Introduction to Probabilistic Methods in PDE
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Lecture-58
Sufficient Condition for Existence of Classical Solution of iIVP

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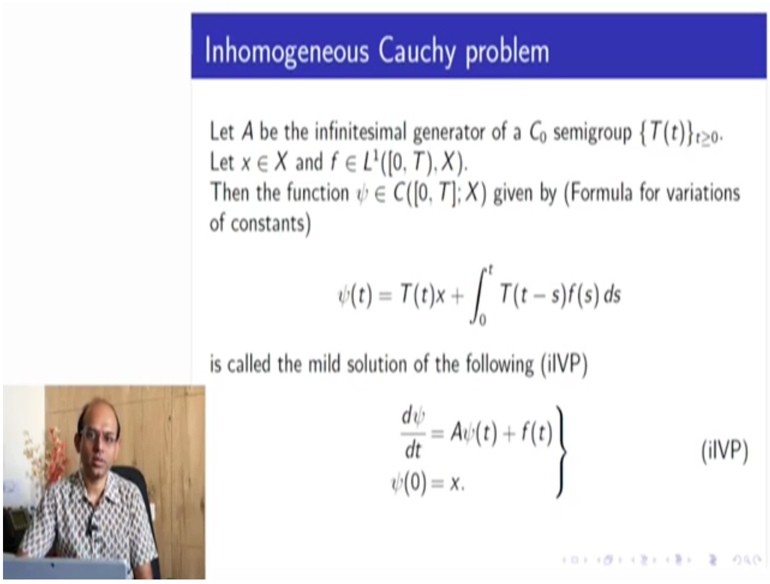

In the last lecture, we have seen initial value problem of homogeneous and inhomogeneous type. So, this is inhomogeneous initial value problem. Here I have d psi dt is equal to A psi t plus f of t, where f is some integrable function, f is in L1 and initial point is I mean data is a point in the banach space, small x is in capital X. For this equation it may not always admit a classical solution nevertheless, what we have seen in the last lecture that one can of course, define one function, this fashion to T t x plus 0 to t capital T of t minus s fs ds where this capital T is the C0 semigroup generated by the operator A.

So, then this function as function of t is called mild solution. So, even if a strong solution does not exist I mean the solution of these classical solution does not exist, if capital A generates a C0 semigroup then one can write down a mild solution of this equation. We have also given the motivation behind the reason why should we call these as a mild solution. So, today we are going to see that we are going to see under what condition on this f one can assure existence of a classical solution. We know that if f is 0 then of course, just for small x in the domain of A, one can get a classical solution correct.

So, here if instead of 0 if f is nonzero, so, for that I mean what type of things do you need? So here from this expression on a mild solution, one can get some idea. So, let us see what idea can I get. I mean, although that apparent idea what we make it needs you know justification and the rigorous proof. So, first intuition is that I would like to have sufficient regularity of this mild solution. Like if psi is a classical solution of this equation, then psi should be differentiable.

So, for that if x I choose from the domain of the operator A then T t x is indeed differentiable. Also we would like to have A psi to be meaningful. Now, if x is from domain of A then Tt x is also in the domain of A then at least this part would be in the domain of A. So, I can put x is in the domain of A then we can get the required smoothness at least for this term but we have another term also. For this term we know that this is just an integration.

So, if we do not go into the details what is written inside the integrand, we know that this is some L1 function. So if you have L1 function and you integrate it, so it is locally integrable. So locally integrable function to integrate 0 to t is of compact set and then it is with time you change, then that is a continuous function. Imagine that this has no t dependence inside, imagine that I have integrand which is independent of t dependence and then if it is just an L1 function and you integrate from 0 to t as a function t here, the whole thing should be continuous.

Now in addition to that if I do not have, if we have is just, instead of just integrable function we have a continuous integrable function. If this is not, there is no t dependence just this is a function of s and it is continuous. Then you know, after integrating 0 to t what do we get is differentiable in t. So, from that one can ask a very valid question, that if I choose my function f this you know this addition data to be continuous, can we expect that the whole integration would be differentiable?

If we can get this that this is differentiable then psi would be differentiable for every x in the domain of A and then this makes sense and when this makes sense, we can make sense of this part also. And then one can get that this psi should be classical solution of IIVP. However, we would see with some example that that is not the case, that it just the mere continuity of f does not give differentiability of this term. Why, because after all, I have another term which also depends on t.

So, at present therefore, you know it is vague, it is not clear that what condition you need to put here for on A, so that this equation would have a classical solution. So, at present we are looking for a set of sufficient conditions on this, on f. Nevertheless, one can put a direct condition I mean entangle with A and f together, that this whole integration is this thing is differentiable. If you put this way, then that would immediately give you know that this equation would have a classical solution., so, this is the thing which we are going to discuss today.

I mean, the set of alternative sufficient conditions on these functions etcetera, we would also give get some other little stronger condition on A for which we would not need to put a condition which is entangle with A and f. Because if I say that this integration is differentiable, then that puts a condition on both A and f together. So, we are going to see examples and you know some theorems results today which would clarify this aspect. So, this part we have already discussed earlier.

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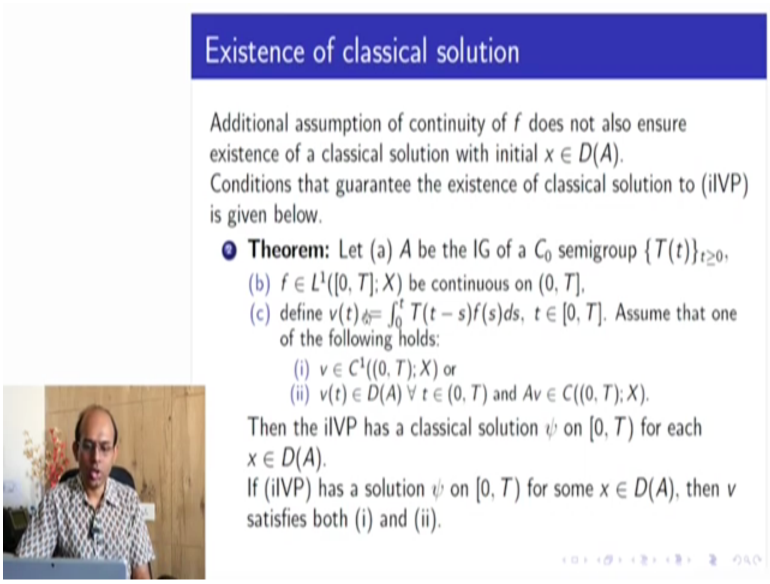

So, existence of classical solution we are going to discuss today. So, as I have discussed just now that additional assumption of continuity of f does not also ensure existence of a classical solution with initial point x in domain of A. I mean, I have not given any example yet but I would surely give an example towards the end of this topic, so, where I would get exactly provide you one particular f and particular choice of x. So where f would be continuous but that equation would not have a classical solution.

So this example I have taken from the book of A Pazzi. So, if this is not a sufficient condition, just continuity of f is not sufficient condition what should be that So, conditions that guarantee the existence of classical solution to IIVP is given below. So, first we assume that condition on A that A be the infinitesimal generator of a C0 semigroup Tt. So, first condition on A, so this is a standard condition that it generates a C0 semigroup.

So, this condition is common in the earlier lecture also, for homogeneous case also you have had these you know assumption on A. Now second condition that f is in L1, so, it is integrable and it is continuous on 0 to T, this is time interval t. So, we are looking at a finite horizon time point. So, then what we do is that we define vt that integration 0 to t capital T of t minus s fs ds. So, this whole thing as a function of t that I call as v of t and we assume that one of the following holds.

One of the following holds of vt, I mean that we have already started discussing that v is continuously differentiable, v is in C 1 of the, on this open interval 0 to capital T. This is one, otherwise we also another alternative assumption is that v of t is in the domain of A, domain of the generator A for each and every t and after applying A , A on v, select A v t, Av that is in, that is continuous function. It is continuous function from this opening to 0 to T to X.

So, assume that A has this standard assumption and on f icontinuity and in addition to that, we assume that this integration is either continuously differentiable or this integration is in the domain of A, it is an additional condition. So under this, we have a conclusion. Then the initial value problem, which is stated in the earlier slide has a classical solution psi on the interval 0 to T for each x in the domain of A.

We cannot expect that you can assure classical solution for any arbitrary initial point because that is even not true for homogeneous case. So we can at most expect x belongs to the domain of A for that, good. So, if the IIVP has a solution psi on interval 0 to T for some x in domain of A, then the reverse is also true. But that then v satisfies both 1 and 2. So, here this says that, this is like a necessary sufficient condition. Given that A generates a C0 semigroup, then this is an if and only if condition.

Because if v satisfies either of this condition, then IIVP has a classical solution. On the other hand, if IIVP has a classical solution, then this v satisfies both these conditions. So this is, this theorem is serving its purpose, it is really giving necessary and sufficient conditions for existence of classical solution of IIVP. However, we are not fully satisfied with this statement because of after all, here this condition on v is like condition on both A and f.

Because a condition on the, I mean this is entangle, correct, this is semigroup and f, both appears here. So semigroup property which depends on A that also appears here. So, we are going to see some other statements, other results also, which do not put condition on v but only put conditions on A f, but those are a little stronger. So, those are not if and only if conditions, those are like sufficient conditions.

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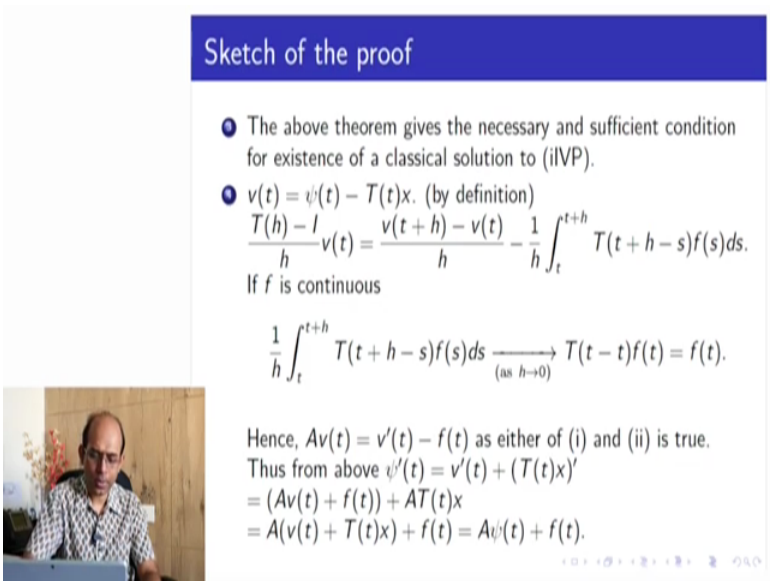

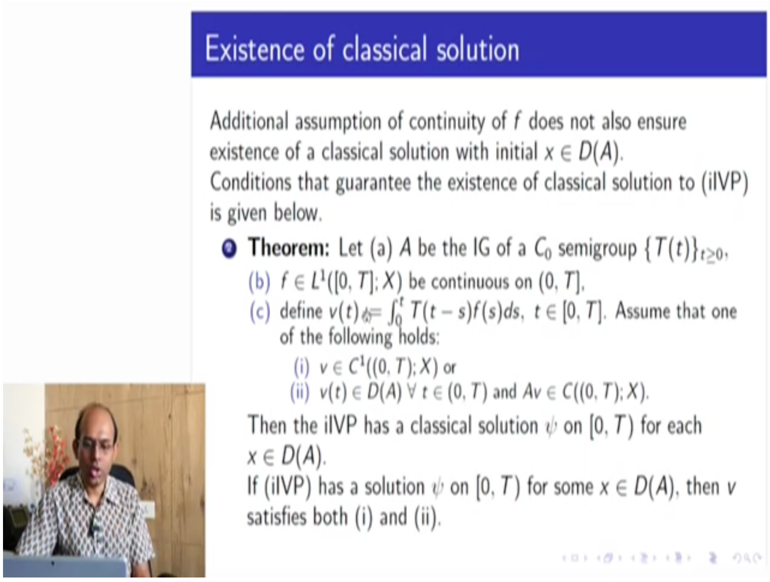

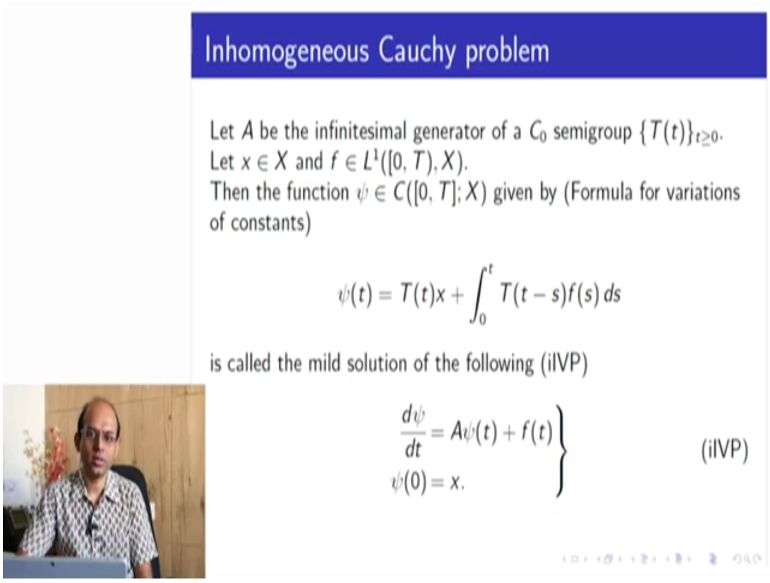

So, before going to another statement, we just see if the sketch of the proof. We are not going to prove this part from here to here. We are going to prove from here to here., so here we from the definition we derive that v of t is equal to psi t minus T t x. why, so let me show this here. So, here v t is equal to this integration and here when we have seen the formula of variation of constants, then psi is equal to T t x plus this integration that we call now v of t.

So psi of t is equal to T t x plus v of t or in other words, that v of t, I mean v of t is equal to psi t minus T tx. So, we get here v of t is equal to psi of t minus T tx from the definition of vt. So now we would like to use either property 1 or 2 to obtain that psi is differentiable, psi is differentiable and not only that psi satisfies that IIVP. So, what do we do is that here we take the semigroup T of h, h is a small positive real number T of h minus identity divided by h, v of t.

So, we do this why do we do this, because there as h trends to 0 this would give me Av, the generator we are going to get. So, T h minus i vt and now recall the definition of vt that is integrations 0 to t capital T of t minus s fs ds. Now, if I operate T of h on this integral, the T of h is the, since it is bound linear operator and then the inside the integrand also we have L1 functions. So, we can put this T of h inside the integration and then inside the integration we already have capital T of t minus s and then this capital T of h together, using the semigroup property, we are going to get capital T of h plus t minus s.

See here that if I have, here this is v, if I have capital T of h on this, so, capital T of h was inside and then capital T of h composition capital T of t minus s together would give me capital T of t plus h minus s here integration 0 to t. So here, I have t plus h minus s, but I have small t here. However, I can add and subtract the integration t to t plus h of this term to get here 0 to t plus h, this term to t plus h minus s and then minus small t to t plus h capital T of t plus h minus s of f of s ds. So one can write down that way, so we do that. Here capital T of h vt.

So, this operation, from that I would get 0 to t plus h this integration of this, which would give me exactly v of t plus h, but I have to also keep in mind the subtraction that is t to t plus h capital T of t plus h minus s fs ds. So, this is all because I have added there so I have to subtract also. So, so that would give me. So, this operation T of h on vt operation gives me vt plus h minus this thing. And then minus identity vt is giving minus vt. So, this left hand side is written this way.

Now here this looks pretty simple, this is basically in the finite difference of vt. So, as I mean if v is differential, continuously differentiable then this exists. Or the second property v is in the domain of A, then also this limit exists. So, either of these distributions exist. So, here on the other hand if f is continuous, so, if f is continuous, then we know that this limit exists. And what is the limit that as h tends to 0 this would be actually the value of the integrand by putting the T plus h I mean h would be 0 there, I will put 0 instead of h.

And then this integration s T to t plus h, so instead of s I have to put small t here. So, this limit of this thing would be the value of just the integrand where h is replaced by 0 and s, the running variable is replaced by t. So, f is continuous, so this part 1 over h t to t plus h, capital T of t plus h minus s fs ds. So, that converges to capital T of, so h is 0, s is t so, t minus t capital T of t minus t f of t. So, this is capital T of 0 that is identity maps.

So, that of f t is f t itself. So, this whole part becomes f of t. So now, since this limit exists and either of these 2 limits exist from the assumption, so the earlier assumption here either of these is true correct. So either of these would give me either of these 2 terms exist, the limit exists. So then what we get the other should also exist, because you know this limit exists this limit exists, so, then this also should exist. Or this exists and this is exists, that will imply that this should also exist, because left hand side limit exists, that mean the right hand side limit also should exist.

So, we would get that this limit exists on both the sides we take the limit, and then what we get from the left hand side is the generator A. So, this is A vt here and this is v prime t here and then this whole thing converges to f t. So, minus sign is there, so minus f of t. So, after taking the limit, we get this as either of 1 or 1, 1 and 2 is true. Next, what we do is that we go back to this equation v of t is equal to psi t minus T t x. Since here we have already obtained this thing.

So here we take the derivative v prime that is equal to psi, I can take the derivative because you know if either of that 1 or 2 exists, so then both exists, both limit exists we have seen here. So v prime is equals to psi prime minus the derivative of Tt x, Tt prime x. So that we write down here by taking the derivative psi t, psi prime is equal to v prime plus Tt x prime. So now here we have an expression for v prime already obtained here that we are going to use now.

So, this is equal to v prime is A v plus ft because this ft will come here, so v prime is equal A v plus f. So, A v plus f, so v prime is written as A v plus f and then T t x prime. So, derivative of T t x is A Ttx, so that we have seen in the beginning of the topic of semigroup theory. So, this I am going to get A Ttx, so here I see the operator A, here is also operator A arises, so, we can take common A from these 2 terms. So, A if we take common, so I would get inside the bracket v of t plus T t x, and then plus f of t.

Now v of t plus T t x is a known quantity what we look at here. So, vt is psi t minus T t x or in other words psi t is going to vt plus T t x. So, this v t plus Ttx is nothing but psi t. So, I write down that is equal to A psi plus f. So, what did I get I got psi prime is equal to A psi plus f, that is the inhomogeneous initial value problem d psi dt is equal to A psi plus f. Let us go back again to the initial problem to see. So, d psi dt is equal to A psi plus f. So, next we come to that example. So, as I have discussed that only continuity of f is insufficient to ensure existence of a classical solution given the initial data is coming from the domain of A. So, this is illustrated by this example.

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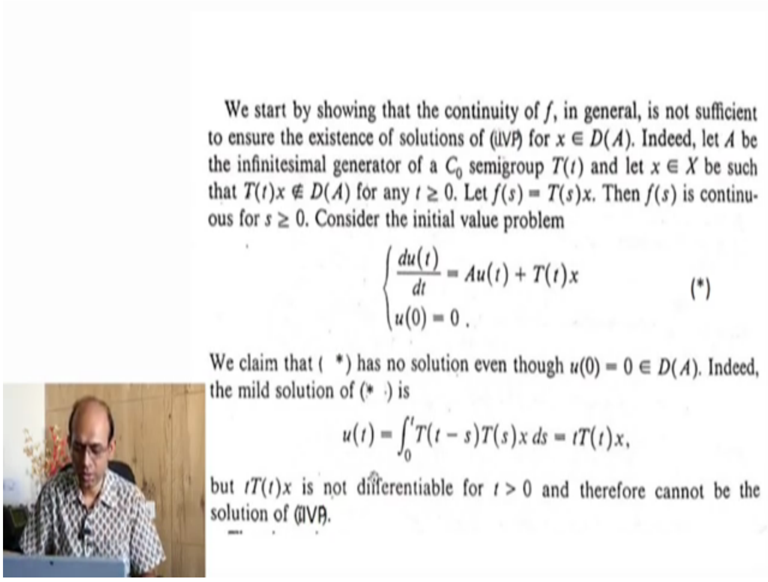

We start by showing that the continuity of f, in general, is not sufficient to ensure the existence of solutions of (IIVP) for $x \in D(A)$. Indeed, let A be the infinitesimal generator of a C_0 semigroup $T(t)$ and let $x \in X$ be such that $T(t)x \notin D(A)$ for any $t \geq 0$. Let $f(s) = T(s)x$. Then $f(s)$ is continuous for $s \geq 0$. Consider the initial value problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + T(t)x \\ u(0) = 0 \,. \end{cases} \qquad (*)$$

We claim that (*) has no solution even though $u(0) = 0 \in D(A)$. Indeed, the mild solution of (*) is

$$u(t) = \int_0^t T(t-s)T(s)x \, ds = tT(t)x,$$

but $tT(t)x$ is not differentiable for $t > 0$ and therefore cannot be the solution of (IIVP).

As I have discussed that this example appears here. So, we start by showing that the continuity of f in general is not sufficient to ensure the existence of solution, that means strong classical solution of the IIVP, inhomogeneous initial value problem, for x in the domain of A. Indeed, let A be the infinitesimal generator of a C0 semigroup Tt. So, assume that is there and small x is in capital X such that Tt x is not in the domain of A. We consider one particular x such that Ttx is not in the domain of A.

For any t greater or equals to 0, let f of s, the data we choose cleverly here, f of s is defined as Ts of x. See, this x is not the initial data, it is just an arbitrary small x such that Tt x is not in the domain of A. So, if x is taken from the domain of A, then Tt x must always be in the domain of A, but otherwise it need not be true. So, if you choose Tt x, if you choose x not from A and ensure that it is chosen such that Ttx is not in the domain of A, so that we consider that type of x.

To construct one f in this fashion f of x is equal to T s of x. So here since capital T is a C0 semi group we would get continuity of f from here, from the definition. So fs is continuous. Now consider the initial value problem, du dt is equal to A ut plus Ttx. So here f of t is replaced by Tt x and initial data is just 0, as I told x is not the initial data, it is taken just to construct one f function and here initial data is taken as just a 0. So 0 is of course in a domain of A, because Th minus I by h of 0, that is 0, so that limit exists etcetera. So 0 is always in the domain of A.

So, we claim that this example equation star has no solution. So, it has no classical solution. So, although that u of 0 is, this 0 is in the domain of A. And here see, I mean the initial data is in the domain of A, function f is also continuous, but still we are going to show that it that it cannot have a classical solution. So, let me, let us see the details it is not very long. So, indeed, the mild solution of star is obtained using the variations of constants.

The formula variation of constants says that u of t is the solution u of t should be equals to T t of 0, Tt of 0 is 0 and then that does not appear, then plus integration 0 to t capital T of t minus s t and then fs d s, fs here is T t x. So, ft is Ttx, so fs is Tsx So, I should write Tsx ds here from the formula. But this is semigroup, the same semigroup. So, we can operate and what we get is the Tt. So, Tt x ds, so, then t minus s T s gives T t So, this inside this integrand becomes independent of the s variable.

So, this integration would be small t times capital T t of x. So, this is ut, so the mild solution of this equation is nothing but small t of T t x, small t times Tt x. So, now question is that can it be a classical solution? The answer is no. Why, because this to become a classical solution, this thing is really should be in the domain of A, however Tt x is not in the domain of A so this is not in the domain of A. So, but Tt x is not differentiable. So, this is not domain of A, so, this is also non differentiable, we have seen that if v of x is, our Tt x is not in domain of A, so, then this is not differentiable.

So, this is clearly not a classical solution of IIVP. So, this is the proof of the statement what we were saying and this also therefore, clarifies our concerns and emphasizes that why do we really need a very good condition, some self sufficient condition for which we required that either of this thing to be true. So, we really required something more than just continuity of f.

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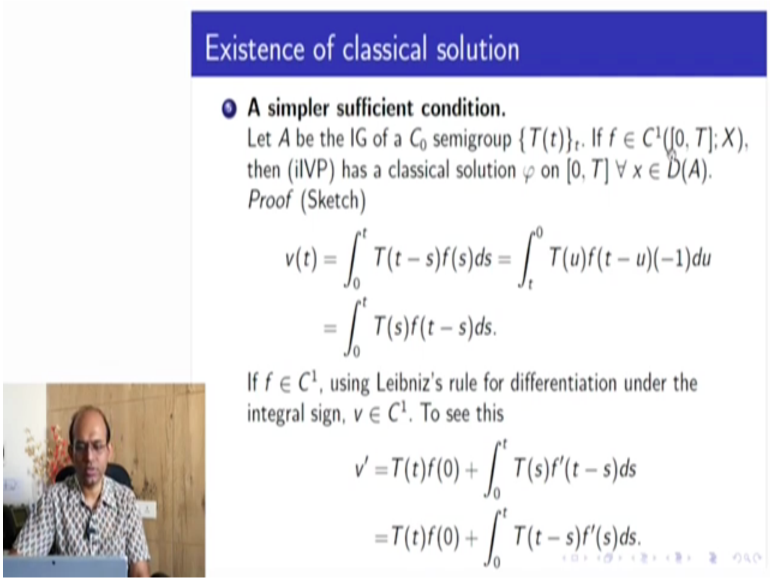

Okay, fine. So, next we go to another theorem, which gives another set of sufficient condition for the existence of classical solution to the IIVP. So, a simpler sufficient condition, let A be the infinitesimal generator of a C0 semi group Tt, if f is itself a continuity differential differentiable function, C1, 0 T to X, f is the banach spaced valued function, is valued function on the time interval. So here f itself is C1, then IIVP has a classical solution on this interval for all x in the domain of A.

So sketch of the proof, so I am just giving a sketch of the proof, not the details one. So what do we need to show, that if since I am using continuously differentiability of A, we need to show that the mild solution is sufficiently smooth and it solves classically. So here we look at v of t. So here we are going to apply the earlier theorem. So if we just show that v of t satisfies either of the conditions, 1 or 2, then we are done. So v of t is integration 0 to t capital T of t minus s fs ds.

So here we now substitute the variable t minus s as u and then t minus s is u so s is t minus u. So it would become capital T of u, f of t minus u, and then when s is 0 u would be t and when s is t u would be 0 and d s is minus of d u. So from there, since minus sign is there, t tends to 0 so we can change the limit these things and remove the minus sign we get 0 to t Ts f t minus s ds. So v of t can be written this way. And now here we see that t dependence appears here and here and f is C1. Now we use Leibnitz rule for differentiation under the integral sign.

So if f is in C1, using differentiation under the sign of integral, we would be able to show this whole thing is also differentiable, whole thing that is v, and then we will be able to use the earlier theorem. However, we see the details that how that Leibnitz rule apply here. So, the derivative of v prime t is equal to. So, now, first we consider this t variable. So, here differentiation of this would be just the integrand where s should be replaced by t. So, you get capital T of t f of t minus t that is f of 0 plus, now, we keep this integrand this fixed and we take differentiation inside.

So, integration 0 to t capital T of s and f t minus s is there, so f prime of t minus s ds. So, this is the application of Leibnitz rule. So, v prime is this and then this thing I can rewrite using exactly the same way, similar manner, by changing the substitution variables, we can write down that as 0 to t integration capital T of t minus s f prime s ds, so f prime exists, etcetera. So, this is the way we can justify that and here why should it exist because f is in C1, 0 to T closed. So, the derivative is continuous on the closed interval, so derivative is bounded. Since it is bounded, so this integration makes sense, this will be finite. So, v prime exists. So, now we can actually apply the earlier theorem where v prime exists, v is C1 and that would lead us that existence of classical solution to the IIVP. So, this is a sufficient condition for existence of IIVP.

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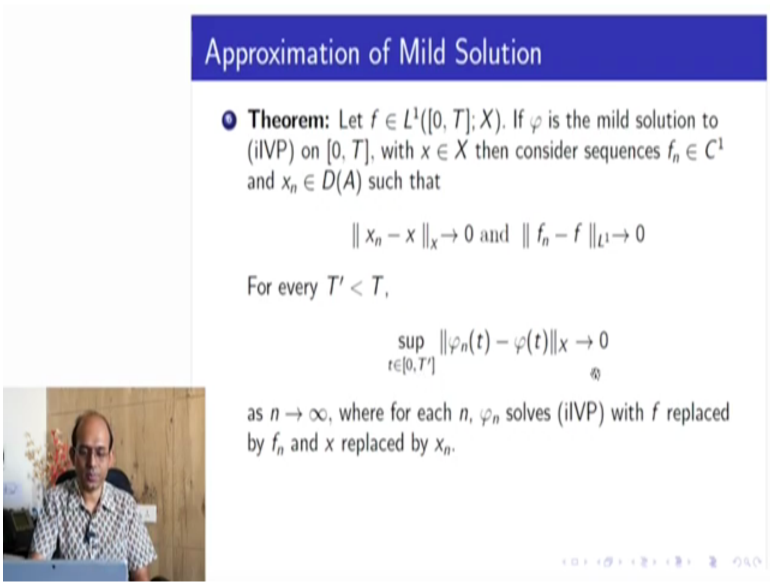

Next we see another very interesting theorem which talks about approximation of mild solution by classical solution of approximated equation. So, remember we have IIVP and in

IIVP, the given initial data x or the inhomogeneous part small f are may not be very good. So x could need not be in the domain of A or f need not be differentiable, etcetera. However, if one has a sequence xn in the domain of A which converges to x, that means x is in the closure of domain of A.

And if one has a sequence fn, which are C1 then converges to f in some sense, then the sequence of initial value problem which we can construct using this fn or xn. So, they would have classical solution and those classical solutions are good approximation or the actual mild solution of the actual initial value problem. So, this is the thing which we need to discuss now. Let f is L1 function, if phi is the mild solution to IIVP on the interval 0 to capital T with small x in capital X.

See a we have not assumed any strong condition on either f or x. So, we can only assure existence of a mild solution if that A is the generator of a C0 semigroup. So, then you get a mild solution. So, then consider sequence fn in C1 and xn belongs to D of A if exists then we consider, such that xn converges to x in the norm, the banach space norm and fn converges to f in L1 norm. Then for every t prime which is less than capital T, if we take phi n, phi n is what, phi n is, where phi n solves IIVP with f replaced by fn x replaced by xn.

So, this phi n which is classical solution of approximated equation and phi is the mild solution of the actual equation, then this solution of the approximated equation is a good approximation of the mild solution in this sense, because the sup norm sense, correct, the converge you know uniformly. So, phi n t minus phi, this is norm over x, supremum 0 to T prime, this whole thing converges to 0, for any T prime which is that capital T we consider and we get it.