Introduction to Probabilistic Methods in PDE
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Lecture 56
Mild solution to homogeneous initial value problem

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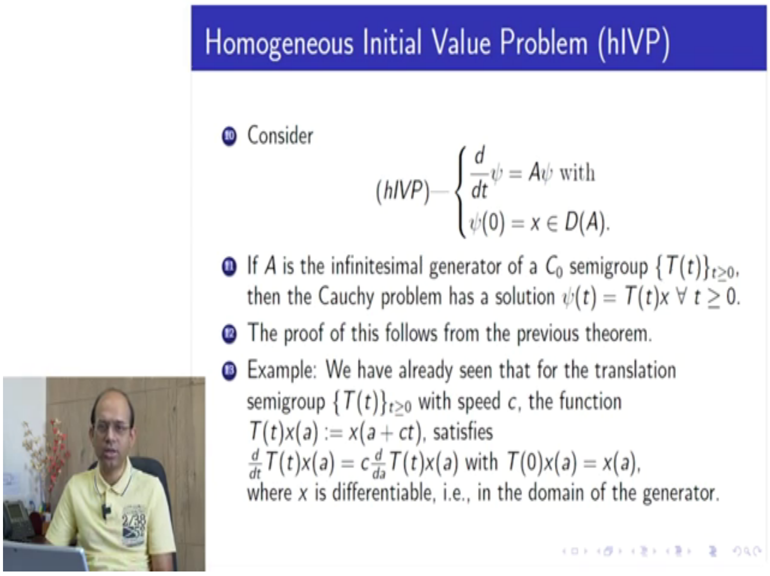

Welcome, so today we are going to see some more results about homogeneous initial value problem. Let us recall quickly the definition of homogeneous initial value problem. So, here in the last lecture, we have seen this homogeneous initial value problem. So, it is written in this fashion that d Psi dt is equal to A Psi with Psi of 0 is equal to x, x is in the domain of A, A is an operator.

So, that operator may not be defined on the whole banach space capital X but maybe somewhere inside that banach space it is defined and then we choose small x from the domain of definition of the operator A, then this problem has the solution T t of x, what is this T t, this capital T is the semigroup generated by the operator capital A.

So, given a homogeneous initial value problem, where capital A is given an initial point is given, one can write down the solution provided one knows what is the semigroup generated by capital A. So, if we know the semigroup generated by the operator capital A, then using that semigroup we can write down T t of x.

So, this is a function of time and this function satisfies this equation that the derivative of T t of x would be A times Psi. So, that was the thing which we have already seen before and then even if x is not in the domain of A, this T t x is well defined, because T t is defined on the whole banach space capital X so, this is still defined. So, that would not be a solution, but would be candidate of the solution.

So, we call that type of solution as that type of function as mild solution, understand? Because here when x is in the domain of A for those particular initial data, the function T t x happens to solve this classically, but if x is not in the domain of A then still T t x is well defined and that T t x we call as mild solution of this equation.

So, the proof of this follows from the previous theorem what we have discussed already in the earlier class and we have seen one example of translation semigroup. For the translation semigroup we have already seen earlier that satisfies this equation that derivative of T t x is equal to c times d da of T t x, that clearly shows that this T t x satisfies the equation where A is replaced by c times derivative.

Next if we ask that when can you assure existence of solution of this equation, then basically, one sufficient condition for that would be that let A generate a semigroup C 0 semigroup because for that we already have seen that we can write down the solution. So, then the question of solving an initial value problem boils down to the following question, that given this operator A n, whether it generates any C 0 semigroup or not. So, we are going to see one result where we are going to list down a set of sufficient conditions on A so that A generates a C 0 semi group, and that would help one to solve a given Cauchy problem.

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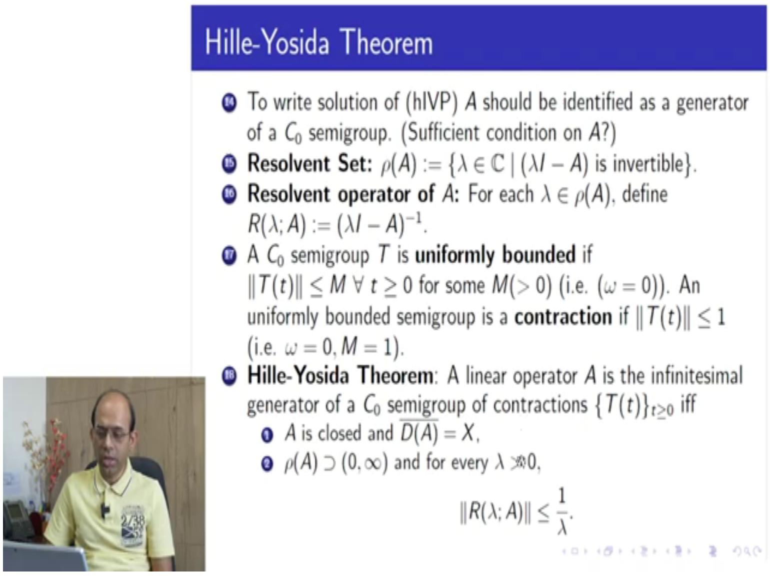

So we go next. So, to write solution of homogeneous initial value problem, A should be identified as a generator of a C 0 semigroup. Now, for that we need some condition on A because every operator A might not generate a C 0 semi group. So, we would look for a set of sufficient conditions.

So resolvent set, so here we are looking at various different definitions, which would be important for describing the Hille-Yosida theorem, which lists some sufficient conditions which assures existence of a C 0 semigroup generated by the upper given operator A. So, here this is the definition resolvent set. So, Rho of A is defined as set of all complex numbers of Lambdas such that Lambda I, I is your identity operator Lambda times I minus A this operator is invertible.

So now, see that if A operator itself is invertible then 0 belongs to Rho A, because then so this is invertible. And this includes all those complex numbers for which this difference operator is invertible so that we call as resolvent set. So, we know that spectrum of an operator is contained in the compliment of the resolvent set because there cannot invert it.

So, resolvent of operator of A so resolvent operator of A is another new notion which you are defining here for each Lambda in Rho A since we have chosen Lambda from Rho A resolvent set that means Lambda I minus A is invertible already. So, we define Lambda I minus A inverse as capital R of Lambda comma A.

So, this operator depends on Lambda and A so that dependence is clarified here by these you know argument Lambda and A. So, instead of writing this whole notation we write down this R Lambda A so, that is called resolvent operator of A. And we are going to see two more definitions here, A C 0 semigroup capital T is called uniformly bounded if norm of T t.

So, what is this? This is subordinate norm, the operator norm in the banach space capital X. So, norm of T t is less than or equal to some positive number capital M for all t non-negative, if that happens if that is true then we call capital T t this semigroup as uniformly bounded semi group.

So, remember that in our earlier lecture, we have seen some theorem that the growth property of the semigroup, the norm of T t is less than or equal to M times e to the power Omega t for some positive M and positive Omega. So here, if we write down this upper bound where the upper bound does not depend on time. So, that means this whole semigroup is just bounded by these number capital M. So, here I write down Omega is equal to 0 case because e to the power Omega t that Omega was, so from this case you can think that is 0 here so, we call this uniformly bounded.

Furthermore if an uniformly bounded semigroup satisfies this condition that this norm of T t is less than or equal to 1, this capital M is equal to 1 value. So, then that semigroup is called contraction, so an uniformly bounded semigroup is a contraction if norm of T t is less than or equal to 1.

So, that is this case Omega is equal to 0 and Capital M is equal to 1. So, contraction mapping I think many of you have already seen in other contexts. So, that this definition I mean quite coincides with that I mean, but sometimes you know contraction definition involves norm of T t strictly less than 1 in some textbooks, so here it differs a little bit.

So, I am following this definition from the book by A. Pezzi. So, here this is Hille-Yosida theorem, Hille-Yosida theorem which I have already discussed that the importance of this theorem because this lists down a set of sufficient conditions on the operator A such that that generates a C 0 semigroup.

A linear operator A is the infinitesimal generator of a C 0 semigroup of contractions if and only if A is closed and closer of domain of A is x. So, this sentence is same as saying that the operator A is densely defined, we say that it is densely defined, why because where it is defined that set is large enough so that the closer of that set is the whole space itself.

So, this is first condition, the second one is that the resolvent set Rho of A contains the interval 0 to infinity, so open 0 to infinity. So, all positive numbers are contained in the resolvent set. So, in other words I can say that for every Lambda positive, Lambda I minus A is invertible.

So, for that case for Lambda positive, I can define R Lambda A because it is in the resolvent set, we require further more one property that the norm of R Lambda A this is the resolvent operator, the norm of the resolvent operator is less than or equals to 1 over Lambda. Or in other words, Lambda times the resolvent operator that is a contraction, I mean, the norm of that is less than or equals to 1. So, Lambda times R Lambda A is a contraction so these are the conditions.

Then we go to Yosida approximation, before going that, let me clarify the implication of this theorem. So, this states that if A is infinitesimal generator of C 0 semigroup then these two things hold. On the other hand, if the operator A satisfies these two conditions, then A can be viewed as an infinitesimal generator of a C 0 semi group of contractions. So, we are not going to prove this theorem, we are just stating this theorem.

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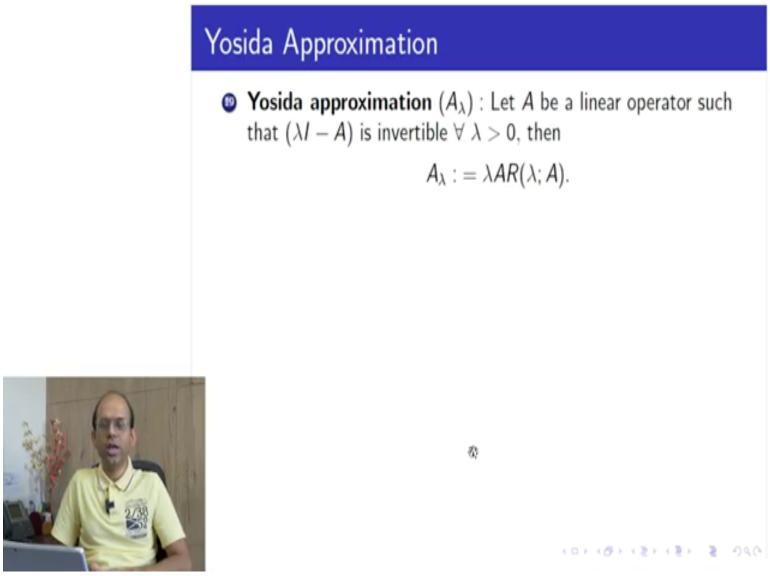

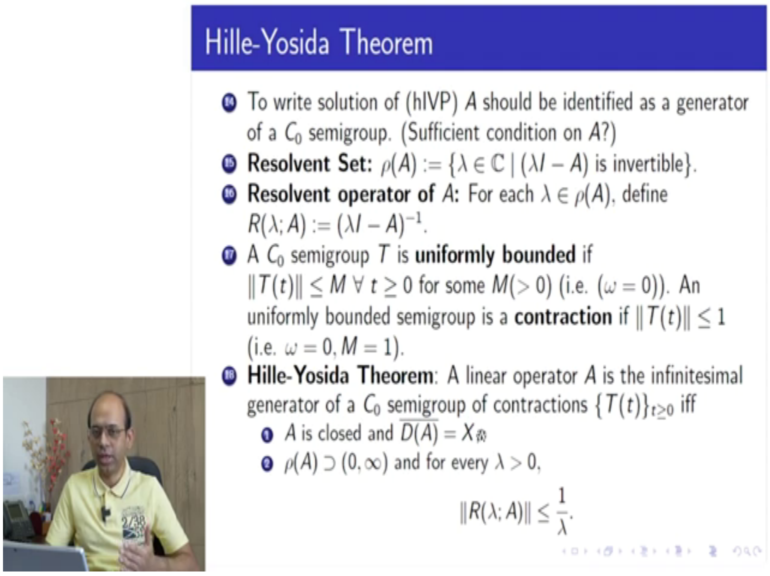

Next we come to Yosida approximation. What is this? This is that given an operator A. So, actually Yosida approximation plays a very key role in proving the Hille-Yosida theorem since we are not presenting the proof I would not be able to show that how this Yosida approximation was used in the proof, but I can actually give intuitive understanding that how is it important here.

So, Yosida approximation basically approximates an operator by a bounded linear operator. So, what we have seen earlier that if your operator A is a bounded linear operator, you can always define e to the power of t times A that forms a semi group and that semigroup is uniformly continuous semi group.

We have also proved that the generator of uniformly continuous semigroup uniquely determines the semi group, so what does it mean? That means that if there are two different uniformly continuous semi groups T and S, then if both have the same generators both would be identical. So, that discussions we have already seen and that lead to one conclusion that for a bounded linear operator A, the semigroup generated by A is easily obtained that is just e to the power of t times A that semigroup is the one.

However, in Hille-Yosida theorem, there A need not be a bounded linear operator, A could be an unbounded operator, if A is an unbounded operator, we just know that norm of T t is can be upper bounded by some exponential map, but as such T t we do not know how to write down because e to the power tA makes no sense there.

So, it is a reasonable question that if A is just an unbounded linear operator, can it be an infinitesimal generator of a C 0 semigroup, then in that for that question one can try to approximate A by a bounded linear operator A Lambda. That A Lambda is just a positive real number as Lambda tends to infinity, A Lambda goes to A in some sense, and which is because if A itself is not a bounded linear operator, then I cannot say that A Lambda converges to A in the operator norm because A itself is infinity.

However, I mean the convergence may not be uniform sense but sense like, point wise like A Lambda x converges to A x kind of thing, for each and every x. But each and every x would be also too strong because A may not be defined for all possible x in the banach space. So, at least A Lambda x converges to A x for x in the domain of A would be desirable.

So, these are the kind of discussions, which indicates that it is not impossible, it is thus possible to actually, you know, approximate a given operator by a sequence of bounded linear operators. But then what? Then for those bounded linear operators A Lambda for each and every A Lambda one can get a semigroup and that semi group is uniformly continuous and that would be easily obtained that will be just e to the power of t times A Lambda.

And then you get a semigroup, you wanted a semigroup coming from A but what you have obtained a semigroup coming from approximation of A. But then that is an approximate semigroup but what is the guarantee that this approximate semigroup is a good approximation of the actual semigroup generated by A.

So, that for that one needs some effort and these conditions are also important. So, with these sufficient conditions work can show indeed, that this approximate semigroup obtained for A Lambda that converges somewhere and where it converges that will ensure that that would be the semigroup generated by the operator A. So, this proof is pretty long, and which has many steps, we are not going to the details of the proof.

So, those who are curious about the outline of the proof, for them, I just told very briefly the basic philosophy behind the proof. So, we understand that the key role played in the proof is by the approximation of A by some bounded linear operator.

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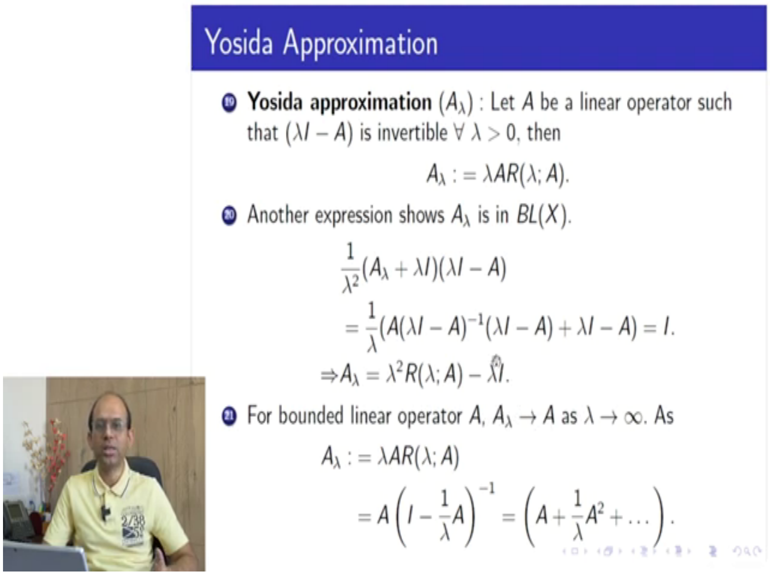

So, we do that here and we call that as Yosida approximation. And we write down A subscript Lambda here. So, let A be a linear operator such that Lambda I minus A is invertible or in other words, I say that for all Lambda positive or in other words, I want to say that, Rho A contains 0 to infinity, because for every Lambda here Lambda I minus A is invertible.

So, consider A is such that it has this property, the property 2 it has. So, Lambda I minus A is invertible for all Lambda positive. Then we define A Lambda by Lambda so this is the definition of that bonded linear operator that Lambda times A composition with resolvent operator of A with the same Lambda value. So, Lambda A R of Lambda A so this is the definition of A Lambda.

But this definition does not immediately imply that A Lambda would be bounded linear operator. So, we need to derive certain things to see that transparently. So, another expression we are going to derive which would show that A Lambda is in B L X. So, what do we do is that we consider this expression 1 over Lambda square times A Lambda plus Lambda I. So, this A Lambda is newly defined approximation, Yosida approximation. So, A Lambda plus Lambda I times Lambda I minus A. So, this A Lambda already has Lambda multiplied, the scalar multiply here also I have this Lambda.

So, this Lambda coming from here and this Lambda here, I take common outside so Lambda divided by Lambda square because 1 over Lambda. And then what I am left with is just A, then composed with R Lambda A, so here A I write, R Lambda A I write in the full expression of that R Lambda is Lambda I minus A inverse.

And then this quantity is getting multiplied with this whole thing, so Lambda I minus A, and then here, I do not have Lambda I, I just have I now because Lambda I have taken outside here. So I times Lambda I minus A would give me again, Lambda I minus A because it is just identity.

So, we would be having this expression 1 over Lambda times inside the bracket, we have A times Lambda I minus A inverse Lambda I minus A plus Lambda I minus A. Now here, Lambda I minus A inverse and Lambda I minus A, they cancel each other and then I will be left with only capital A here.

So, capital A here and minus capital A here, they would cancel each other. So, I would be left with only Lambda I, but here 1 over Lambda is already there so only I, the identity operator is remaining. So, this Lambda I minus A I Invert both sides, now, this is the left hand side, right hand side is just identity.

So, left hand side, I would have 1 over Lambda square A Lambda plus Lambda I is equal to R Lambda A because you know this inverse is R Lambda A. So, then I take Lambda square both side, I multiply Lambda square, so A Lambda plus Lambda I would be Lambda square times R Lambda A, but left hand side I have A Lambda plus Lambda I.

So, this plus Lambda A, I take on the right hand side, so I would be left with A Lambda is equal to Lambda square R Lambda A minus Lambda I. So, this is another expression. So, this is considered as definition of A Lambda which does not say anything that whether it is bounded linear operator or not, but here this expression gives a very clear thing because A Lambda is equal to Lambda square R Lambda A minus Lambda I.

And here in the Yosida theorem, in the first step we prove that R Lambda A the norm of this is less than 1 over Lambda so, this is a bounded linear operator so, its norm is finite. So, this is a bounded in linear operator and then multiply with scalar and subtracted by another bounded linear operator.

So, what I get as A Lambda is a bounded linear operator. So, this clarifies that A Lambda is a bounded linear operator, but I have not yet shown that why should we consider this at all, an approximation of A, although the name is Yosida approximation, but it is not a very clear that how is it approximating. So, here is one special case for which we show the very basic heuristic here for bounded linear operator A.

So, special case when A is itself a bounded linear operator, then I show that A Lambda converges to A in the operator norm. Why did we choose a bonded operator because otherwise this makes no sense, I mean so, if A itself has infinite norm, there is no way one can show that the sequence of operator converges to A in the operator norm.

So, it is a bounded linear operator A for that we show that A Lambda defining this fashion converges to A as Lambda tends to infinity. The proof is very simple, as A Lambda it is defined by Lambda times A times R Lambda A here, and R Lambda A is Lambda I minus A inverse that is also written here and then you multiply Lambda inside.

So, then you get identity minus 1 over Lambda A inverse because, when you put Lambda inside you get 1 over Lambda actually multiplied inside. So, here let Lambda becomes Lambda divided by Lambda is just 1 and here you get 1 over Lambda, Lambda A inverse and then this I minus 1 over Lambda A inverse this I can write down as a series expansion, why should I be able to do that, see the because Lambda we are considering sufficiently large where Lambda tends to infinity.

So, I can always choose one Lambda larger than the norm of A, then this would be like we know that whenever I have an operator whose norm is less than 1 so, I minus that operator's inverse can be written as expansion of I plus B plus B square plus B cube, etc.

So, we do that we write down that I plus you know 1 over Lambda A plus 1 over Lambda square A square, etc and then this A was already there outside so we multiply A also inside so I becomes A here so, we get A plus A by Lambda because A square by Lambda plus A cube by Lambda square, etc.

Now, this if we take norm of these terms these are decreasing like geometric progression GP series because A by Lambda is less than 1 and the square etc, cube etc those terms appear there. So, so that converges and then as Lambda tends to infinity this term goes to 0. So, all these term would go to 0 as Lambda tends to infinity so, what we would be left with is just capital A, or in other words, A Lambda converges to A as Lambda tends to infinity.

However, I mean this proof is pretty simple and intuitive, but in the actual case, I mean this Yosida approximation works even if A is not a bounded linear operator, but then this approximation is surely not in this sense, not in operator norm sense, but in the point wise sense.

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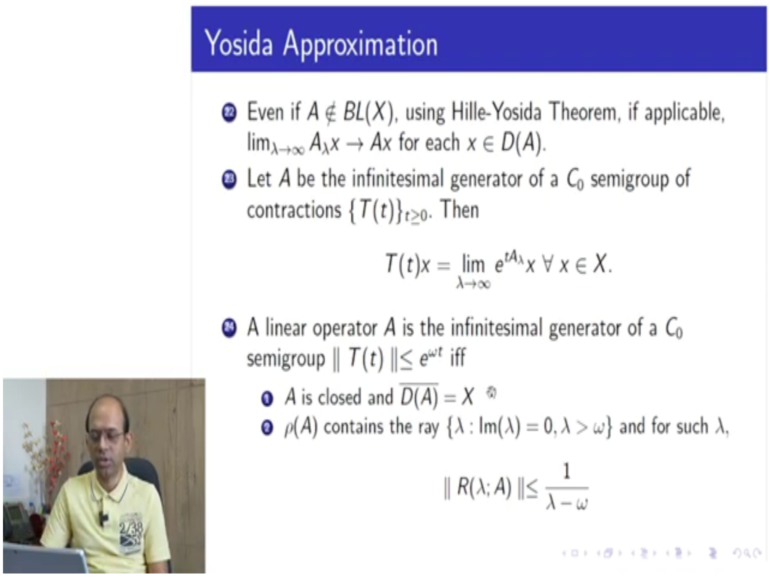

So, clearly state that even if A is not in B L X is not a bounded linear operator on the banach space X using Hille-Yosida theorem, if applicable. Why applicable, because Hile-Yosida theorem requires some sufficient conditions on the operator A which is applicable then, limit Lambda tends to infinity A Lambda x converges to, so I should have written equal to here because I have already limit here.

So, equal to A x for each x in the domain of A is the best what we can expect because I cannot expect that I would get this equality for each and every x in the banach space capital X because this right hand side may not be meaningful because A may not be defined on the whole banach space.

So, wherever it is defined there we would be able to get this convergence. So, that is the success Yosida approximation. Now, this result is used also in the Hille-Yosida theorem. So, let A be the infinitesimal generator of a C 0 semi group so, if it is so of semi group of contractions then this A Lambda the way we have defined e to the power t A Lambda I can always get because this is also an uniformly continuous semigroup generated by A Lambda and then e to the power t A Lambda x that converges this to T t x.

So, this is the part which says that the approximated semigroup is also a good approximation of the semigroup generated by the operator capital A. So, for each and every x in the banach space capital X one gets this convergence.

So, a linear operator A is the so this is another result so, we are not going to see the detail proof of these results, we are just stating these results. A linear operator A is the infinitesimal generator of a C 0 semigroup T t which satisfies the norm of T t is less than or equals to e to the power Omega t if and only if some conditions, that we are going to see.

Here, I like to before showing these conditions I would like to mention that this is an extension of Hille-Yoshida theorem because in Hille-Yoshida theorem your semigroup was you are looking for only in the class of contractions. So, norm of T t is less than or equals to 1 bounded, but here we are asking the linear, when A is given possibly unbounded linear operator and then under what conditions on A it generates a C 0 semigroup which is satisfied this condition that less than equals to e to the power Omega t so is a larger class.

So, this is if and only if the condition is that as before A is a closed linear operator. So, closed linear operator means the graph of A is a closed subset in the product space, so A is closed and A is densely defined. The closer of the domain of A is the full space X. Second is that the resolvent operator Rho A since you know this we are talking about the broader class. So, I have to deal with a relaxed condition here.

So, Rho of A contains the ray, which ray that the set of all Lambda in the complex plain such that imaginary part is 0. So, that means these are all real Lambda and Lambda is greater than Omega. So, Omega onward more than Omega. So, Rho of A contains so Omega onwards that ray for such Lambda that is the for Lambda which is in this class Lambda greater than Omega, the norm of R Lambda A.

So, if we choose this Lambda from this set, the norm of R Lambda A is finite, so this is a bounded linear operator not only that that is less than or equal to 1 over Lambda minus Omega. So, this is an extension of Hille-Yoshida theorem for contraction. So, generally when we say Hille-Yoshida theorem we also know sometimes refer this whole collection of theorems.

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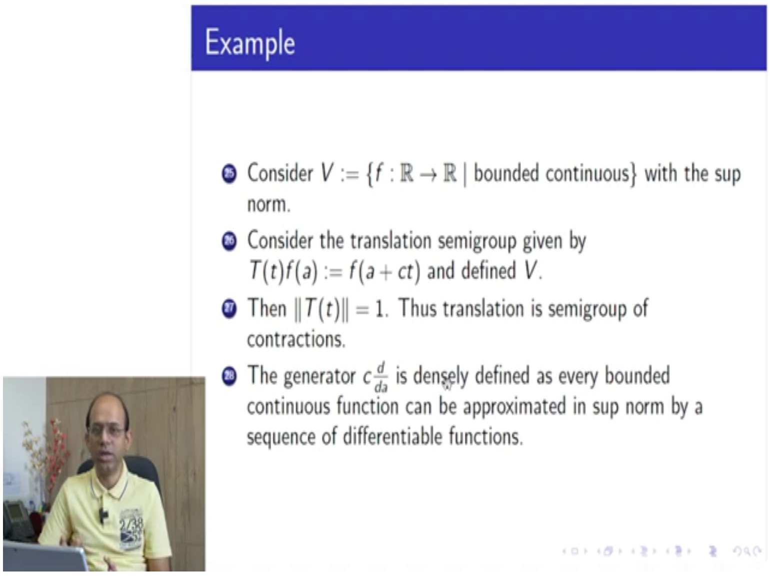

Now, consider so we need to talk about little bit about some examples. Otherwise things are becoming a little more abstract. So, consider the space V is a set of all functions f real valued functions R to R which are bounded continuous with the sup norm. Consider the translation semigroup given by T t of f a defined as f of a plus c times t, so this is a translation semigroup we have already seen the definition.

So, this is defined on V defined so, then for this particular semigroup we can ask that is it a contraction? Yes it is, we can show that norm of T t is equal to 1, how do you show that? See, we would consider f from I mean with norm of f is equal to 1 like you unit sphere of these V. So, that means, where supremum of f is equal to 1.

So, in that case T t of f if we take the sup norm of that, that will be same as norm of sup norm of f of a plus c t because when we take sup norm we range A over all possible real number so that anyways covers a plus c t also, it does not matter what t you have chosen. So, sup norm of T t of f is same as f of a plus c t norm of f. So, operator norm would be just one because norm of f if I am choosing on the unit sphere of the banach space that norm of f I am choosing 1, so I get 1 here.

So, thus the translation semigroup is a semigroup of contractions. Now the generator, generator of the translations semigroup we have already seen earlier is constant time the derivative. So, this derivative operator is surely not defined on the whole space V, why because this is space bounded continuous functions. So, all the members need not be differentiable. So, this derivative is not defined on the whole space V. So, this operator is defined only on the set of all functions which are differentiable.

However, this operator is densely defined. Why, because every bounded continuous function can be approximated in supremum by a sequence of differentiable functions. Even if we have a very sharp points where it is not differentiable we can have a smooth version there and can keep it very close to sup norm, I can decrease to 0 as the sequencing progresses.

So, that is the comment here that for this particular example, this is the way we understand that the operators are need not to be defined everywhere but is densely defined. So, this justifies the conditions what we are talking about here and the concerns what we are keeping in our mind, the relevance of those.

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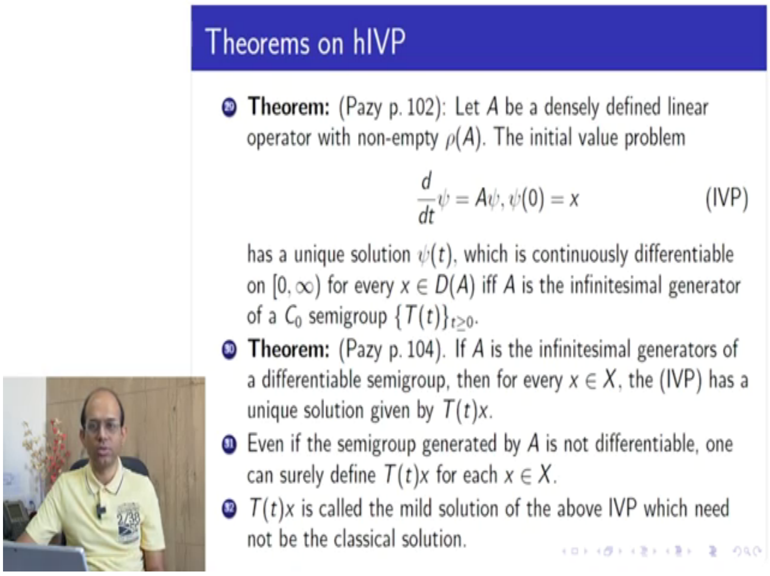

Next we come to theorems on Homogeneous initial value problem. So, these two theorems are taken from the A Pazzs book. So, I have written down the page numbers of this theorem in the book so that the curious students who would like to see the detail proof of this theorem can refer to this.

So, let A be a densely defined linear operator with non-empty resolvent set, the initial value problem. d dt Psi is equal to A Psi and Psi of 0 is equal to x that IVP we call has a unique solution Psi t which is continuously differentiable on close 0 to infinity for every x in D of A if and only if, A is infinitesimal generator of a C 0 semi group T t.

So, what is the difference between this theorem and the theorem what we have already quoted, the difference is in earlier theorem what we have seen that if A this operator A generates a C 0 semigroup then that constitutes the solution of this initial value problem. But this theorem says the other side also is if and only if thing that this equation has a unique solution Psi t which is continuously differentiable basically it is talking about the classical solution for every x in the domain of A if and only if A is an infinitesimal generator of C 0 semi group. So, that means that if A not an infinitesimal generator of a C 0 semi group, this equation does not have a classical solution also for x in from the domain of A.

So, remember that here for this theorem we really require x to be in the domain of A, if x is not in the domain of A, T t x does not solve it, it is considered just as a mild solution, not a classical solution. So, I go to the next theorem. So, this appears in number 104, if A is the infinitesimal generator of a differentiable semi group.

So, here we are putting more condition on the semigroup, a differentiable semigroup so, that means it is a very restricted class of operators A, then for every x in capital X, since we are choosing only a very special type of operator A, so we can get a nice result that is that for every small x in the whole banach space capital X, the IVP has a unique solution given by T t x.

So, this is as expected that of course, if we select from a smaller class we can get a better result, but this is the theorem. So, what is differential semigroup that means T t x that is the differentiable. Even if the semigroup generated by A is not differentiable, so then we cannot say that T t x is the solution, but one can surely define T t x even if A is not differentiable, one can still define T t x then that T t x would not be a classical solution, but we call that time T t x is a mild solution of the above initial value problem, which need not be the classical solution. Thank you very much.