Introduction to Probabilistic Methods in PDE.
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Lecture 61
Non-autonomous evolution problem and mild/generalized solution.

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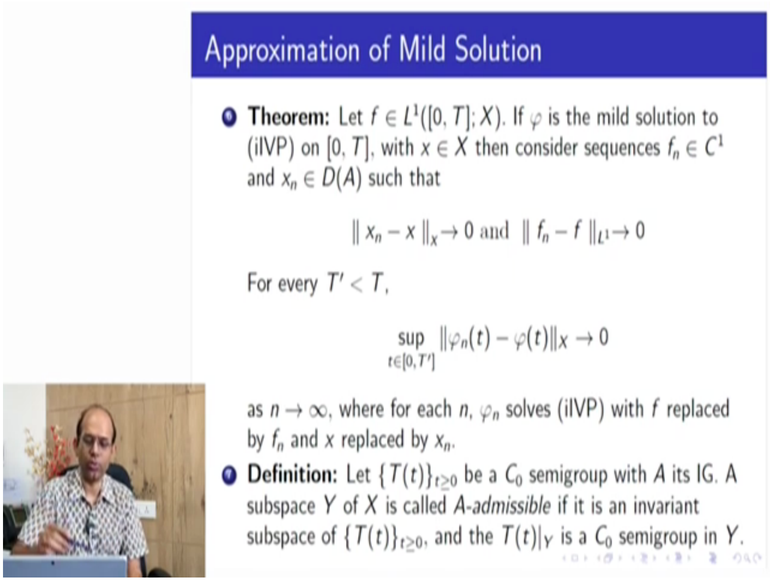

Next we are going to discuss evolution problem. What is that? So here remember that the equations whatever we have discussed, homogeneous or inhomogeneous there the operator was time independent. The operator was not a function of time. Now, if operator is a function of time, then the notion, the way we are discussing that A generates a semigroup etc does not necessarily extend, because every time t the A changes. So, we need a different settings, we need a new set of definitions, which you should basically generalize the notion of semigroup.

So, like A which is not a function of t is a special case of this evolution problem where A is a function of t. So, whatever we are going to introduce for that general class, if we restrict we should get similar result, what we have obtained for homogeneous or inhomogeneous initial value problems. So, for this we need some definitions in the beginning. So, we start with this definition.

Let Tt be a C0 semigroup with A its infinitesimal generator, a subspace Y of X is called A-admissible if it is an invariant subspace of the semigroup, Tt semigroup and the restriction of the semigroup group Tt on the subspace Y is C0 semigroup in Y. So, here a semigroup, a subspace Y would be called A admissible if it is invariant subspace of Tt that means, Tt of Y would belong into Y. So, it is invariance. So, Tt of Y does not leave the subspace Y, it does not go outside.

So, Tt of X is of course inside X but if we take a subspace of X, Y, it is not necessary that the image of Tt and image of those points, the subspace under Tt would also be in that subspace. But okay, so if we have such subspace Y the image of Y under Tt is inside y for each and every t and furthermore, if we also have that Tt restricted on Y. So, that thing is a C0 semigroup in Y. So, if we just restrict the, I mean if we just consider the same Banach space norm on Y. So, you get a norm linear space there and Y need not be a complete space in general.

So a norm linear space there and in that norm linear space, if you know this, Tt restricted to Y is a C0 semigroup in Y, if that thing also is true, then we call Y as A-admissible. Why A? Because Tt is obtained from A only correct because A is the generator, A is the operator which appears in the Cauchy problem, from A whatever the semigroup Tt you have obtained and then we are looking at the subspace of the Banach space X, which satisfies two conditions, one is that, that Y is invariant subspace of Tt and Tt restricted Y is the C0 semigroup in Y. So, if both are true, then we call Y as A-admissible.

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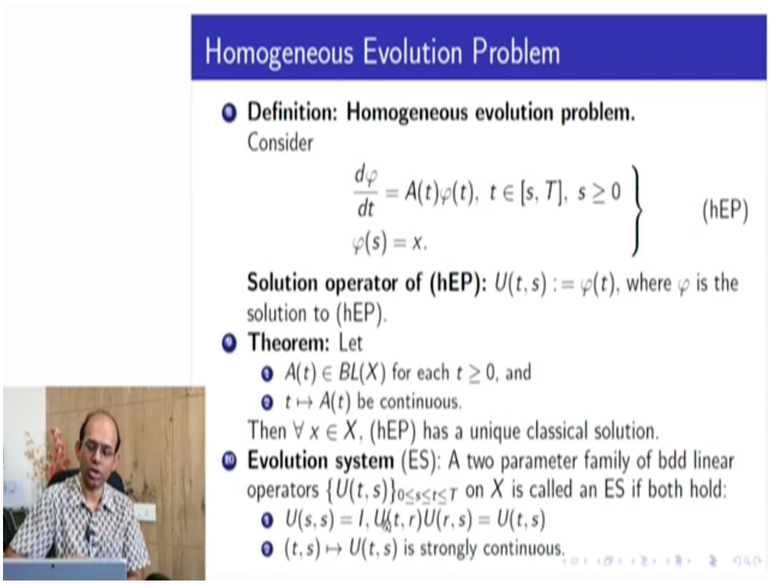

Next, this is the definition of homogeneous evolution problem. So, here consider d phi dt is equal to A of t phi t. So, here it is homogeneous, why because I do not have plus f, I do not have any data here. However, here the operator A is a function of time, time varying function, is a non autonomous system.

So, we call these problem as evolution problem, we do not call it anymore initial value problem. Although so something, but here see I mean, I there for every time these things are changing. So, s need not be always 0. So, small t belongs to s to capital T s greater or equals to 0 some s and phi of s is equal to x.

So, at time s, it is x and then we are starting the equation. So, we need here some more definition. So, instead of semigroup we are going to introduce this and we are going to use this two parameter family of operators, the semigroup was a single parameter family of bounded near operators on the Banach space.

Here instead we are going to take two parameter family, one is the starting point and another is the ending point. So, U t s, U t s that is defined as phi of t where phi is the solution of hEP, I call this hEP because h for homogeneous EP for evolution problem. Ut s is equal to phi of t. Now this theorem we quote here.

So, let A t is a bound linear operator for each and every t. So, this is a very, very special case, very restrictive case, which we actually have visited earlier for the initial value problem, when A is bound linear operator then it generates a uniformly continuous semigroup and that is can also be written as e to the power of t times A and then the solution of these thing is also very clear, it was just the semigroup itself.

So, but here for this case, if A is a time varying operator, A of t is BL X for each t and then t to At is also assumed to be continuous, both of these conditions are assumed, then for each and every x, small x and capital X as before, because before also we could add assure existence of classical solution, even if small x is not in the domain of A. But here actually for matrices etc, the domain which are bounded linear operator. So, their domain is in the full thing.

So, then for all x in capital X, the homogeneous evolution problem has a unique classical solution. Now, we introduce this evolution system from U, a two parameter family of bounded linear operators. So, here it is just a particular cooking up, but we are going to find out, not a particular example, but we need to discuss that, what when should we call a two parameter family of operators as evolution system.

So, these are the properties, a two parameter family of bounded linear operators U t s where s and t is between 0 to capital T on X is called an evolution system or ES, if following both hold. What first is that U s s is equal to I and U t r composition U r s is equal to U t s. So, let us take some particular subclass where we can actually justify these relations.

So, imagine that if a At was a very special, it was independent of time t. So, then this would give me that capital T of t minus s, capital T of t minus s and then if you have that, U t s is equal to capital T of t minus s, then U s s is capital T of s minus s, s minus is 0. So, capital T of 0 is equal to identity.

So, U s s is its identity. So, in the case where U t s is equal to capital T of t minus s where T is a C0 semigroup, then this can also be justified here U of t comma r is capital T of t minus r and then U of r comma s is capital T of r minus s. So, here we have capital T of t minus r, here we have capital T of r minus s, then using the semigroup property, we are going to get capital T of t minus r plus r minus s that is capital T of t minus s. But capital T of t minus s is U of t , comma s.

We can justify this also, for this very special case where U t s is capital T of t minus s for some semigroup capital T. But that is a very special case, all the evolution system need not have such kind of representation. Second thing is that as a function of t and s, t s to U t s is strongly continuous.

What is the meaning of strongly continuous? That means that if I have U t s x where x is coming from the Banach space. So, then this function, this map, here left hand side I have the product space timing in s, t s and right hand side have Banach space U t s x and there we have the Banach space norm. So, if that map is continuous, then we call this map is strongly continuous. So, under these two conditions, if U is satisfies these two conditions, then you call U as a evolution system.

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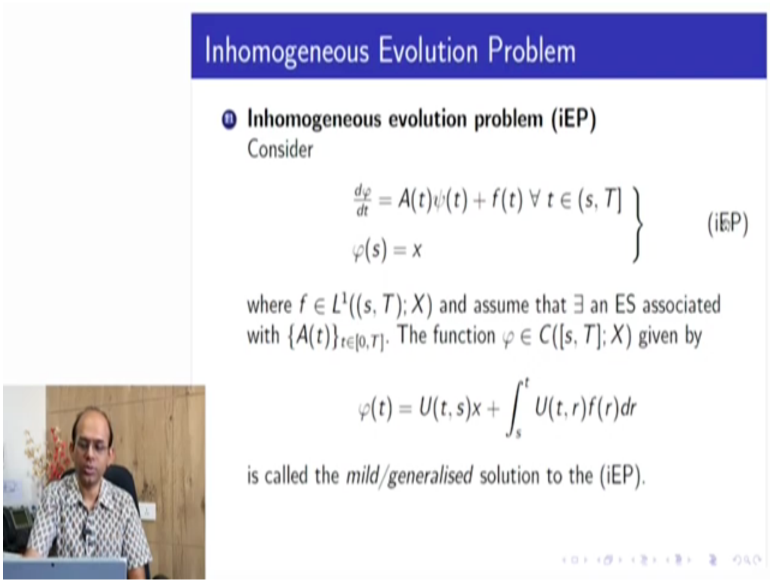

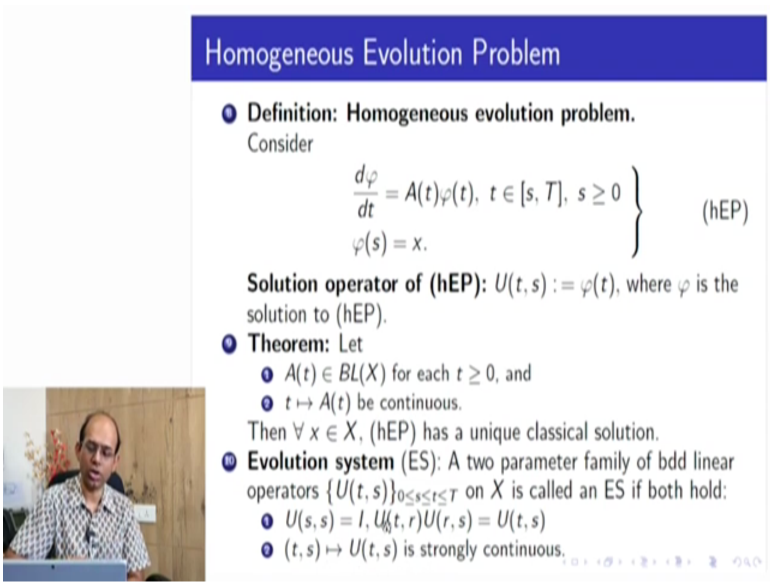

Now, we state this equation, so d phi dt is equal to A t phi t this thing. So, here it is inhomogeneous, so remember here we do not have f here. So therefore, it is homogeneous evolution problem. Now, we state inhomogeneous version of that, inhomogeneous evolution problem. So, you consider d phi dt is equal to A t psi t, here it should be phi.

So, same phi here both the sides, d phi dt is equal to A t phi t plus f of t for all t in s to capital T closed and phi of s is equal to x. So, here I have inhomogeneity. So, where f as before is an integrable function, f is in L1. So, in this time interval s because s is the starting point here, s to capital T, so s to capital T to X, it is in L1 and assume that, there exists an evolution system associated with the family of operators A t.

Earlier for initial value problem, we had only one single operator A. But now we have a family of operators A, because A is itself a function of t and now for this family of operators if we assume there exists an evolution system. So, what is the meaning of associated with A t that thing, that means that I mean there exists one evolution system U such that, for which when we take the derivative, we are going to get A times U, we are going to show in the next lecture the detailed definition of these that when we should call a evolution system is associated with A t.

The function phi which is continuous functions C s T into X given by phi t is equal to U t s x plus integration s to t U t r f r dr. So, this is called the mild or the generalized solution to the homogeneous evolution problem. So, earlier case the U t s, U t s x. So, here the U t x appears. So U t s x is the, so x is missing here.

So, U t s x is the mild solution of this equation. So, next we would study when can we associate an evolution system with a given family of operators A t. So as earlier it, we had A in the homogeneous or inhomogeneous initial value problem and then Hille-Yosida theorem could assure for certain cases that we can actually associate a C0 semigroup there and when we can associate that C0 semigroup capital T, was very useful to write down the solution of those Cauchy problems.

Here also I have, instead of a single operator A, I have a time dependent family of A and for that, we would also have a proper notion of evolution system and then only thing is that when, under what sufficient condition on A we can assure is that there, we can associate one evolution system.

Why is that question interesting because if we can do that, then using that evolution system we can write down this extension of formula of variations of constants to write down a mild solution of the equation and under some strong conditions on x and f etc, one can assure existence of classical solution also. So, let me stop here now. Thank you very much.