**Introduction to Probabilistic Methods in PDE**
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**Lecture-60**
**Feynman-Kac formula and the formula of variations of constants**

Welcome, we will see the relation between Feynman-Kac formula and the formula for variations of constants. So, these 2 things we are going to see and we are going to build their relation using some new definitions.

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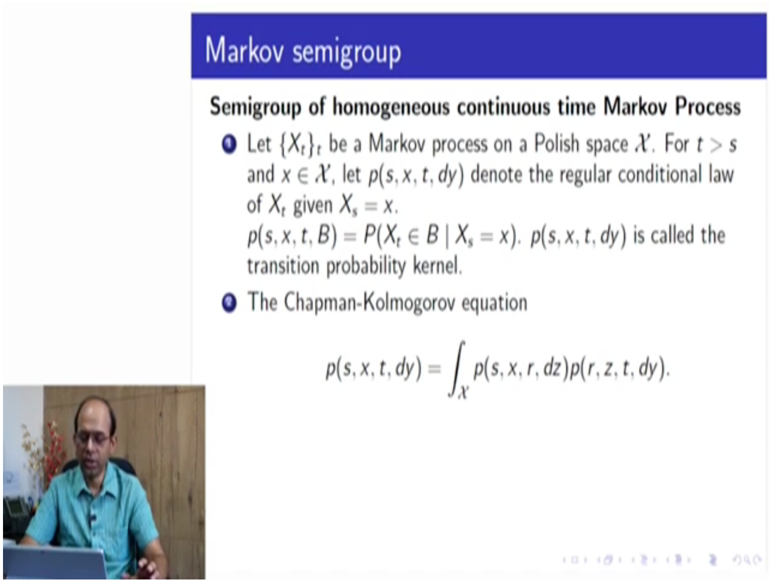


So, here we are going to talk about Markov semigroup, it is not that we have never discussed it, but we have discussed it always in disguise and we have discussed in very briefly for some special cases. So, I will try here to give broader settings for that. So, I mean the background, let me tell that when we have introduced semigroup of operators that time we also have given some examples, the example from transition semigroup, examples from finite state, continuous time, Markov chain etc.

So, there we have already you know shown what is the semigroup for corresponding dynamics and then also we have obtained the generator there. So, but here we are treating for a general broader setting. So, X t t be a Markov process on a Polish space script X. What is this a Polish space, Polish space is a complete separable metrics space. So here we assume that t is Less than s. So, for t more than s and small x in script X, we introduce this notation, small p, so p of s x t dy, p s x t dy.

So s and t is like time like parameter, and x and y is the space like values. So this denotes the regular conditional law of Xt given Xs is equal to small x. So dy is like the measure, the measure it cost on the range set of the random variable, which random variable we are talking about Xt. X is the Markov chain and for that Markov chain at time t, it is a random variable. And that random variable gives the law or the probability distribution of the random variable. And that distribution is this, when given that at time s which is earlier than t the Markov chain is at position small x.

So, this is the kernel. So, let me clarify again. So, p s x t B where B is any Borel subset of the Polish space script X, then p s x t B denotes the probability that X t belongs to B given Xs is equal to small x. So, this p sxt dy is called a transition probability kernel. Now, Chapman Kolmogorov equation is the equation which we obtain for Markov chains transition kernels So, that equation is stated below that p s x t d y, that means the law at time t given that at time s it is at x is equal to integration p s x r dz, so r is the timeline in between s and t, p r zt dy.

So this is the probability distribution of the Markov chain at time t given its location at r, rth time on z. So its location at time r is z given that what is the distribution and this is saying that given the Markov chain is at x at time s what is the distribution of this at time r. So, you are integrating this with respect to z. So, I mean this is little vague way of writing but I should have written small z belongs to these to denote that I am integrating it to z.

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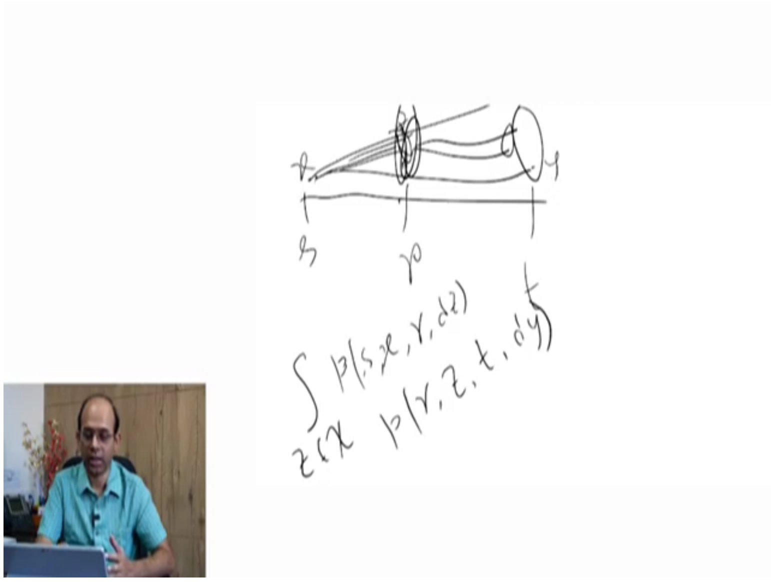

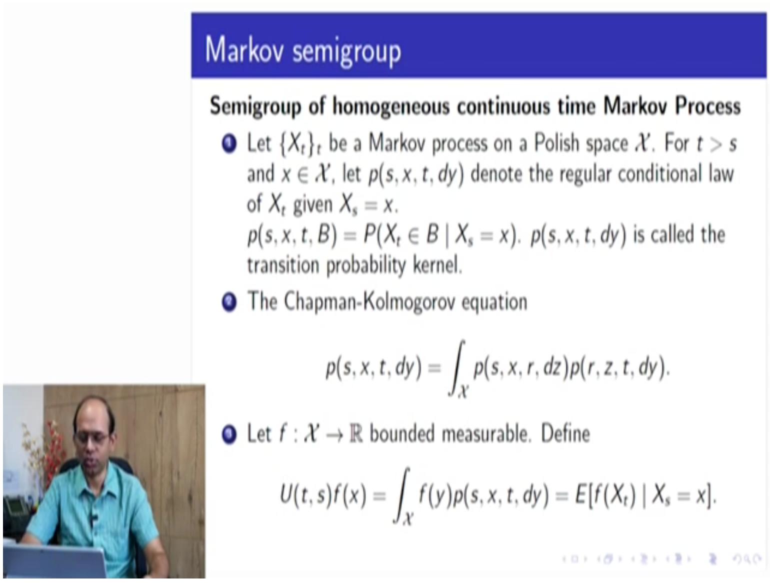


So, that means that this s, this is time r, this is time t. So, here the Markov chain is at x, here the Markov chain is at some place z and here it is at some place y. So, we would like to understand that the distribution of this given at s it is x. So, in between we come here that we find out from here to here. That is p s x r dz gives me the probability coming from here and from here to here that is p r z t dy and then we integrate with all possible z so z belongs to this, with all possible z. Then that would give me you know the, the in between from s to t time.

So, that is the idea. I mean this is possibly something which you have all seen in some course in probability theory stochastic processes before taking this course. So, this is Chapman Kolmogorov equation p s x t dy is equal to p s x r dz p r z t dy where integration is with respect to z variable. Now let f is a bounded measurable map, real valued bounded measurable on the state space of the Markov chain. Now why did it take bounded measurable I must take because I am going to integration, but why I take bounded because to ensure that integration exists is finite.

So, Ut s fx, we define U, a new operator to go with the family of operators U, U s fx is defined as f of y p s x t dy. So, it is just integration with respect to the probability measure of f. So, this is nothing but expectation correct because expectation, because this is like a density, this if you take dy, dy here divided by dy and dy, that will give you the density if that exists. So, this is like a density here. And so, this integration is just the expectation. So, this is actually expectation of f, what is y, y is the location of the process at time t. So, this is f of X capital X subscript t at time t given that at time s at small x, Xs is equal to small x.

So, this is the interpretation in terms of expectation. So, this is an operator surely, why, because given a function f, which is bounded measurable, I get another function of x because this whole thing is also a function of x. And that is also a bounded measurable function, because f is bounded by capital M, the whole thing would also be bounded by capital M. So that also says that U is actually a contraction, is a contractive map. So, the operator norm of U is less than or equals to 1.

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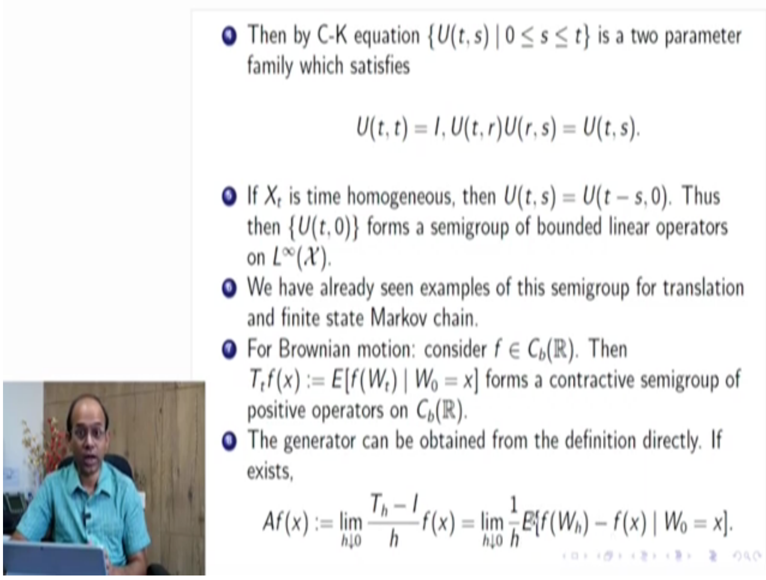

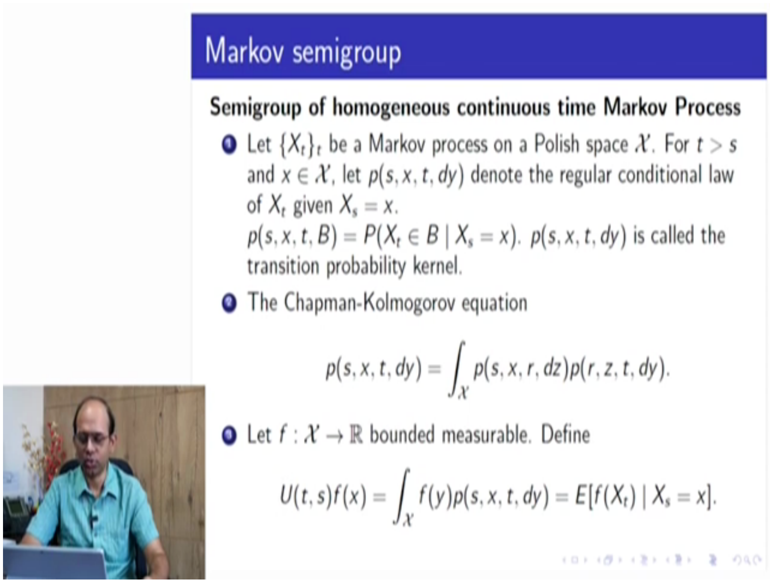


So, we are now going to see more properties of these 2 parameter family U. So, U t is, where s is less than or equal to small t, so, this is a 2 parameter family. So, then we use the Chapman Kolmogorov equation, see I mean Chapman Kolmogorov we can easily do because you know that I mean in between t and s if you put r there and then we take a conditional expectation here, so we can use the tower property of expectation. And from that whatever we are going to obtain, if you want to write down in terms of integration, we would come up with this type of you know integrations.

And then using Chapman Kolmogorov equation, you can do that. So, we get Ut t is equal to identity, why because at time t if the information is given, so, if you take X t is equals to small x, then expectation of f of X t is f of x, because X t is given, there is no randomness here. So, f of X t given X t is equal to x is f of x. So, then this whole thing is the same, that means this operator is just the identity operator. So, Ut t f is f itself. And then we get further that Ut r U r s is equal to Ut s. So here that means that given s time the r is the future and the given rt is the further future.

So that thing is Ut s so you would get all these things. Now if Xt is in addition to the above, is time homogeneous then we do not need to bother much about the 2 parameter family, we can reduce to one parameter family. Why is it so, because here expectation f of X t given Xs equal to small x is the same as expectation f of X t minus s given X0 is equal to small x. So, does not matter that where it starts from, the location only matters, because it is time homogeneous.

So the time homogeneous case is, I mean much simpler and for this simple case, we can write down U of t, s as U of t minus s, 0, as I have discussed in the earlier slide. So, thus, this Ut 0, so 0 is fixed and then we vary t, forms a semigroup of bounded linear operators. See, I mean here, so U,t,r would be U t minus r s, 0 and this would be U r minus s, 0 and this will be U, t minus s, 0. So r minus s and t minus r if you add with t minus r plus r minus s, that is t minus s you get t minus s. So it shows the semigroup property.

So, however this Ut minus s, Ut 0 forms a semigroup of bounded linear operators on these bounded measurable function on the state Space X on L infinity X. So, this is the banach space we are considering for this setting. Now, we have already seen examples of this semigroup for translation and a finite Markov chain. So, we are going to see now some more example for Brownian motion. For Brownian motion, what should be the semigroup we and what should be the generator. We have some you know anticipation that in Brownian motion the half times Laplacian was coming again and again very crucially. So, that was also an operator and that also appears in the martingale problem.

So, we anticipate that okay possibly that should be the generator, but we have never discussed it. So, let us do it now more clearly. So, for Brownian motion consider f is in, instead of the bounded measurable, so here I am taking continuous bounded continuous function on R. So, consider f is in, in the bounded continuous function on R then Tt f x. So, as our earlier slide earlier page, we write down the expectation f of W t given W0 is equal to x. So, this is the operator Tt. So since I know the Brownian motion is actually time homogeneous Markov chain, so I am not going to the 2 parameter family directly I am starting with the one parameter family. So, this is the one parameter family I am just starting with 0 here and Ttx.

So this forms a contractive semigroup I mean, because it does not need a separate proof, from the earlier steps, we know that this forms a semigroup. And not only that, the norm is less than or equal to 1 this semigroups, so it is a contractive semigroup and also it is a positive operator. Why is it a positive operator, because if f is non negative, the expectation is also non negative. So this operator takes a positive function to positive function. So it is a positive operator on the CbR.

So the next question is that what should we its generator. So the generator can be obtained from the very definition. If exists for some x we write down A f x, if A is the generator, A f x is defined as limit h tends to 0 T h minus identity divided by h f of x. So, here we know that what is Th is the semigroup, the semigroup is the expectation of this form. So, we write down that so Th fx is expectation of f of W h given W 0 equals to x. So, write down that expectation f of W h given W0 is equal to x and here minus identity, so that gives me only f x.

So, we have these difference. Now, we have to compute this difference. How do you compute it? We have seen in the stochastic calculus part, that we can, so when f is smooth enough, we can apply Ito's formula, Ito's formula is a perfect thing to find out the increment of function of the stochastic process. So, here f is of course, in general need not have that sufficient smoothness because f is only continuous bounded continuous function. Nevertheless, we, here for finding out this value, we take a much smaller subspace of CbR, much smaller subspace.

So, where we can actually compute it, when we can compute it, we can get a form of capital A, which is valid for at least some smaller class. And then for other cases, we do not care much about that where the limit would exist, what would be the thing. Because, I mean, when f is like Cc infinity functions continuously I mean, infinitely many differentiable on compact support or even we just take C2b or say C2c, I mean so it is a twice differentiable continuously with compact support. And then also we can actually apply Ito's formula and then we can find out this expectation that becomes finite etcetera. So you can do that and we get some form and then you say that okay the generator restricted on that class how good functions has this form. So and then we call that infinitesimal generator of the process W.

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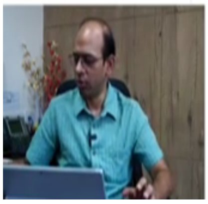
- If $f \in C_c^2$ then as an application of Ito's formula $Af(x) = \frac{1}{2}f''(x)$, i.e., $A = \frac{1}{2}\frac{d^2}{dx^2}$.
- For a similar reason, $\frac{1}{2}\Delta$ is the infinitesimal generator of the multidimensional Brownian motion.
- Let $\mathcal{A}$ be the operator associated with a time homogeneous (SDE), whose weak solution is denoted by $\{X_t\}$. Then we know that $M^f$ given by $M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A}_s f(X_s)ds$ is in $\mathcal{M}_2^c$. Therefore,

$$\begin{aligned} Af(x) &= \lim_{h\downarrow 0}\frac{1}{h}E[f(X_h) - f(x) \mid X_0 = x] \\ &= \lim_{h\downarrow 0}\frac{1}{h}E\left[\int_0^h \mathcal{A}f(X_s)ds \mid X_0 = x\right] \\ &= \mathcal{A}f(x). \end{aligned}$$

Thus $A = \mathcal{A}$, infinitesimal generator of strong Markov $\{X_t\}$.



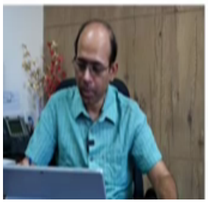
$$f(W_h) = f(W_0) + \int_0^h f'(W_s)dW_s + \frac{1}{2}\int_0^h f''(W_s)ds$$

as $x = W_0$

$$\frac{1}{h}E(f(W_h) - f(x)) = \frac{1}{h}E\left[\int_0^h f'(W_s)dW_s\right] \quad f \in C_c^2$$

$$+ \frac{1}{h}E\left[\frac{1}{2}\int_0^h f''(W_s)ds\right]$$

$$\to E\,\frac{1}{2}f''(W_0) = \frac{1}{2}f''(x).$$



- Then by C-K equation $\{U(t,s) \mid 0 \le s \le t\}$ is a two parameter family which satisfies

$$U(t,t) = I, U(t,r)U(r,s) = U(t,s).$$

- If $X_t$ is time homogeneous, then $U(t,s) = U(t-s,0)$. Thus then $\{U(t,0)\}$ forms a semigroup of bounded linear operators on $L^\infty(\mathcal{X})$.
- We have already seen examples of this semigroup for translation and finite state Markov chain.
- For Brownian motion: consider $f \in C_b(\mathbb{R})$. Then $T_t f(x) := E[f(W_t) \mid W_0 = x]$ forms a contractive semigroup of positive operators on $C_b(\mathbb{R})$.
- The generator can be obtained from the definition directly. If exists,

$$Af(x) := \lim_{h\downarrow 0}\frac{T_h - I}{h}f(x) = \lim_{h\downarrow 0}\frac{1}{h}E[f(W_h) - f(x) \mid W_0 = x].$$

So we do that here, if f is C2c, twice contrast differentiable with compact support. Then as an application of Ito's formula, we can find out A f x is equal to half times f double prime x. Do you remember how do we do that. So let us recall very quickly, f of W h is equal to f of W 0 plus integration 0 to h f prime W s d W s plus half times 0 to h f double prime W s and then quadratic variation of W d of that, that is s itself d s. So, now if I want to find out expectation of f of W h minus f of W 0, here given W 0 is equal to x. So, f of x here, so as x is equal to W 0, so this is equal to expectation so I close the expectation here.

Expectation integration 0 to h, f prime W s dW s. So this is one expectation plus the remaining part, that is also another expectation I break in 2 parts for the ease of this. Half is there, 0 to h, f double prime of W s ds. Now if I have 1 over h both sides so I write down here 1 over h here. Now, here in this integration is 0, this expectation is 0, why, because this integration here f prime is bounded, because f is C2b correct. f belongs to C2c that means twice continuous differentiable with compact support, so all the derivatives are bounded.

And then is bounded and then the bound function is integrating with respect to a martingale you get a martingales expectation with mean 0. So, this expectation is just 0. So, you are left with only this. So this is the part which appears in a slide and then this part should remain and then as h tends to 0, so this thing would be, so here 0 to h f double prime Ws d s and as h tends to 0, this whole integration would be (expectation), this is converse to expectation of half times f double prime of W 0. So that is, this is not random, this is deterministic, W0 is x. So this is half times f double prime x.

So, we go back to our discussion. So A f, A f which is a f of Wh minus Wx, 1 over h and expectation. So, that becomes half times f double prime x. Or in other words, we say that the operator capital A is half times d 2 dx 2. So, now for a similar reason half times Laplacian is the infinitesimal generator of the multi-dimensional Brownian motion. So, here I have just given one-dimensional case, for multi-dimensional also same reason applies. So, we get half times Laplacian. Laplacian is the I mean del 2 del xi 2 where the sum over i, i is equal to 1 to d, where d is the number of the dimension.

Next, we go further the general case. So, let script A be the operator associated with a time homogeneous SDE. So, which we have already seen earlier when we have discussed strong and weak solutions of SDE we have discussed the operator associated with the SDE. So, if script A is the operator associated with a time homogeneous SDE whose weak solution we denote by Xt. So weak solution is actually the triplet is Xt W t and probability space and filtration, but here we are suppressing those things.

So just writing down the X t. So if that is the case, then we have seen a theorem, we are just recollecting that M f, there is a process given by Mf t at time t is defined as f of X t minus f of X0 minus integration 0 to t As. So here A is independent of time so I do not should not write down, you know s dependent because we are talking about time homogeneous case. So this s is redundant. So integration 0 to t A fX s d s. So, this whole thing as a function of t, this forms a process and then stochastic process we denote by Mft and then this stochastic process is a continuous square integrable martingale.

So, this we have seen earlier. Now in the expression of the generator of the semigroup also we have this difference. So, f X t minus f of X0 can be written as this Mf t plus this part goes there. So, therefore, A fx, where A is the generator of the, so this X is actually forms a strong Markov process. Because when I have the time homogeneous SDE, the solution gives me a becomes a stronger process with respect to the filtration what is there.

So for that we can also discuss what is the Markov semigroup and what is the infinitesimal generator of that Markov semigroup. So that we denote by capital A. So, see this A is just the capital A the upper case. But this is a math Cali, the script A. So, these 2 are different at present, differently this A is associated with the SDE. So, that is actually coming from SDE coefficients. If coefficients involve b and sigma, so this is A also involves those b, sigma those things.

So, I am not recollecting the whole definition because that was done earlier, you can surely go back to the earlier video lectures and slides to see the definition of script A. So now this A fx, so far we are discussing now here like these generators, these are just 1 over h expectation of f of X h minus f of x given X0 is equal to x, to small x. Exactly the same manner as we have obtained the infinitesimal small generator of Brownian motion. So now instead of Brownian motion we are talking about some general stochastic process which is weak solution of some stochastic differential equation.

So, now this f of X h minus f of x, from here we know that okay that can be written as in terms of M and then this part, M plus this part. But expectation of M would be 0 correct because M is just a 0 mean square integrable continuous martingale. So, so that part would not appear here So, this difference expectation would be just expectation of this integration. So, 0 to h, because h is there, t and t is there, now here h is there, so integration 0 to h A f X s ds, that should be this, given X0 is equal to small x.

Now here we have 1 over h and we want to take a limit h tends to 0, downward to 0. So here same rule applies that since this is a, I mean here we assume f to be function which is in the domain of A and not only that, A f you know it is a nice thing I mean it is giving a continuous function. So, for that if it is giving a continuous function, and then this thing is also bounded, so we have all these things. So for that we get that this limit would be that 0 to h and 1 over h as h tends to 0 would be just this integrand and where s is evaluated at 0.

So A f X 0, so X 0 is small x, so the answer would be A f x. So, capital A f x is equal to script A f is x. So, capital A is equal to script A as long as your small f is a good function. So, here it is sufficient to consider C2c. So, here I mean throughout this slide I assume that f is twice continuous differentiable and having compact support. So A is script A, so we call, therefore this is infinitesimal generator of the strong Markov process Xt. So, here by this slide we have connected 2 different notions.

Infinitesimal generators of semigroup what we were introducing in a deterministic setting from the book of A Pazzi. And another thing that when we are studying stochastic calculus, stochastic differential equations and then we have came up with 1 operator which is called, this operators associated with that SDE. And now what we see that from that solution of the stochastic differential equation, if we find out the semigroup that is nothing but, that is very easy, that is written in terms of conditional expectation and from that semigroup if we find out the infinitesimal generator then that infinitesimal generator, that operator is same as the operator associated with the SDE. So, this gives the connection. So, since this is connecting these 2 different operators, perhaps that Feynman-Kac formula what we have obtained there, is also connected with the mild solutions what we are writing down here. So, this is the thing which I am now going to discuss next.

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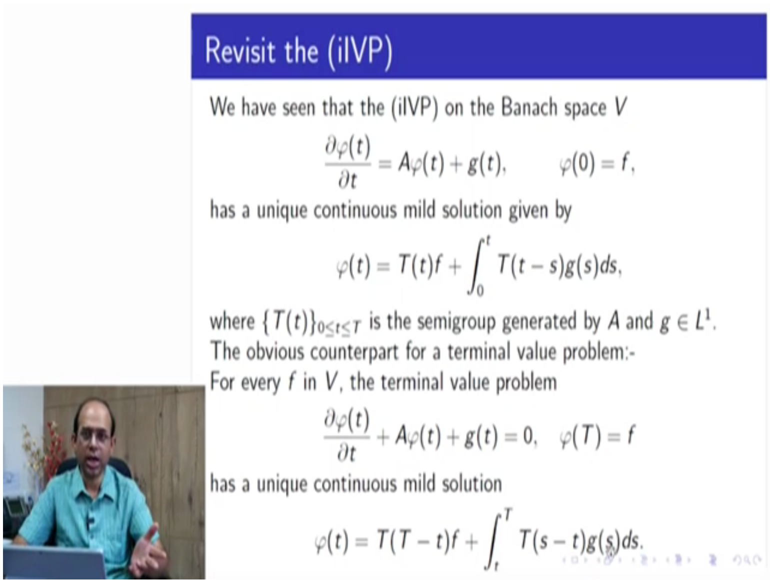


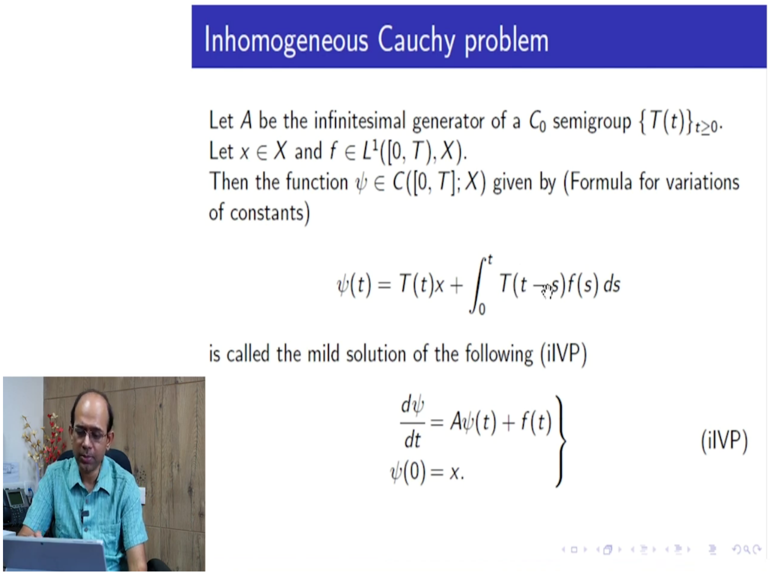


So, continue revisit, so we are revisiting that inhomogeneous initial value problem. We have seen that the inhomogeneous initial value problem in the deterministic setting, on the banach space V. So here I am now changing notations here a little bit so just to make both compatible and to compare these 2 things. So, this ddt phi t is equal to A phi t plus gt. So, this is the inhomogeneous initial value problem. You remember there we have used f function, now we are using g function.

And initial condition of phi of 0, we have used small x, but I am writing small f. Why, because these members of the Banach space I would like to write down as a function, as a function because this is basically Banach space which we are considering, these are like functions on the state space of the Markov process. So has a unique continuous mild solution and that is given by the formula variation of constants, that is phi of t is equal to Ttf plus integration 0 to t capital T of t minus s g s ds. So, this is the slide earlier we had this IIVP d psi dt is equal to A psi plus f,  psi 0 is equal to small x.

Its solution, mild solution was written as psi, this is psi, this solution mild solution Ttx the initial thing and integration 0 to t and then the semigroup t minus s fs ds, f appears over here. So, we are doing that thing here, so Tt minus s gs, g is here and initially 0. So, this is borrowed from the earlier lecture just last lecture. So a Tt is the semigroup generated by A and this g should be in L1, otherwise this integration may not make sense, so L1. So, we have spent reasonable time and this discussion on these things.

So, the obvious counterpart for a terminal value problem, why should we talk about terminal value problem, because Feynman-Kac formula what we have quoted is for terminal value problems. So, here if we just change the time direction, then what should we get? I mean that is very elementary way so we can do that. For every f in V the terminal value problem, so you write down the values for terminal value problem, d phi d t plus A phi t plus gt is equal to 0 phi capital T is equals to small f.

So if we change the time direction, then this comes here correct. So, the negative sign comes here. So d phi dt plus A phi plus gt, and then phi of capital T is small f, instead of 0 it is now terminal time capital T. I hope this notation would not be confusing, here I am writing terminal time as capital T and I am also writing the semigroup capital T, because semigroup always comes with the parenthesis starting where for these final time there is no parenthesis which starts here.

So, has a unique continuous mild solution, phi of t is equal to capital T, this is the semigroup capital T of capital T minus t, this is the final time so instead of small t I should write capital T minus small t because I changed the time direction. Instead of 0 to t we have small t to capital T here and instead of capital T of t minus s, we have capital T of s minus t gs ds. So, this is the time changed version. So, now this equation can be compared with the equation what we have seen for Feynman-Kac formula.

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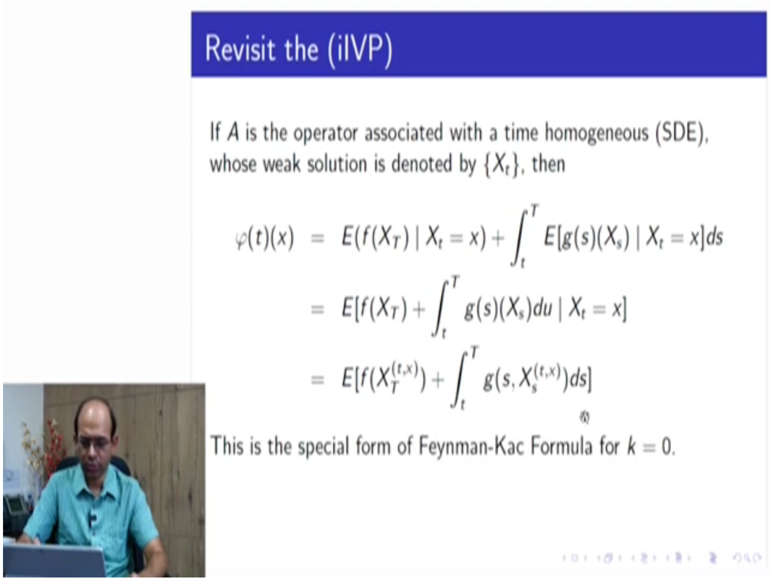


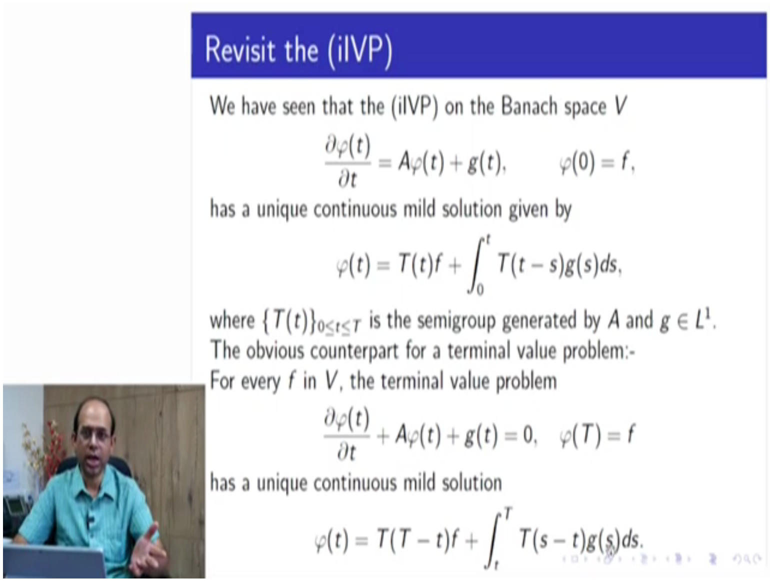


So, we do that, if A is the operator, so if A is the operator associated with a time homogeneous SDE whose weak solution is denoted by Xt, then phi t, why I am writing small x, small x is just a point. So, I mean point in the state space of the process Xt. So, you remember that here capital V is actually a function on the state space, set of all functions on a state space. So those functions that you can take bounded measurable or bounded continuous functions with sup norms. So the bounded continuous function with sup norms gives me a Banach space capital V.

So, so here every you know f gt for each t, phi t are member here. So since this is a member here, so this can be evaluated at some points small x, which is a point in the state space of the process. So we write down phi t of small x is expectation. So here, whatever is written here, you know, whatever is written here I am writing down in the next page here. Here phi t of small x, now capital T should be replaced by the conditional expectation, the capital T of T minus t f. If I do that, I would get exactly this thing, expectation of f of X capital T given Xt is equal to small x, because this capital T minus t times difference.

So, this is the expectation f of X capital T given Xt and small x is the evaluation, phi t of x is equal to expectation of f of capital XT given Xt is equal to small x plus. Then integration small t to capital T, capital T of s minus t gs. So, this we can write down as expectation of g of s,x s given Xt is equal to small x. So, so that we do here expectation of g s of Xs given X t is equal to small x.

So that semigroup of expression I am writing this way, and then integration with respect to s small t to capital T. So now we now simplify a little bit here. So there are 2 expectations are there, this integration is there, so, we write down expectation outside. So, as you mean all these things expectations exist, then we can do this. So expectation and then this is like you know interchangeable, so, you can do that. Expectation of f of X capital T plus integration small t to capital T g s Xs t sorry, this should not be u, this should be s here, same s given Xt is equal to x.

So, this continuous expectation is outside of these 2 expressions. So, that addition of 2 terms f of Xt and then this. So this is equal to expectation of f. So now given Xt is equal to small x, so that I write down as, I mean, the notation for SDE, because the SDE which starts at time t, at small x that we we write down f of X superscript t x given X subscript capital T. And here it would be also X superscript t, x because it was the starting point is x at time t, and s. And instead of g of s Xs, I am writing g of s, Xs because this is a function of 2 variables.

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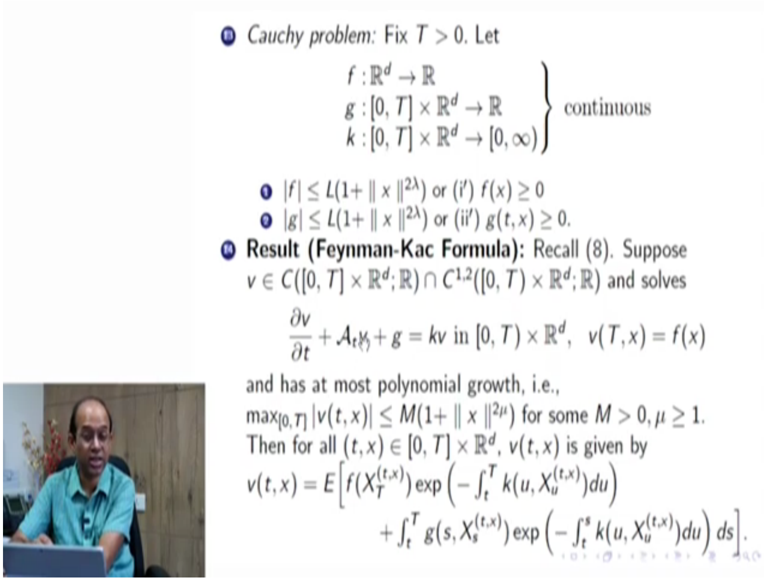


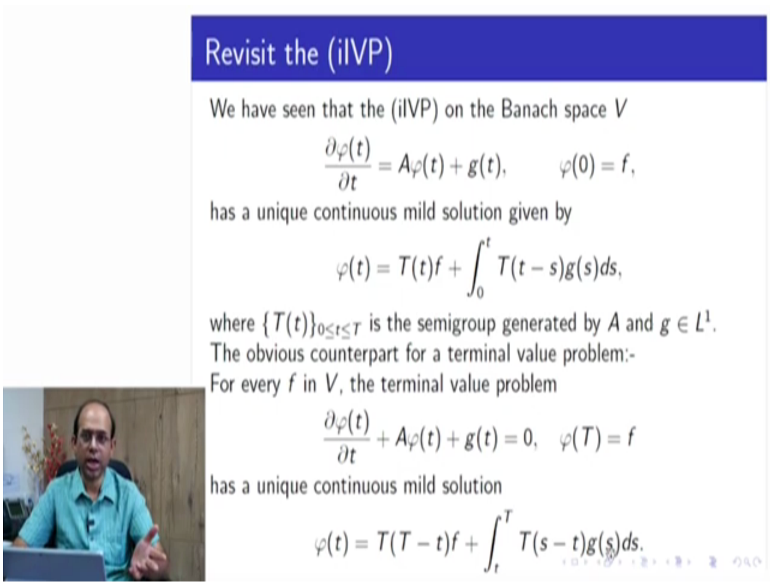


So, this formula, let us compare this formula with this one. Here we have written del v del t plus A tv plus g is equal to kv. So, imagine for our case k is 0, k is not here. So, dv dt plus A v and A is time homogeneous, so, A subscript t does not appear. So, here this is our equation d phi dt plus A phi plus g is equal to 0 and this is d phi dv dt plus A v plus g is equal to 0, k is equal to 0 case. So, there the solution is written as v is equal to expectation of f of X capital T, e to the power of this, but k is 0, so e to the power 0 is 1, so this does not appear. f of X T appears plus integration small t to T g of s Xs and e to the power of 0 is 1 so only these 2 terms. So, we also have obtained only these 2 terms.

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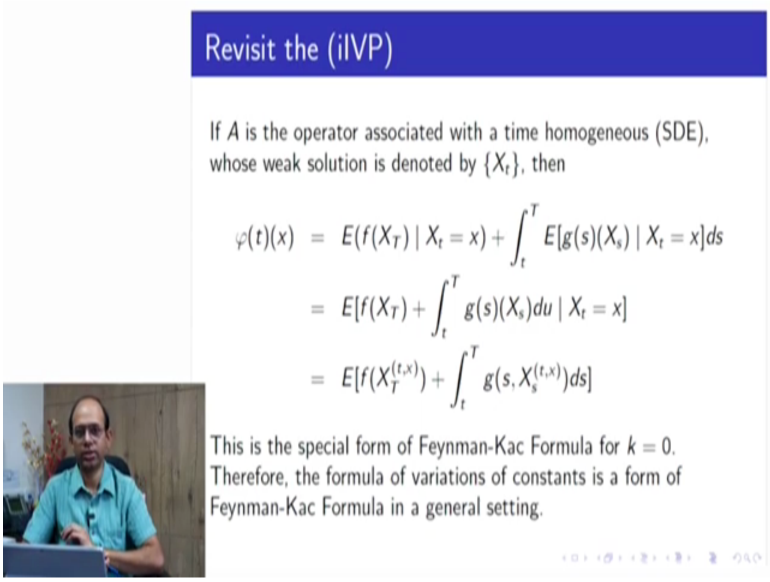


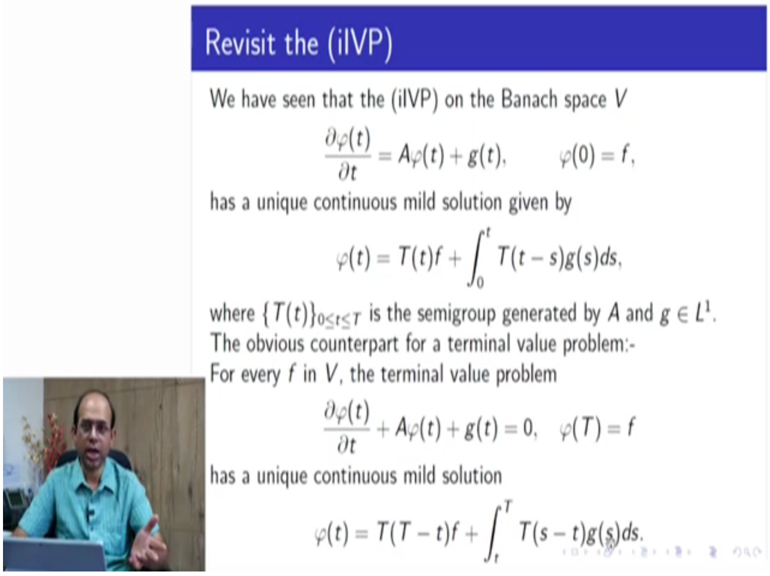


So, expectation and of f of X T plus small t to capital T g of this ds. This we have obtained from the formula of variation of constants, because this we have started from here, this is the formula version constants for the initial value, we have just time changed to the terminal value. And then we have just further assumed that the operator is associated with a time homogeneous SDE. And for that we get a Markov solution Markov chain also. And from that we just write down and what we obtain is also a particular special case of Feynman-Kac formula. So, this is the special form of Feynman-Kac formula for k is equal to 0. So, therefore, the formula of variations of constants is a form of Feynman-Kac formula in a general setting. So by this I end to this lecture. Thank you very much.