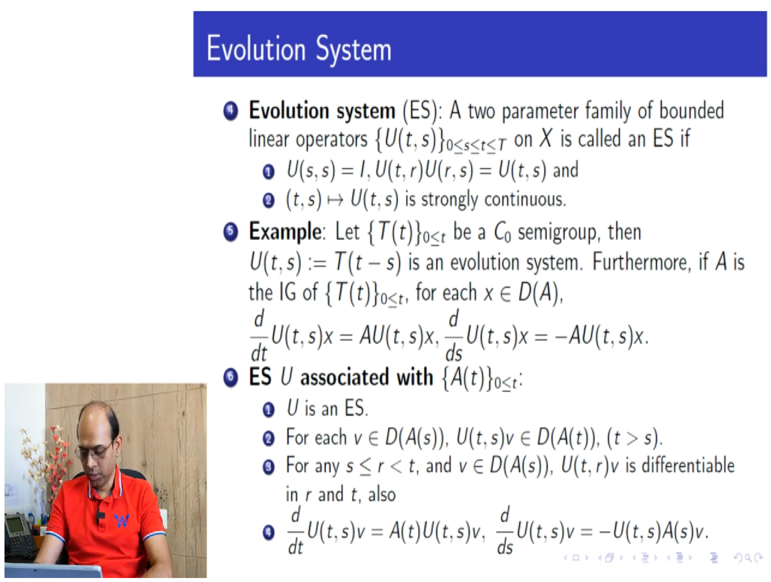

Next, we introduce what is evolution system, evolution system or in short ES this is a two parameter family of bounded linear operators. So, U t s, we write down this way and this kind of yours two parameter family of operators would be called ES or evolution system. If following two conditions true that U s s is equal to I identity operator.

When there is no initial and final times are same, then it is just identity and U t r U r s is equal to U t s, this is like generalization of the similar property. That we have discussed in quite detail and another thing is that regularity property that t s to U t s is strongly continuous, what does it mean? That means t s to U t x the point in Banach space. So, that map is continuous with respect to the variables t and s.

So, example that if we have a C0 similar Tt and you define U t s as capital T of t minus s. So, then this forms evolution system. So, this is an example further more if capital A is the infinitesimal generator of a C0 semi group T t for each small x in the domain of A, we can write down, we can get the following things, that derivative of U of U t s x for this U t. We are going to get A U t s x and derivative with respect to A s, if we do, then are going to get minus A U t s x this is trivially obtained because you know we know that derivative of T t x is equal to A t t x.

So, just we are applying the, because we are applying this relation here to obtain this expression. Next point is that evolution system U associated with A t. So, this is the thing we are going to see now today that what do you mean by evolution system associated with family of operators. See there is no single operator, the family operators, t greater or equals 0.

So, first thing is that, this U should be an evolution system a two parameter family which satisfies these conditions. So, it should be evolution system and then for each small v in the domain of A s remember here A is not a single one, there are plenty of A. So, one should be very careful that which A we are talking about.

So, here time s is the initial time, so we put s here. So, if v is in the domain of A s and then U t s v that you should also belongs to domain of A of t. So, if v belongs to in the domain of A s then U t s v which is also member in the Banach space. So, that is actually solution that is phi t. So, that should I means we do not require it to belong to the domain of the same operator. The domain of the operator A of t, so this condition actually puts some condition on family of operators A t also because in A arbitrary family operator A t you may not be able to get anything, any such existence.

So, anyway so we are going to talk about the sufficient conditions the A t for which one can assure existence of such usual system that is the separate topic. We are going to see today also that thing. But here we are talking about when this things are true, then we say that this capital U is associated with family of operators A t.

So, the third point is that for s and r, s less than or equals to r. So, here we require that the expresion of the derivative, that how A and U are related. So, this is the derivative the d dt of U t s v is equal to A t U t s v. So, its look like almost like what is written above, but of course there is some difference because here A was constant, A are not a function of time but here it depends on time.

So, d dt of U t s v is equal to A t U s v is one condition another thing is that derivative with respect of s variable, then we know that capital T t minus s minus sign comes here. So, I should expect a minor negative sign here. So, negative sign U t s A s v.

So, it is important notice this difference that here comes A first and then U, here U comes first and then A and here also one must be careful that the condition that here the second property was crucially used that when v is in domain of A s then U t s v is in domain of A t. Others requires, others wise this expression makes no sense, because if v is in the domain of A s which was assumed here then U t s, if that is not in the domain of A, then these is does not makes sense.

Anyway, so and here there is no such trouble because v was anyway in the domain of A. So, it makes sense A s v, U is anyway define on the whole Banach space. So, this is the definition of evolution system associated with family of operators in A t. So, I mean why are there, I mean two different definitions, I mean so this is just definition of evolution system. When can we call a family of bounded linear operators and evolution system, we can call when this is true. But this is a different question here, here we are asking that, given a family of operators A t when can you say that a particular evolution system is associated with that family of operators.

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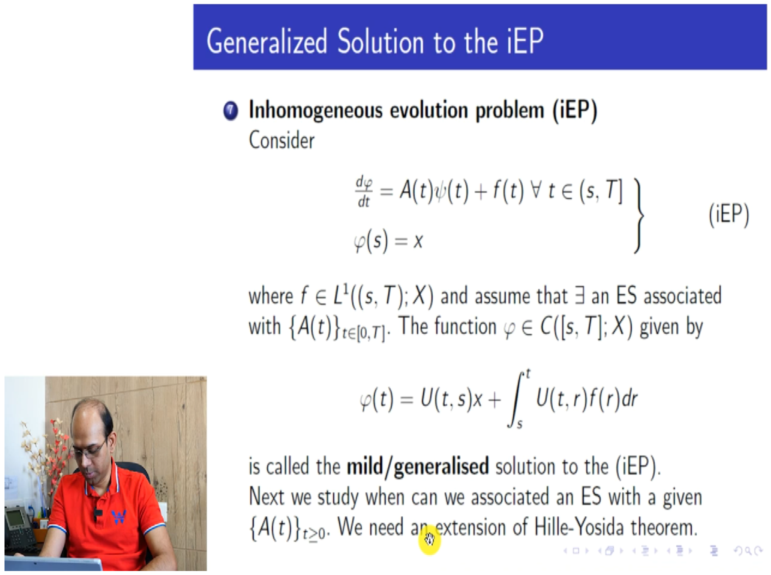

Go to the next slide here this, slide we have seen in the last lecture that inhomogeneous evolution problem iEP in short. So, this there is typo here should I have written here phi, d phi dt is equal to A t phi t f t, phi t plus ft and then phi s equal to x. So, this is true for all , s to capital T.

So, this is the equation and here, you call this, inhomogeneous because of the presence of small f and here f is assume to be L1 as before because when we have discussed the inhomogeneous initial value problem iIVP there also we assumed f to be in L1. So, we cannot drop any assumption there because it is more generalization we can, in certain cases we have to put more condition but not less for existence of mild solution or classical solution whatever we discuss.

Where f is in L1 and assume that there exist an evolution system associated with A t. So, then the function phi which is a continuous function on the time domain to Banach space X given by phi t is equal to U t s x integration to 0 to t U t r f r dr. So, this is called the mild generalized solution. So, it is very easy to recognize that this is nothing but, the extension formula for variation of constants. So, this things also we have discussed in the last lecture. The next to study, when can we associate an evolution system with a given A t, that is a main important thing. So this is like you know a statement similar to the Hille-Yosida theorem.

So, when given a iEP or you know hEP or in other words given any evolution problem, we would be given a family of operators At and then to write down its mild solution, we first need to know, whether we can find evolution system as such with this family of operators. If we can then we are going to see that we can write down here, say for example, assume that there exist an evolution system then we can write down the mild solution using the evolution system.

So, for this question whatever answer we are going to get that would give us a sufficient condition. Set of sufficient conditions under which given family operators would admit existence of the evolution system associated with this and we would, you know assume those sufficient conditions throughout for our development.

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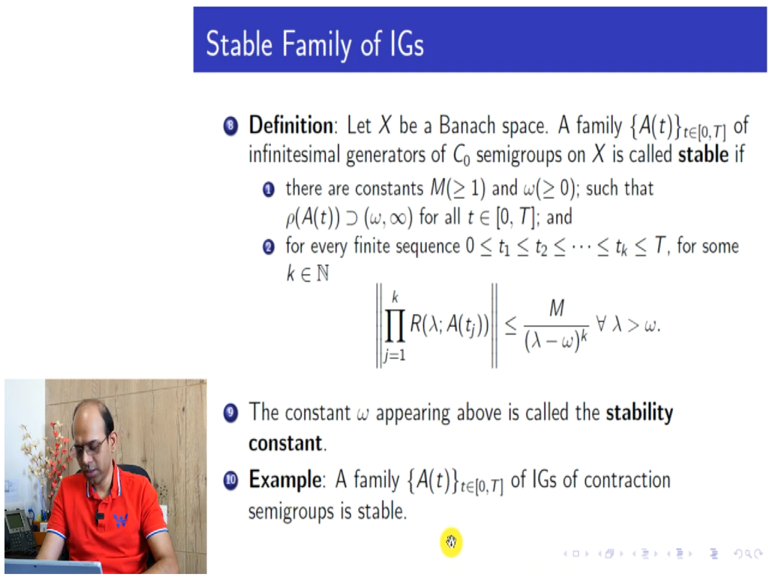

So, here are some more new definitions are recalled here. So here, let X be a Banach space a family A t of infinitesimal generator of C0 semi group on X is called stable if there are constants capital M which is not less than one and omega which is not negative, so greater or equals to 0, such that the resolving set of A t includes this interval open interval omega to infinity.

This includes this for each and every t, between 0 to capital T. So this is saying that, okay I mean but omega is some omega. So, that means at least there exist I mean, this row for each and every t A t would have the resolvent set, but the intersection of these includes at least one ray, ray we mean that omega to infinity this set.

This is one condition and another condition is that for every finite sequence t1, t2, tk k is some natural number, where this t1 and tk is less than or equal to capital T. So, between 0 to capital T we have considering a partition, basically some finite sequence increasing sequence and then for this t1 to tk we construct this product.

So, this is this should be understood or arrange in a following manner R lambda A t1, then composition R lambda A t2 actually, this is the smaller numbers come on the right. So, in a particular order, in the order of this time because these operators need not commute each other. So, this product should be understood in a particular order.

So, after fixing this order then integral norm and then that is less than equal to capital M divided by lambda minus omega to the power k and this true for all lambda greater than omega. So, surely this is an additional condition on the family A t, because this should be true satisfied you know for every lambda greater than omega.

Omega would be a fixed number which does not depend on t and then so here k1, k is any arbitrary natural number. So, this thing should be true. So, this puts a condition and if a family satisfies these two conditions we call that family a stable family, a stable family. The constant omega appearing above is called also the stability constant. So, in Hille-Yosida theorem also we have seen similar kind of you know upper bound there, this is the operator.

So, example a family A t, so this of infinitesimal generators of contraction semi groups is stable. Because contraction semi groups, this capital M becomes 1 and then the omega also becomes 0 because there all these A ts etc, those are norms always less than or equals to 1. So, there one gets that this would be a stable family of semi groups.

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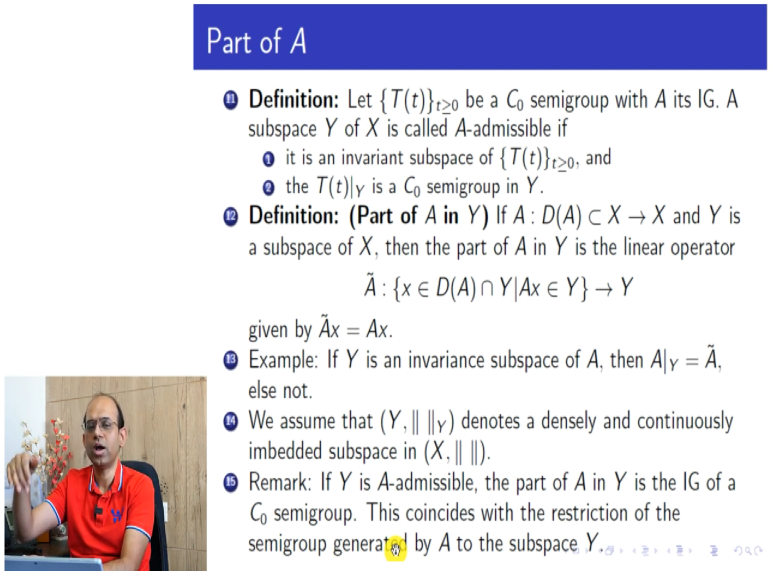

Now another new notion what we call as part of it is some sort of restriction of an operator in a subspace, where it is not strictly speaking a restriction but a little more, one has to be careful that restriction not only the domain but also we require a desire property on the range space also.

So we define in this manner, so let T t be a C0 semi group with A as its infinitesimal generator a subspace of Y of X is called A-admissible, A-admissible if it is an invariant subspace of T t. So, that means that from any point Y if we take, if we take a point from Y, small y say and if T t of y would also be in Y. So, this is invariant subspace second is that T t restricted to Y is a C0 semi group in Y. So, that is that means that from Y it takes value to Y itself, C0 semi group on Y.

So, definition of part of A in Y, so what is this, this is if capital A is a map is it bound, is a linear map on X, but not on full X, but on some domain of A. So, which is subspace of X. So, A from D A to X and Y is a subspace of capital A X. Then the part of A in Y it is you know sounds quite similar to the restriction.

Because A was defined on some domain of A and Y is another subspace of X, Y need not be strictly you know subset of D domain of A. Y is some other linear subspace. So, part of A in Y is the linear operator A tilde given by A tilde X is equal to A x what is A tilde? Where is it defined? It is defined for all point x which is in the domain of A and intersection of Y.

So, the point x which is common in the subspace Y and the domain of A and such that the image of Ax under A is also in Y. So, if we choose x from Y also in domain of A and then we look at the image if Ax is also in Y. So, those points, so this would surely be subset of domain of A and Y and but the good point is that A tilde takes point x from here to Y itself. So, it is going to Y itself.

So, this is given by A tilde x is equal to Ax. So, it is like you know exactly like the A map, but the domain should be carefully chosen, that we call as a part of A in Y. So example, if Y is an invariance subspace of capital A, then A restricted to Y is precisely A tilde otherwise it is not true. Next we assume that Y denotes a densely and continuously imbedded subspace in X. So, here in this above discussion point 12 we were discussing some Y is some, subspace.

But here now onward we are going put a particular condition on Y, we are going to put that. This Y denotes a densely and continuously imbedded subspace. What is the meaning of densely defined? That means the space Y under this norm, what were the topology if it was closer than or that is X and then what is the meaning of continuously imbedded?

That means the identity map from Y to X, where Y is subset of X I can take identity map from Y to X in the sense that the domain we take the topology the norm of Y and the right hand you know the codomain we take the topology coming from the norm on X and then with this two topologies, so the identity map becomes continuous. So, using these two if these two are true then we say that densely and continuously imbedded subspace of Y and we are going to assume that, throughout we are going to assume that.

So, before we are going to next slide, we look at this remark, if Y is A-admissible, the part of A in Y is the infinitesimal generator of a C0 semi group itself and this coincides with the restriction of the semigroup generated by A to the subspace Y. So what does it mean? It is like a diagram commutes kind of things, it is a saying that.

So, if I have the operator A here on the domain of A in the Banach space X and then from that this is that A assume that generates a C0 group T t and from this A again with the given Y we can get a part of A and then the part of A is an operator from that also I can get another semigroup generated by this operator. So, assume that all of this you can do, then here I have a semigroup Tt, here I have a T delta say and then this is saying that the restriction of Tt on Y is precisely T tilde. So, that is there remark.

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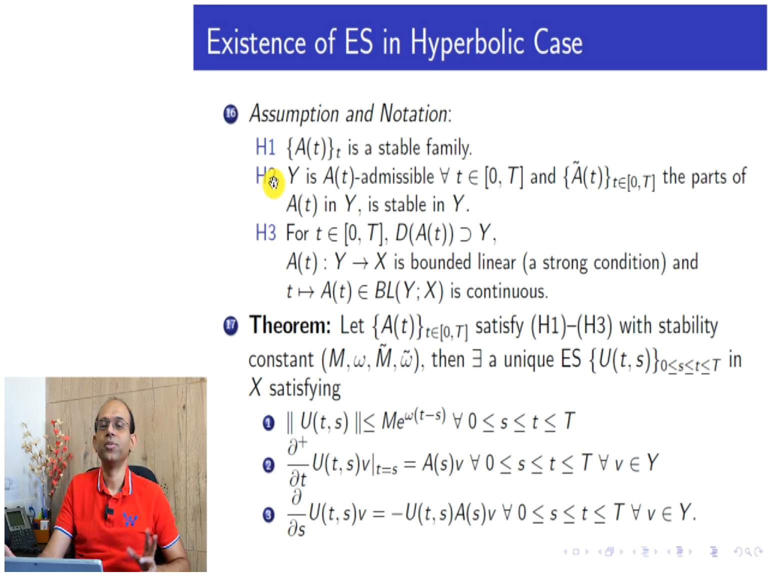

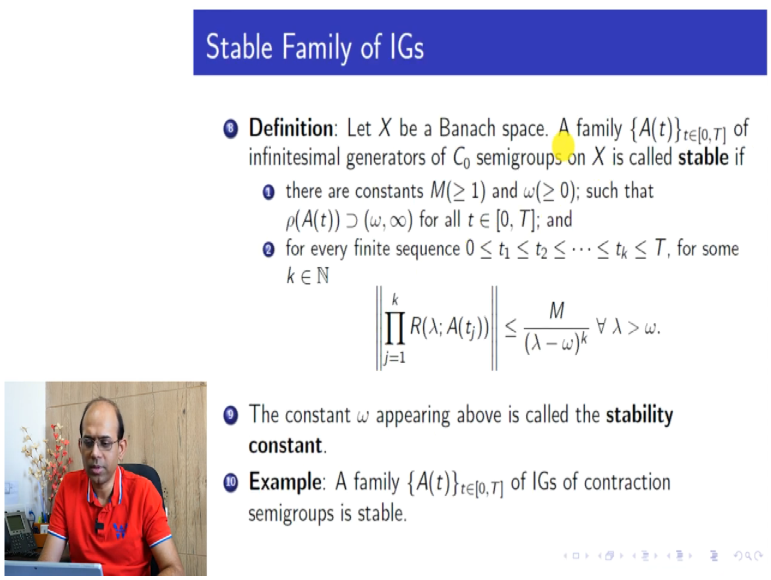

The next we talk about Existence of Evolution System in Hyperbolic Case. Assumption and Notations. So, there are three conditions we are going to stick to, so that is called Hyperbolic Case, I mean that is that name actually arises from the application of this theory in different branches of partial differential equations. So, here there are three conditions I am reading first one first, A of t is a stable family. So, this stable family I have just discussed here, so this is a stable family, a family of operators is stable if these two conditions are true.

So and omega is called stability constant. Here first is that A t is a stable family. So, that is one condition. H2 is that Y is A t admissible for all t. So, we assume that Y is this and A tilde t which is a part of A and Y that is a stable family. So, H1 says A t is a stable family, A2 says that even its parts in Y that is also stable in Y and Y is A t admissible. Third condition is that for every t non-negative 0, capital T time, Y is included in the domain of A t or whatever time you have, 0 to the whole horizon here.

So and then A t from Y to X, I cannot talk about A t from Y to X because you know domain of A t includes capital Y. So, for every member in capital Y A t of that member is well defined. So, A t from Y to X is bounded linear and the time t to A t is a continuous map. So, this is important because if A t is not bounded linear then I cannot talk about continuity of, in this sense. So, it is bounded linear operator and then this is continuous in the norm topology.

So, under the norms topology. So, this is H1 H2 to H3. So, if any family of generators, family of operators A t, satisfies these three conditions all together then we call that family is satisfying the hyperbolic case. So then theorem, so let A t is satisfy H1 to H3, H1, H2, H3 with stability constant M omega M tilde omega tilde.

Why do we write down four instead of two because these capital M and omega are stability constant is coming from the operator A and M tilde omega tilde is coming from its part in Y. So A tilde coming for A tilde. Then there exist, so under this condition there exists a unique Evolution System capital U in X, unique evolution system.

So this H1, H2, H3 with these three conditions there is unique evolution system and satisfying some usual properties that norm of U t s is less than or equals to capital M times e to the power omega t minus s. So, these are direct extensions what is true even for the semi group of operators Tt and another thing is the derivative here since there are two variables, so I should be very careful with respect to who whichever one you are differentiating. So, d plus dt, so the derivative with respect to time t but positive derivative and then put t is equal to s, so positive slope, positive slide slope.

So that is equal to A s v, why does should it not depend on time t? Because here time t is equal to s is substituted, t is equal to s has been substituted, so this is A s v. Third property is that the partial derivative with respect to s variable.

So, as I have said here earlier that for the special case where A is time autonomous, there this is like capital T of t minus s and then d to the minus sign one should expect a negative sign here also is equal to minus U of capital A of v. So, this is a theorem which talks about existence of evolution system, and it I mean without any doubt it clarifies that this H1, H2, H3 are formed a set of sufficient conditions.

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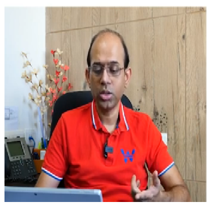
Y-valued solution

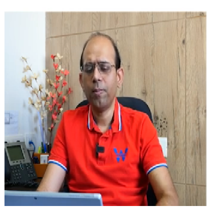
18. **Remark:** Unfortunately we do not know any simple conditions that guarantee the existence of a classical solution to (iEP); in the hyperbolic case even if $f \equiv 0$.
19. **Definition:** Y-valued solution. A function $u \in C([s,T],Y) \cap C^1([s,T],X)$ which satisfies (iEP) is called *Y-valued solution* of (iEP).
20. **Theorem:** If (H1)–(H3) are true and $f \in C([s,T];X)$ and (iEP) has Y-valued solution, then that is unique and identical to the mild solution.
21. **Theorem:** Let $\{A(t)\}_{t\in[0,T]}$ satisfy (H1)–(H3) and let $\{U(t,s)\}_{0\leq s\leq t\leq T}$ be the ES. If
    - (a) $U(t,s)Y \subset Y \; \forall \; 0 \leq s \leq t \leq T$
    - (b) for $\forall \; v \in Y$, $(t,s) \mapsto U(t,s)v$ is continuous in Y,
    - (c) and $f \in C([s,T];Y)$,

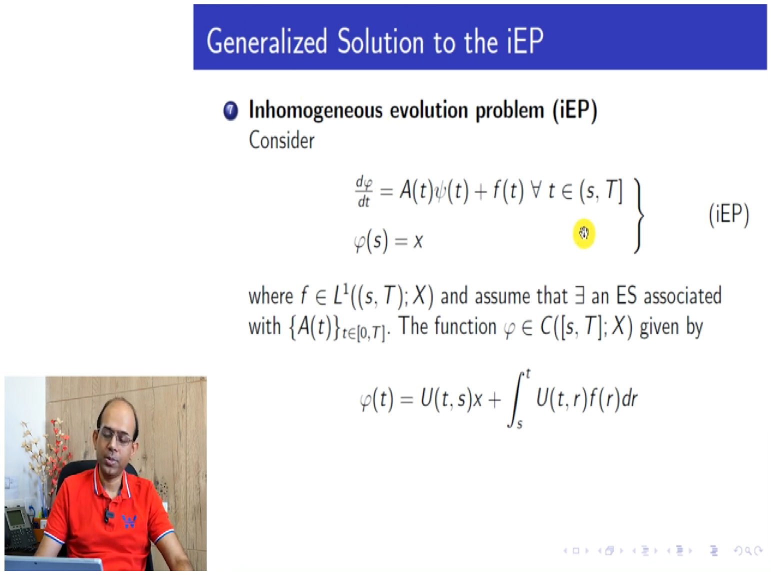
    then for each $x \in Y$, (iEP) has a unique Y-valued solution which is identical to the mild solution.



Existence of ES in Hyperbolic Case

16. *Assumption and Notation*:
    - H1 $\{A(t)\}_t$ is a stable family.
    - H2 Y is $A(t)$-admissible $\forall \; t \in [0,T]$ and $\{\tilde{A}(t)\}_{t\in[0,T]}$ the parts of $A(t)$ in Y, is stable in Y.
    - H3 For $t \in [0,T]$, $D(A(t)) \supset Y$, $A(t): Y \to X$ is bounded linear (a strong condition) and $t \mapsto A(t) \in BL(Y;X)$ is continuous.
17. **Theorem:** Let $\{A(t)\}_{t\in[0,T]}$ satisfy (H1)–(H3) with stability constant $(M,\omega,\tilde{M},\tilde{\omega})$, then $\exists$ a unique ES $\{U(t,s)\}_{0\leq s\leq t\leq T}$ in X satisfying
    1. $\| U(t,s) \| \leq Me^{\omega(t-s)} \; \forall \; 0 \leq s \leq t \leq T$
    2. $\frac{\partial^+}{\partial t}U(t,s)v|_{t=s} = A(s)v \; \forall \; 0 \leq s \leq t \leq T \; \forall \; v \in Y$
    3. $\frac{\partial}{\partial s}U(t,s)v = -U(t,s)A(s)v \; \forall \; 0 \leq s \leq t \leq T \; \forall \; v \in Y$.





Next we come to another definition, so this is called Y-valued solution. So, here first thing is that given inhomogenous evolution problem, we have already seen that we, what should be the expression of mild solution and then we asked the question that well while writing the mild solution, I really do require an evolution system and then under what condition on the family of operators, one can assure existence of some evolution system and for that last theorem answers to that questions that if the evolution systems satisfies H1, H2, H3 then there exist one evolution system.

So, to solve one inhomogeneous evolution problem one just needs to check this H1, H2, H3 if one is lucky and the those conditions are true then one can get evolution system from earlier theorem and then one can apply the theorem for writing down the mild solution of iEP. Using some the extension of formula for variation of constants.

But by this we are settling down the question of existence or uniqueness of mild solution. How should I write down that, but the question about classical solution has yet not been discussed. The thing is that for classical solution one needs much more stronger condition, it is difficult to find out a particular sufficient and necessary and sufficient condition for, on the family of operators At, so that the equation would have a unique classical solution.

So, here this remark says, something here, that unfortunately we do not any simple conditions that guarantee the existence of a classical solution to iEP, inhomogeneous evolution problem. In the hyperbolic case even if small f is equal to 0. So, hyperbolic case means H1, H2, H3. To those appear to live strong, however those are not that stuff. So for example, A t is bounded linear operator, so this is very strong.

But one should also look at that if it is X to X then just really strong, but here I have Y. So Y may have some different norm also. As long as Y remains continuously and densely imbedded in capital X. So, A t is bounded linear operator here with these norms choice. So, now you go to the next definition that what is Y valued solution? So, what are we doing here? Since answer to the question of classical solution is difficult, we are looking at some another class of solutions which is also classical solution, but that requires little more condition that the solution is in Y.

So, we will let us see the detailed definition here Y-valued solution. A function u as a function of time, so continuous function from s to capital T to Y and then also to once continuously differentiable on s T to X. So, a function u which satisfies the inhomogenous evolution problem is called Y-valued solution of iEP. So, main thing is that here these function should be solution iEP and second thing is that it should be continuous function of Y and it should be differentiable also.

But for differentiability, we do not take a norm coming from Y but norm of X. Theorem, so H1 to H3 are true, so that is our assumption. If H1, H2, H3 if these hyperbolic case and small f is continuous function, a continuous function on these s to T. So, when you say that it is some continuous function on a compact set here, it is s to T. So, that means it is automatically bounded and a bounded function with respect to the Lebesgue measure is integrable.

So L1 property you do not need to say separately, just f is continuous clarifies that on the compact set. So, if H1, H3 are true and f is a continuous function and the inhomogeneous initial, the inhomogeneous evolution problem has Y-values solution. So, here we are assuming if it has Y-valued solution, then the conclusion is that, that the solution is unique and that can be written as a mild solution.

So, and identical to the mild solution. What is a mild solution we are talking about? We are taking about this relation phi t is equal to U t s x plus integration small s to t U t r f r dr. So, this is the mild solution, a generalized solution to the iEP. So, here we mean that under that condition

H1, H2, H3 and if he is continuous and one would get that, the solution is identical to the mild solution, the one which is an extension of the formula of variation of constants.

So, there is one theorem the, another theorem says that. So, in this theorem we have assumed existence of Y-valued solution. But here we are going give sufficient condition for that also. So, let A t satisfies H1 to H3, so since its satisfies, this family of operators satisfies H1, H2, H3. So, that assures resistance of U the evolution sustained. So, that means that we can write down the mild solution and then if in addition to that U also satisfy the following tow conditions, following two conditions that U t s of Y is a subset of Y.

So, basically it is saying that Y is an invariant subspace under U. So, image of Y under U t s is in Y itself for all time for all s and t. b is that for all small v in Y. So, point in Y or it must have small subset correct. So, t s to U t s v is continuous in Y. So, this is continuous of course in X because this is coming from the evolution system. But, we are asking that this continuous in Y, so that is an additional condition.

So if these two conditions are true for this evolution system U and another additional condition on f that f is continuous function. So, with these three conditions, then for each small x in capital Y, the iEP inhomogeneous evolution problem has a unique Y-valued solution. So, that means this is a sufficient condition, sufficient condition. So, this suffices that the existence of unique Y-valued solution which is identical to the mild solution. So, let me stop here, thank you.