Introduction to Probabilistic Methods in PDE
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Lecture-57
Mild Solution to Inhomogeneous Initial Value Problem

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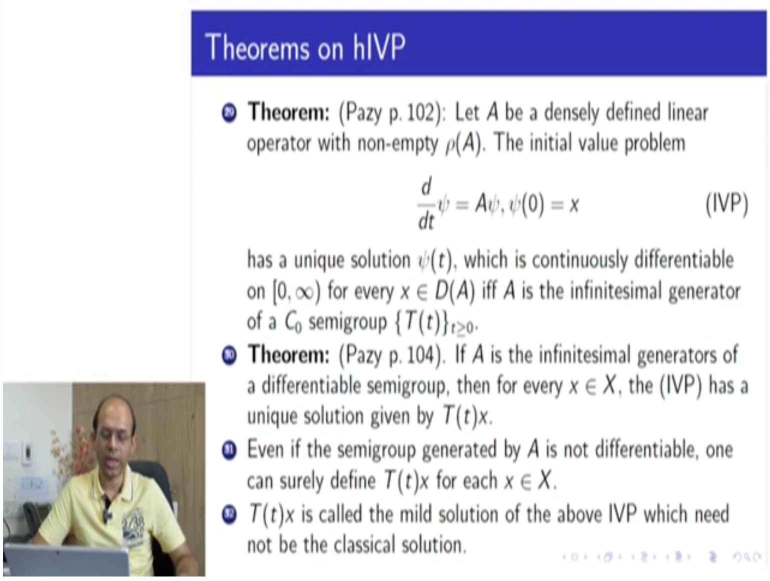

Welcome, today we are going to see Inhomogeneous Initial Value Problem. In the earlier lecture, we have seen the homogeneous initial value problem, we have stated couple of theorems also for that. We have also discussed about Hille-Yosida theorem, which lists necessary and sufficient condition for the operator A to be infinitesimal smooth generator of a C0 semigroup. And that theorem is useful because, given the operator, if we can identify the C0 semigroup, we can write down the solution of this equation using the semigroup and the solution turns out to be Ttx. So, here if we do not have just this but we have some more term on the right hand side, so ddt of psi is equal to A psi plus some another member in the Banach space or maybe function of time. So, the Banach space evaluated function of time, see if we have more data here then how to find out solution. So, that is the question we are going to address now.

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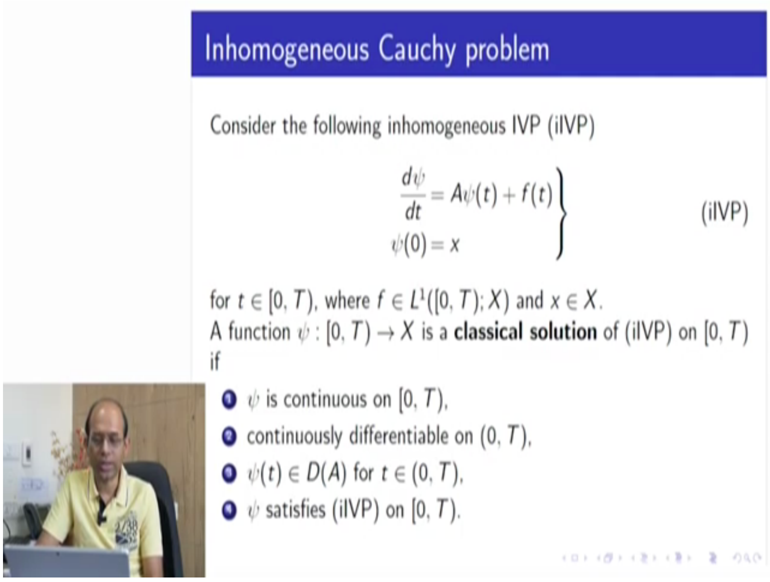

And we call that as inhomogeneous initial value problem. So, consider the following inhomogeneous initial value problem. So, why do you say inhomogeneous because I have some more term here, earlier I did not have, earlier it was just f is equal to 0. So, the equation was a linear equation. But now, it is no more just a linear equation because here we have some more term here f of t. But however it is affine linear in some sense because this is just a constant, it does not depend on the solution psi. So, here a d psi dt is equal to a psi t plus f of t and the solution psi at 0 is equal to small x.

So, initial point is small x, small x is a member in theBanach space and then over time. So, it looks like this ODE, but it is not ODE, you understand because A could be differential operator. Because this for every time psi t is a member of Banach space. So, it could be for every t psi t could be a function of another variable say of variable A. So, it is basically a Cauchy problem, is a PDE where this A when A is a differential operator we get a partial differential equation.

So for this problem we define first what we mean by a classical solution. So for small t, we considered this problem for time between 0 to say capital T. Here this capital T is different from the same capital T, I am sorry about using the same symbol in both the places, but that is really unambiguous why because whenever I write down a semi group I write in capital T and then in the parentheses small t.

And if we write down the, sometimes we write down capital T as a semi group also without parentheses but then we write down very clearly that let capital T be a semi group. So, however here this capital T comes, so when nothing is mentioned then it is just a constant. So, this use of the same thing two different places is really not ambiguous, however so, it needs a careful reading. So, for all small t between closed, from closed 0 to open capital T, we study this problem.

So, where small f is one L1 function, integrable, that the area under the curve f of f is finite, (infi) the mod of f is integrable on 0 to capital T. And we also choose small x from any point in capital X. so a function psi which is a function from 0 to capital T, this to X is called a classical solution of this I I VP inhomogeneous initial value problem on closer to open to T if the following conditions are true. The psi is continuous on this interval close 0 to open T, it is differentiable in the interior.

So continuously differentiable on 0 to T, otherwise I cannot justify the meaning of the left hand term, because we need d psi dt to be well defined classically. And then psi t also should be inside the domain of A otherwise this thing is not meaningful. So, psi t is in the domain of A for all t in that, I mean open 0 to capital T. So, if I include 0 here psi of 0 is x, then we are just asking that x also to be in the domain of A, but here x is in the capital X. So, we exclude 0 from here. So, psi t is in the domain of A for t positive 0 to capital T. So, this is the, these are 3 conditions we need also the most important fourth condition that it satisfies. Psi satisfies this equation, on this time interval. If all these things are true, then you call psi as the classical solution of IIVP.

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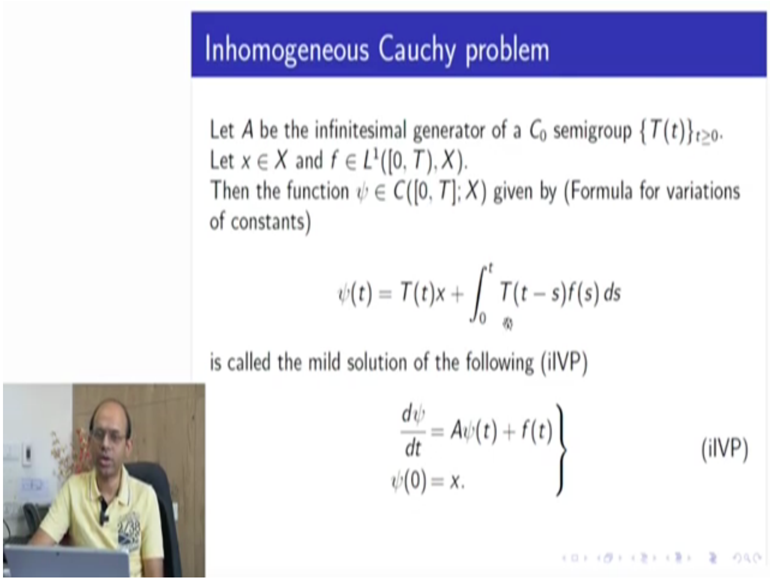

Now, we assume that A is infinitesimal generator of a C0 semigroup. It need not be the case, so given any IIVP that operator which appears on the right hand side need not be generator of a C0 semigroup, but we are considering the special case where A is the, an infinitesimal generator of a C0 semigroup and we also consider small x in capital X and f as before the integrable, then the function psi, which is defined by the following formula, which would be in the continuous function.

So, this is defined as psi of t is equal to capital T t of x plus integration 0 to small t capital T t minus s f s ds here these you know integration. So, here you can view these as you know, integration 0 to open t. So, T t minus s f s ds. So, this whole thing is a function of small t, it also depends on s and then this semi group is the one which is generated by the operator A. So, Tt is, I mean A is infinitesimal generator of the semigroup T.

So, from that PDE with 3 different information A, f and x, all these things are utilized here and then you construct this psi, this psi is called the mild solution of the IIVP which I have written it again here, d psi dt is equal to A psi t plus f t psi 0 is equal to x. So, this function which is a continuous function, why is it so, because Tt x is continuous that we have seen in the beginning of the topic of semigroup Ttx is continuous in time t and then f is a L1 function and this is a C0 semigroup.

So, here this Tt f is you know L1 loc, so, this integration is 0 to t this whole integration, this integral would be continuous in t variable. So, this is the reason that psi t would be continuous on the closed interval 0 to capital T though it is different here. So, this is called mild solution. Now, we have seen for HIVP that if psi t is defined as Tt of x, so then that is called mild solution for the homogeneous initial value problem, here, we have additional term f.

And then this expression actually if you put f is equal to 0, then this is IIVP because HIVP that is homogeneous initial value problem then the mild solution as it is described here also coincides with the mild solution of the HIVP because this whole term becomes 0. If f is equal to 0, the whole term becomes 0, then it also coincides. So, it is just a generalization, but that does not necessarily justify why should this be called a mild solution. So, in the next slide we are going to answer to this question.

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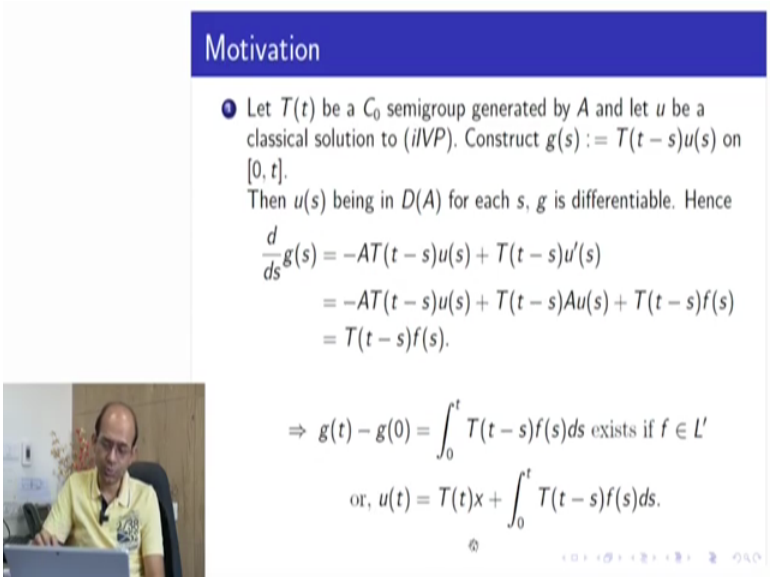

So, let capital T of t be a C0 semi group generated by A and let u be a classical solution to the IIVP. So assume that that IIVP has the classical solution, we call that u and that IIVP involves operator A, which is infinitesimal generator of a C0 semigroup capital T t. So, now we construct g function, where g of s, time s, where s is you know a point between 0 to t. So, T of s is defined to be T t minus s u s, for s is equal to 0 u0 is x and then is T t minus 0, so T t x basically. So, when s is equal to 0, g 0 is equal to T t x good. But when s is called small t, then it would be ut because u t and T of t minus t ,T of t, minus that means T of 0, T of 0 is identity.

So, g of t would be u t and g of 0 is x correct. Now Tt x, g of 0 is Ttx. So, we understand what is this g s. Then u is being in domain of A because you know u we have assumed to be a classical solution. So, the property of the classical solution is that it should be in a domain of A So, u being in domain of A for each s so g is differentiable because since u is in the domain of A we have seen that the semigroup operated on the point which is the domain of the generator so, that is differentiable.

So, g is differentiable in the interior open 0 to t. So, hence we take the derivative, we take derivative here both sides with respect to the s variable. So d d s of gs, dg ds is equal to here I have 2 terms Tt minus s u s, so this term also involves s, this term also involves s. So first we use the chain rule. So, we take the derivative here. So derivative of Tt is A Tt that we know. But here minus sign is there so we will get minus A Tt minus s and u s as before.

Next, we keep the first term as it and the second term, we differentiate, differentiate the second term, so plus Tt minus s u prime s. However, u solves that IIVP classically. So we know that what is the expression of u prime because u prime appears on the left hand side and that is equal to A u plus f which appears on the right hand side. So, that we are going to use. So, here as before this minus A T t minus s u s appears here plus T t minus s and u prime s is written as A u s plus f s and Tt minus s is operated both the sides.

So, Tt minus s A u s plus T t minus s f s. So, here I have minus A Tt minus s u s plus Tt minus s Au s. I can interchange between A and T here, because after all u is in the domain of A, so we can do that. And then these two are just you know, negative to each other. So, they cancel and then what we are left with is just Tt minus s fs. So, hereafter obtaining this, we integrate both sides. So, when you integrate both sides or the intervals 0 to small t, so the left hand side becomes gt minus g 0 and the right hand side becomes integration, 0 to t, T t minus s fs ds.

So this is of course finite, why, because f is in L1 and this is a C0 semi group, so here we get 0 to t T t minus, or 0 is bound linear operator, so we get integration 0 to t T, t minus fs ds exists if you know f is in L1. So this exists, f is in L1. So now, here, we know ds is equal to T t minus s u s and we have already seen the g of t is just u t and g of 0 is T t of x. So use that. So g of t is u t. So I write down u t here, g of 0 is T t of x, so I take g of 0 on the right hand side and get T t of x here and then plus integration 0 to t capital T of t minus s f of s ds.

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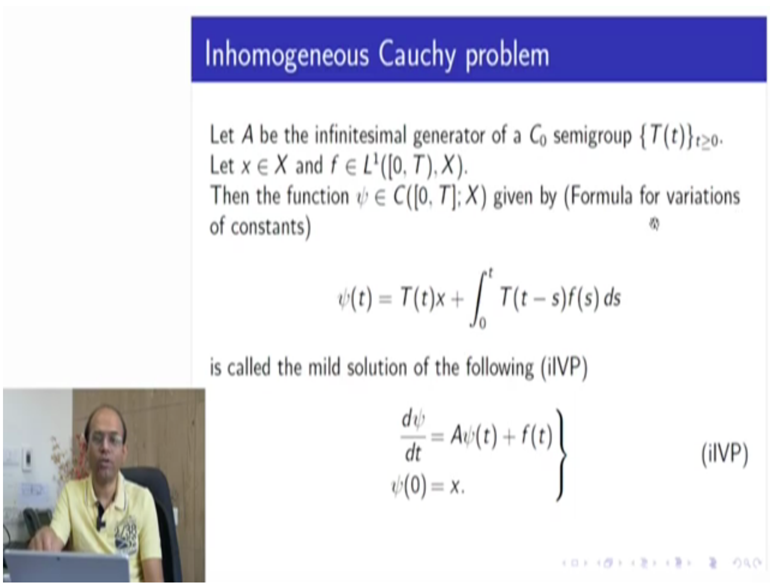

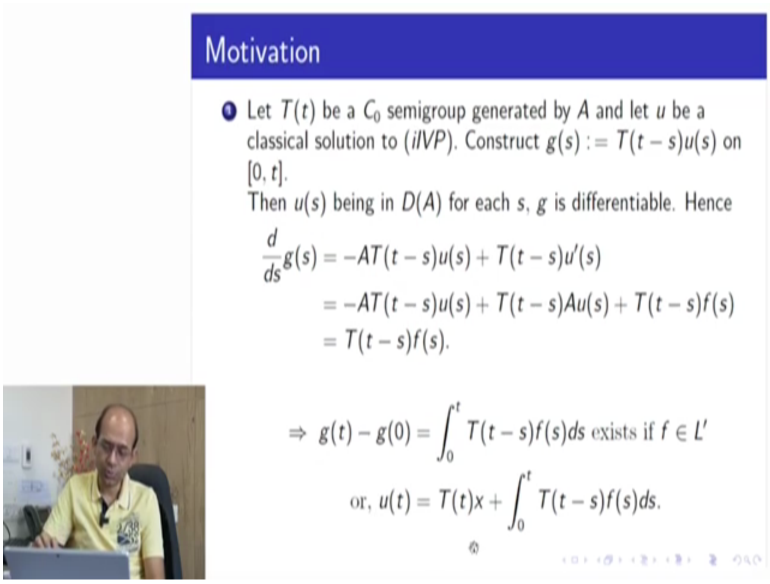

So, you obtain this formula, which appears before. So, psi t is equal to Tt x plus integration of 0 to t Tt minus s fs ds. So, this is called formula for variations of constants. And here in this derivation, we have clarified that this formula is actually for the solution, because u was taken to be the classical solution. So, if the IIVP has a classical solution and the operator which appears on the right hand side generates a C0 semigroup, then the solution, the classical solution has this expression.

But now, if we look at this expression, it clearly appears that this is generalization of the mild solutions that we have introduced for homogeneous initial value problem where small f was 0. And we also recognize that this right hand side is well defined for even if the equation possibly does not have a classical solution. So, like and x need not be in the domain of A, I will we can always define Tt of x as long as the operator which appears to the right hand side that A generates a C0 semigroup.

As long as the operator on the right hand side generates a semigroup, we can write down this expression, which is therefore a candidate of the solution. And therefore, we call this as a mild solution. So, this proof also clarifies another thing that for this IIVP if it has a classical solution, then that is unique because that means you would have this expression provided that A generates a C0 semigroup. And that as I have mentioned that even if A is not a bounded linear operator, but if it generates all these properties, if there are 2 different C0 semigroups T and S having the same infinitesimal generator, they both should be identical.

So, it is unique because A would generate a unique semigroup and then the classical solution would have this expression. So, even if it has, you assumed, then you prove by contradiction even if you assume that IIVP have 2 different solutions say u and v, then v of 2 would also be this. So, that would necessarily imply that u and v are the same. So, that proves that IIVP has a unique solution. So, this discussion gives us various different aspects. The first thing is that IIVP has a unique classical solution. Second thing is that the expression of that in this fashion is well defined even if x is not in the domain of A.

So, if x is not in domain of A, it is not a classical solution because you know we cannot even assure that is differentiable. So surely you can say it is not a classical solution. But still it makes I mean still one can write down this. So, that would be a candidate of the solution or we can call that it is a mild solution. So, by this thing, one thing is clear that we can introduce a more generalized notion of solution or the equation which is easy to write down and we call that as a mild solution, sometimes we also call that as generalized solution.

Sometimes we also call that as a generalized solution and only thing is very important here is that to write down this way is that the operator which appears in the PDE in the equation in the Cauchy problem should generate a C0 semigroup. And that question that whether it really generates a C0 semigroup or not, can be answered using Hille-Yosida theorem, Hille-Yosida theorem lists down the sufficient conditions under which an operator generates a C0 semigroup.

So, we understand now the importance of the Hille-Yosida theorem, application of that in solving equations. Also we understand that for 2 different classes of equation, one is homogeneous initial value problem and there is inhomogeneous initial value problem, for both we can define a notion of generalized solutions or mild solutions and we have also written down the expressions of mild solutions. Thank you very much.