Introduction to Probabilistic Methods in PDE
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Lecture-59
Tutorial on Resolvent operator

Welcome, today we are going to discuss a few things which is of tutorial type not much new concepts would be discussed today. However, some results which are nice and useful would be shown today about the resolvent operator.

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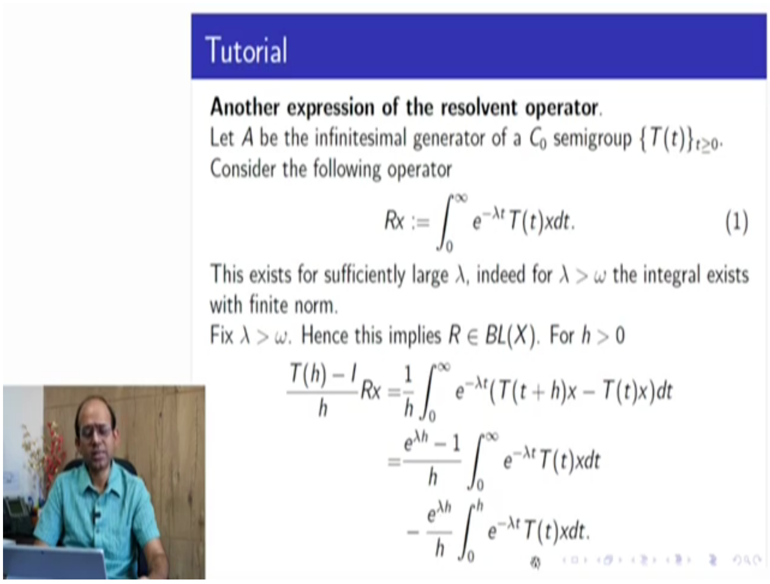

So we start from here. So here I am first showing you another expression of the resultant operator. So let A be the infinitesimal generator of C0 semi group Tt, consider the following operator R, where R is an operator on that Banach space, on which the Tt is the bound linear operator is the semigroup of bounded linear operator. This R is an operator which is defined in this way R applied on x small x is a member in Banach space capital X. So, R applied on X is defined as integration from 0 to infinity e to the power minus lambda t capital T of t x dt.

So and this integration t running from 0 to infinity. So here which lambda is it. So, lambda is some positive real number. So, this is something very familiar expression, we have seen this such an expression in some other courses or in some bachelor degree courses. So, this integration is called Laplace transformation. So, this is Laplace transformation of this function. So, now, if you consider this is a banach space valued function of time t and then this R is a Laplace transformation of that, so, Tt x takes value in the banach space capital X, for every x small x you get this integration.

However, the question arises why should this integration exist. So, this exists for sufficiently large lambda. Why is it so, because we have seen when we consider a C0 semigroup capital T when you have studied the growth property of capital T. So, there is one lecture dedicated to that. So, in that what you have seen that norm of Tt, that can be upper bounded by one exponential function. So, we have written in the following manner that capital M times e to the power of omega t.

So, so now, so far norm is concerned here. So normalize, this can be upper bounded by m times e to the power of omega t and if lambda is larger than omega then integrand after taking the norm. So, this is norm of this integrand will be upper bounded by one constant. Not only constant that will decay if lambda is more than 1, more than omega that will decay to 0. So, exponential decay that should have.

And hence this integration would be finite. So, this, that means this so for sufficiently large lambdas, that lambda more than omega. So then this integration is finite. So this is well-defined in the strong sense, in the absolute sense. So, indeed for lambda more than omega the integral exists with finite norm, norm would be finite. So, what does it imply, that implies that R would have finite norm. So, R would be and then we can ask that whether the operator norm of R is finite or not.

So, of course, you can take x on the unit sphere of banach space and take supremum of that, but whatever norm X you take, norm is equal to 1 take you know that that is upper bounded by capital N times e to the power of something which is goes to 0. So, that will be finite. So this immediately implies that R has finite operator norm. So from here we can conclude that this operator R defined in this way, the Laplace transformation of dt is a bounded linear operator. Why linear, because linear is very clear because it is acting linearly from here.

So, now onward we fix the lambda which is more than omega, we fix lambda more than omega. So, hence this implies R is a bounded linear operator on X. Now, what we do is that we will try to find out whether this R is in the domain of definition of the generator of Tt. So, that we are going to check and then we are going to check whether this R can be identified as the resolvent operator. So we would come to that point later. So first we check this. So to check whether R is in the domain of the definition or Rx is in the domain of the definition of generator, what do we need to do, we need to take this fraction Th minus identity by h.

And let h tends to 0. And if that limit exists then we can say that the range of R is in the domain of operator A, where A is the generator of Tt. So we do this. So we take h positive and then we take the fraction Th minus identity divided by h times R, I mean operated on R x. Now, this thing we are writing down here, so 1 over h I write down here and then Th and then R, Th compose R, composition R, so that if I put Th here th R, Th here, so, I would get a capital T of h plus t here.

And then so e to the power minus lambda t capital T of t plus hx. So, that we are going to get and then identity map is there, so, minus R, so it is exactly the same. So, it is e to the minus lambda T t x, e to the minus lambda t Tt x, this is coming from this I identity operator. And 1 of our h is outside. So, this expression, left-hand side expression is written on the right hand side. Now, what we do is the following. What we do, we write down using some substitution of variable.

So, instead of lambda t if I write down lambda t plus h, then I have to subtract also h, t plus h minus h that way. So lambda t plus h if you do and then we replace t plus h as t. So then instead of it would start from 0, it would start from h to infinity. So, instead of starting h to infinity, we write down 0 to infinity. So then I have to add some 0 to h. So, that thing we do here. So, here we add here h and then minus h minus minus sign plus sign, so plus e to the lambda h we take here.

So e to the lambda h if we do then it is like e to the lambda t plus h and then t plus h here, but the h to infinity, but then again using the substitution variable. So we want to write on this as 0 to infinity and then some term would be required. So, 0 to h term would be required. So, we write down 0 to h, 0 to h e to the minus lambda t Tt x dt. So, for this term we do and then the minus 1 term, where is it coming from? It is coming from exactly this, e to the power minus lambda t Tt x dt, 0 to infinity. So, this is exactly e to the minus lambda t Tt x dt 0 to infinity. So, this minus 1 term is coming from here.

And then just to manage this term, I had to take something common here and then something, subtract something here. So this is a very convenient expression on the right hand side what we have obtained. Why is it so, because now I can take, see I mean, this integration is independent of h only this term and for this we can take limit h tends to 0, we know that limit exists. Why, because e to the power lambda h minus 1 divided by h, I can write down as lambda h and lambda here.

So, then it is like e to the power x minus 1 divided by x form. And as the limit x tends to 0, that goes to 1 we know that. So what would be the result of the limit of this, that would be just lambda. So, lambda would be the limit of this expression and then this would remain as it is. And for this, I have e to the power lambda h. So, this numerator, as h tends to 0 goes to 1 and then this 1 over h and then this integration as h tends to 0, it converges to the integrand where it is evaluated t is equal to 0. So, because this is near the neighbourhood of 0, which is like a derivative of the antiderivative, so it is the same thing. So, from here we are going to get e to the power of minus 0 that is 1, T 0, this is identity x. So this would give me just minus x actually.

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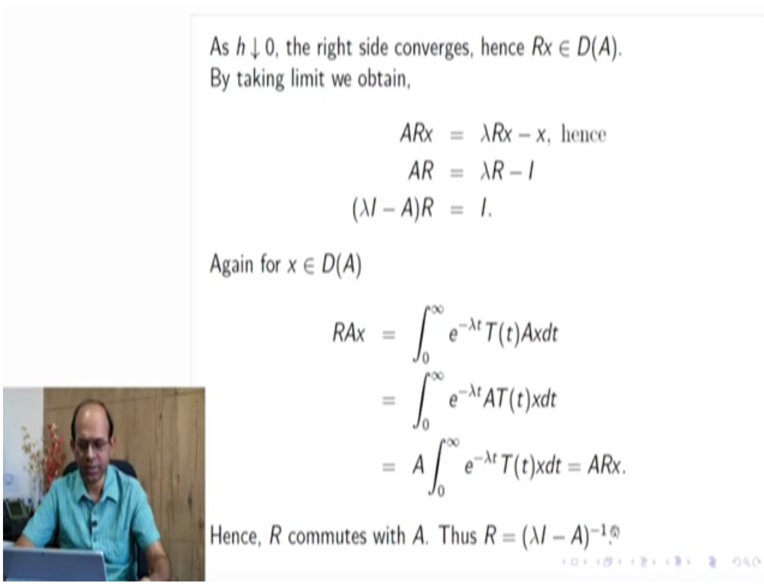

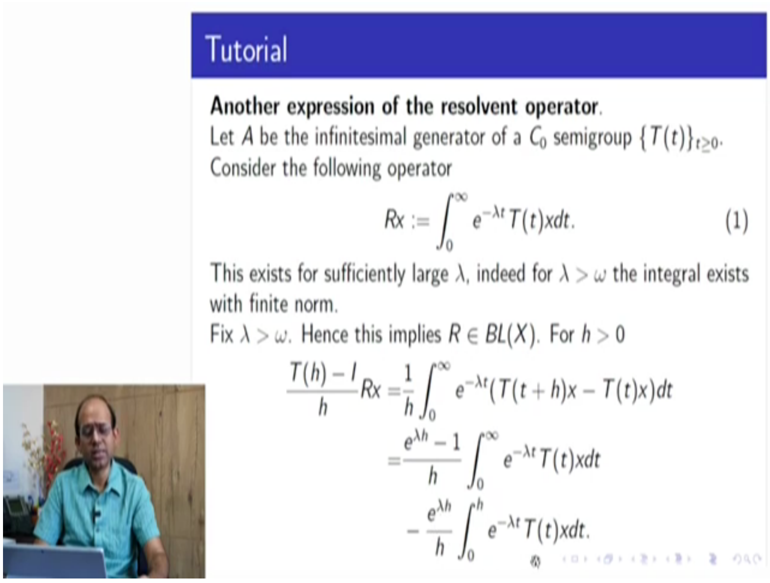

So, as h tends to 0 that right-hand side converges, hence left hand side is in the domain of A and then left hand side, you know is this thing would be it would cause to capital A. So AR x is equal to lambda times R x minus x. Why? Because this is going to give me lambda. As I have mentioned, this is going to give me x. And then this inside thing is from definition is R x. So right hand side is becoming lambda times R x minus x. So, this is true for all possible x small x.

So, we can conclude that the operator AR is equal to lambda R minus identity. So, now if we wish to take R common from these things, so I take lambda R on the left hand side. So then this lambda I minus A times R is equal to identity and then cancel negative sign, we get this expression. So, what does it imply, it implies that R appears like a right inverse of lambda I minus A. So, recall the definition of resolvent operator. Resolvent operator was the inverse of lambda I minus A. So, this is an indication that this, but we need to show this is also the right inverse.

So for that what do we need, we need to show that A and R commute each other, but for that we, because A is not defined on the whole domain. So, one has to be careful that what is the meaning of commuting each other. So, when the small x is in the domain of A, for that AR is same as RA. So in that sense. So, here we go, again for x in the domain of A, R A x equal to integration 0 to infinity e to the power of minus lambda t Tt Ax dt. How did we get it because, just from the definition, if RAx, I mean just R applied on A x so I have written A x here.

Now we know that Tt and A, they commute each other. So, Tt A is same as A Tt and then this whole thing so Tt x and then integration 0 to infinity this thing is in the domain of A, so A comes out here. So, A operating on 0 to infinity e to the power minus lambda t Tt x dt and this whole thing is R x so, is equal to AR x. So, this clarifies that AR and A commute so, as long as small x is in the domain of A. So hence R commutes with A. So, here also we can do the commuting things and we can conclude that R is equal to lambda I minus A inverse.

So this proves that the resolvent operator which is given as lambda I minus A inverse. So, that has another alternative expression and that expression happens to be very interesting, that is just nothing but the Laplace transformation of the semigroup. And here you know only thing is that where you talk about when it exists or when lambda is more than omega then it is, for that reason,.

And that is also is not a additional assumption because to define R resolvent you also need lambda I minus A is invertible in the first place, so that means you need lambda to come from the set of resolvent set. So, this is basically equivalent conditions, there is no additional condition for this expression and for this reason actually in many authors take this as the definition of the resolvent operator. Unlike us, I mean we have taken lambda I minus A inverse as the definition of resolvent operator. Now we go to the next topic.