Introduction to Probabilistic Methods in PDE
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Lecture 55
Homogeneous initial value problem

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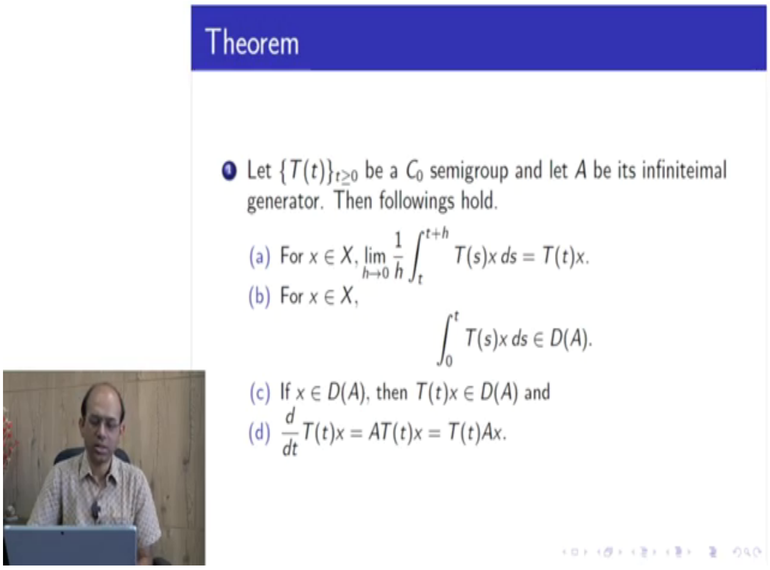

So, we will see another theorem today that let us take capital T, this semigroup be a C 0 semigroup and let A be its infinitesimal generator then following hold. For any x in capital X, capital X is the banach space. So, capital Tt is a C 0 semigroup on this banach capital X. So, for any small x in capital X limit h tends to 0, 1 over h integrations small t to t plus h capital T of s x d s is equal to capital T of small t x.

So, this criteria is I mean this property is immediate actually because of the continuity, if this is a continuous function and that is true because it is a C 0 semigroup. So, if it is continuous then of course, if you integrate T to t plus h and then that integration you divide by h and take limit h tends to 0, you would get a value at t so, that is actually immediate.

Now, property b, for any small x in capital X, this is integration 0 to small t T of s x d s this integration. So, this would be a vector point in capital X because small x is there, T s is the linear operator, you are getting a vector in capital X on that you are doing this integrations 0 to small t so, it will be actually a vector in this.

So, but this statement is saying that something more, it is saying that this vector is not only in x, it is also in the domain of A, why, because domain of A could be a subset of capital X, need not be everywhere. So, this is saying that after integrating the application of the semigroup on a vector, whatever vector is there in capital X.

And then if you integrate some, then that integrated one whatever you get, that would lie inside the subspace of the domain of the generator. So, this needs I mean this is not obvious clearly, is one can think that since integration is there possibly some kind of sort of smoothness has occurred so we are getting some sort of regularity.

Indeed that is the kind of argument we have, I mean that is the kind of reason, but we need to prove it in systematically it is in the domain of operator A. Now, this result is saying that if small x vector is in the domain of A that means, it is not anywhere in capital X but actually in the subspace of that the domain of A if it is there.

So, then we can say that capital T t of x is also in the domain of A. I mean this is something which is not true otherwise, when small x is not in the domain of A but in capital X, we cannot conclude this. However, small x in the domain of A, then application of the semigroup on small x is in the domain of A.

So, one can compare these two statements, here in this statement b, we got that for any small x in capital X, the application of the semigroup on the vector small x, but integrated on any interval 0 to t, it does not matter what t is, it is arbitrarily small or large, whatever, then that would lie in the domain of A.

But if we want something more that we want, not the integration but just the T t of x to be in the domain of A, then we need to assume that x is in D of A. And so if x is in D of A T t x is D of A, not only that we can also conclude that the derivative of T t x because you know T t x is in domain of A.

So, we can talk about the derivative d dt derivative of T t x that would be equal to A T t x, I can talk about AT t x because T t x in is in domain of A since T t x is in the domain of the operator A. So, A of T t x is meaningful. So, this is equal to A T t x, and not only that, the value is same as this vector is same as T t of A x. This is also meaningful because small x is in the domain of A, so A x makes sense, so we get this thing.

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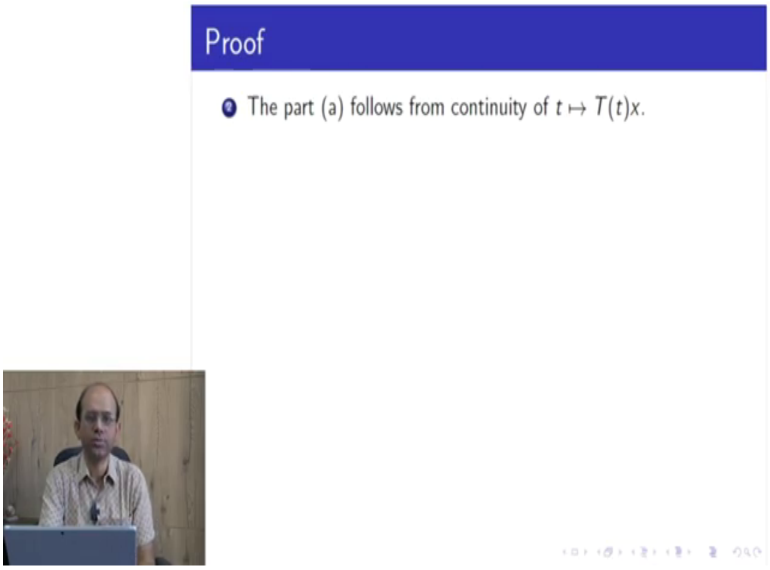

Next, we start the proof the part A, so as I have earlier mentioned this part A is easy. This is trivial actually, this is followed by the C 0 semi group property because this is a continuous map for any continuous map this is true, so this follows from continuity of this map small t to capital T t of x.

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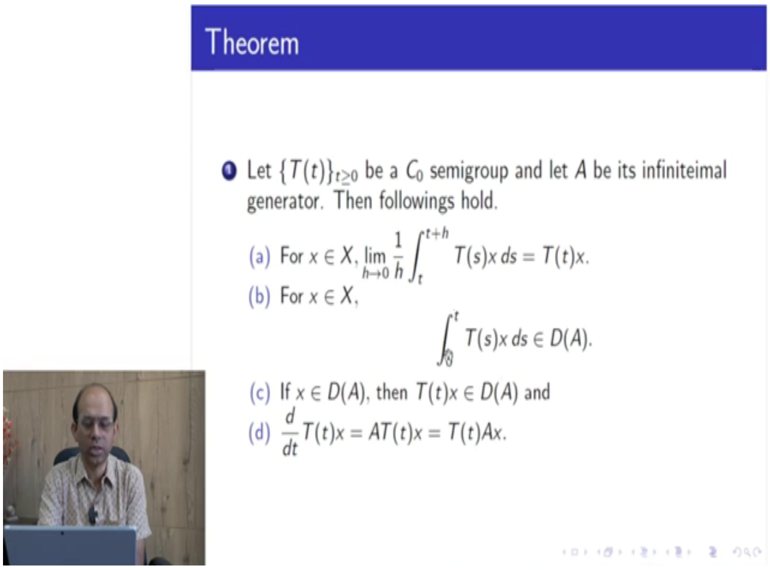

Now for part b, we need to show that this whole thing is in the domain of the operator. To show that this thing is in the domain of operator, we need to apply this T h minus I divided by h that operator and we have to show that that limit exists, that is the way to show something is in the domain of operator so, we do that.

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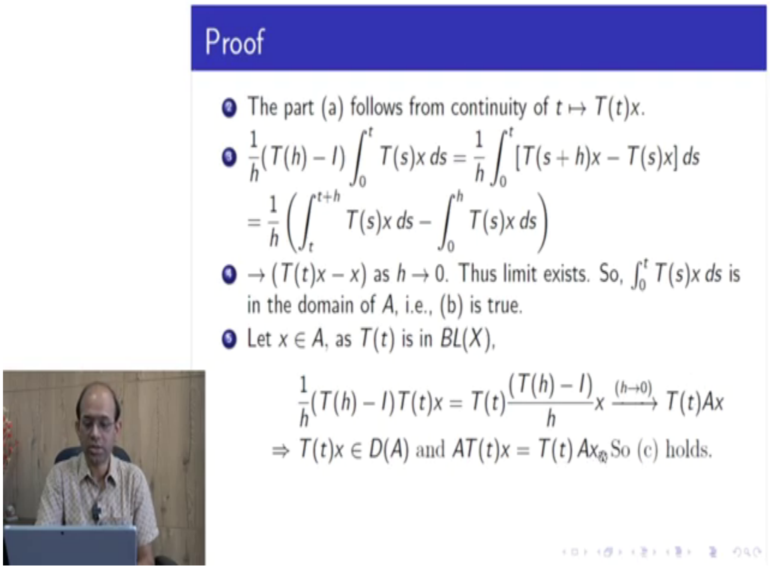

We take this fraction **one** over h T of h minus identity. So, this fraction, so this operator we apply on this vector 0 to small t T s x ds and then how after applying this what do we get is that this 1 over h T of s plus h, this s is there, h is there, the composition of T s and T h is T of s plus h x minus identity is minus T s.

So, this operator composed with T s gives me T of s plus h x minus T s x ds. So, from here now I can write down this as two different integration, one is from say s plus h x ds. So, here I can do substitution of variable to make it h 2 T plus h s T s, another is 0 to t. So, when h to t plus h that we can write down as 0 to t plus h by adding subtracting 0 to h this thing, 0 to h T s x ds. So, if this integration 0 to h if I add here and then subtract, I get 0 to t plus h and here it is 0 to t. So, that difference would give me t to t plus h T s x ds.

So, the way to get this is that write down these difference, as you know two, difference two different integrations and then subtract, those integration and then you get into 1 of over h t to t plus h T s x d s minus 0 to h T s x d s. So here, what we have obtained is that a very small time intervals sub interval t to t plus h, here also time intervals 0 to h, but 1 over h is also there.

So, now we apply the first part a. So, part a so let us say this is continuous so this limit converges to T t of x. So, this converges to T t of x and this converges to T 0 of x but T 0 is identity so it converges to x. So, we get that to this thing converges to T t of x minus x as h tends to 0.

So, what did you get? We have obtained that this limit exists as h tends to 0 this limit exists. So, since this limit exists that means, this and from the definition of the domain of operator or the generator of the semigroup we know that this vector is in the domain of the infinitesimal generator. So, integration 0 to small t T s x ds is in the domain of A so that is the Proof of b part.

Next, we see the proof of Part c, let small x is in that so this is short notation, I mean small x is in the domain of A. So, like dom of A, domain of the operate A as T t is in B L X that is you know bounded linear operator of X, so we can get the following thing that here c, in the C part let us recall what we are going to prove. We are going to prove that if x is in the domain of A, then T t of x is also in the domain of A.

So, we need to apply that fraction here and need to show that the limit exists. So, we do that. So, **one** over h T h minus I exactly as we have done before. 1 of over h T h minus identity of T t to x, we need to show that this limit exists as h tends to 0. So, this thing I can rewrite as T of t because you know there they commute these operators, T of t times T of h minus I divided by h x.

So, now, since x is in the domain of A, so, this limit exists as h tends to 0, so this limit goes to A x and here T t is bounded linear operator, if this goes to A x T t of this thing would go to T t of A x due to continuity of T, so as h tends to 0, this thing goes to T t of A x. So, what does it imply, it implies that T t x this thing, this you know this whole limit converges so that means T t of x is in the domain of A. And not only that, here that when this limit exists so, this left hand side is nothing but A of T t x, so A of T t x is equal to right hand side that is T t of A x.

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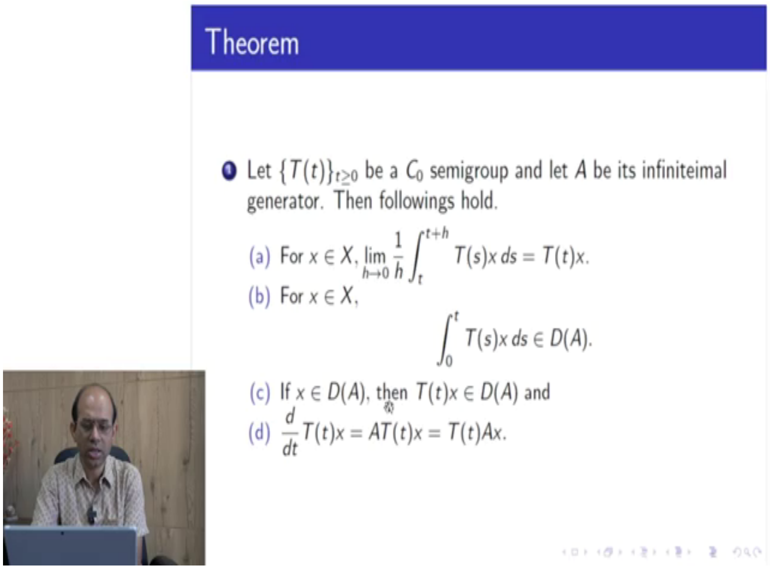

So, that is the condition of this that if x belongs to D A then T t x belongs to D A and in the d part also like you know A T t x is going to T t x so, this part also partly you have proved however, we have not proved that that is actually derivative. So, if we can prove only one part of this, other part will be obvious why because already have shown that these two things are equal.

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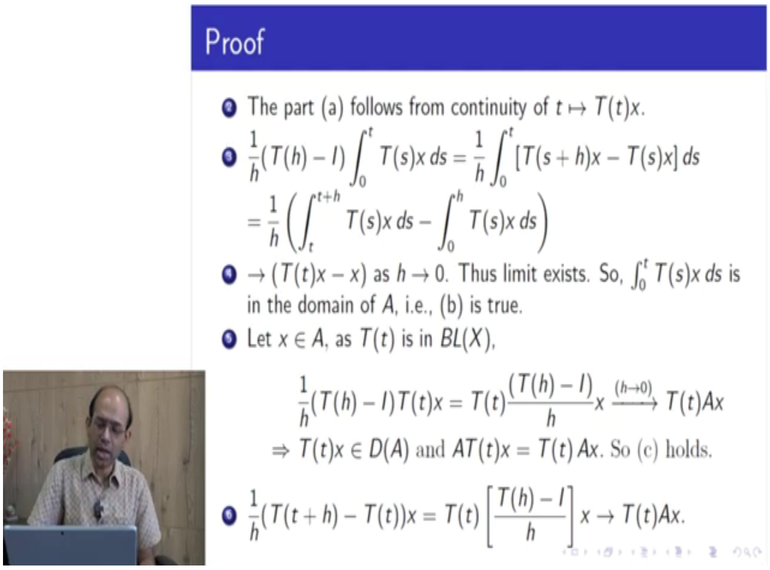

So, the A T t x is equal to T t A x. Now, for the d part what we do is that we take 1 over h T of t plus h minus T t x. Why do we do that? Because we need to show it is differentiable T t x is differentiable, to show it is differentiable we use the first principle. The first principle we take to show some function is differentiable we need to take the rate of change.

So, here we take first positive side, right hand derivative, so t plus h, h is positive here. So, capital T of t plus h minus T t of x divided by h. Now, here since capital T is a semigroup so capital T of t plus h is equal to capital T t composition T h. So, we can take capital T of t common. So, after taking capital T of t what we are left with is capital T of h from here, and here I would be left with only identity matrix identity operator.

So, after taking T of t common we have T of h minus identity divided by h. However, since x is in the domain of A, so this limit exists and this converges to A x, and T t is a bounded linear operator that means it is continuous. So, this thing would converge to T t of A x. So, what we have obtained is that this thing converges to the T t of A x this limit converges, since this limit converges we cannot straight away say that this is differentiable because this is right hand side limit h is essentially taken as positive.

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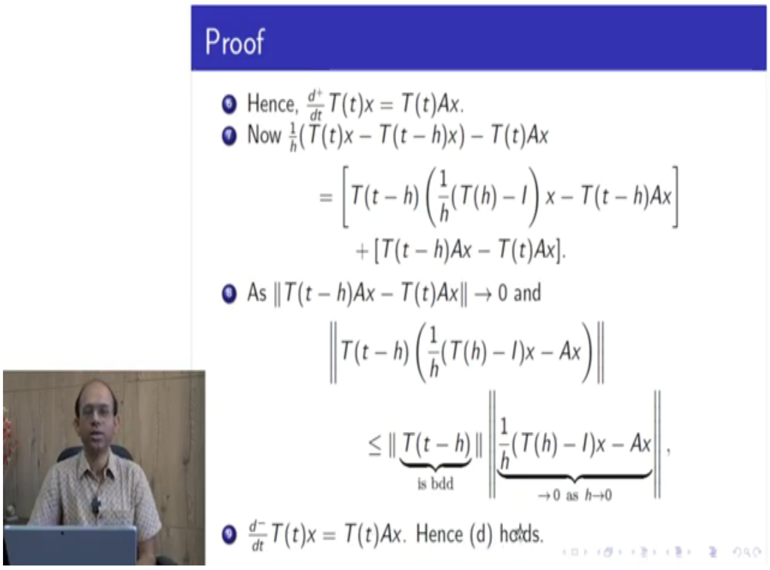

So, we can conclude that the right hand side derivative d dt plus of T t x exists and that is equal to T t of A x so T t A x. Now, we should consider the left hand side derivative also. So, we do that, so for left hand side derivative what we do is that **one** over h T t of x minus T of t minus h x for any positive small t we write on this.

Of course, small t cannot be less than h or 0, then it would be negative, but that is not defined here. So this is, I mean written only for small t, which is h or more. So, does not matter I mean, one should view it in the following manner, you consider any positive small t and for that positive small t, you find out one h, which is even smaller than that, and then consider this difference.

Make sense? Good. So, 1 over h of this T t x minus T of t minus h x minus T t of A x. So, what do we need to do, we need to show that as limit h tends to 0 this thing goes to T t of A x. So, for that what we do we just take the difference and want to show that this whole thing goes to 0.

So, this difference is rewritten as, so, here we take common T t minus h, so, since we have small t here, so, small t is t minus h plus h, so t minus h we take common and then we are using semigroup property. So, capital T of t minus h taken common, so, here we have capital T of h here and then minus identity from here and then whole thing.

So, this there is a typo here. So, this 1 over h should have been outside here and this is an extra bracket here. So, this whole thing is that fraction what we are looking for, times x minus capital. So, here I had T t of A x, instead of writing T t of A x I add and subtract this quantity that is T of t minus h A x so here T minus, T of t, but I am writing T of t minus h. Since I have subtracted it, I must add also, so I add here T of t minus h A x. So, the same thing I have subtracted and then added and then I have this minus T t of A x. So, I have from these I have these two differences.

Now, we see that, this difference, capital T of t minus h A x minus capital T t of A x. So this difference so the norm of this goes to 0, why is it so, because x is in the domain of A, so A x is a member in the banach space and then this member is fixed here both the places and here capital T is a C 0 semi group. So, as h tends to 0, so, this thing converge to here. So, norm of these things goes to 0.

So, here from this we know that this part this norm this part would be 0. So, now norm if we apply all the sides, so norm of this thing is less than or equals to norm and then use the triangle inequality norm of this plus norm of this. So, this part goes to 0, just I need to show that this norm also goes to 0.

So, let us look at this capital T of t minus h here and here. Here it is correctly written, capital T h minus identity bracket times h minus here I have taken T of t minus h common, so I have only A x here, so from left hand side we have taken common T of t minus h both sides, so I have this thing.

So, now norm of composition of operators is less than or equal to product of the norms. So, the using that we write down this is less than or equal to norm of capital T of t minus h times norm of this quantity, norm of 1 over h T h minus identity x divided by h minus A x. However, x is in the domain of A that means, as limit h tends to 0 this thing converges and converges to A x and A x is here so this converges to here so, that means this whole thing converges to 0 as h tends to 0 in this norm.

So, this is not the operator norm, this is the banach space norm. So, in this norm this goes to 0, so that was our goal. So now, we know these terms norm of this term also goes to 0, norm of this term also goes to 0, so norm of this term also goes to 0 or in other words that this is the left hand side derivative, so d dt minus of T t x.

So that exists because that limit converges and where does it converge this derivative? This is the answer T t of A x. So, right hand side derivative also exists and that is T t x, left hand side exists that is T t A x. So, the derivative exists so hence d holds. So, that is the end of the proof of the theorem.

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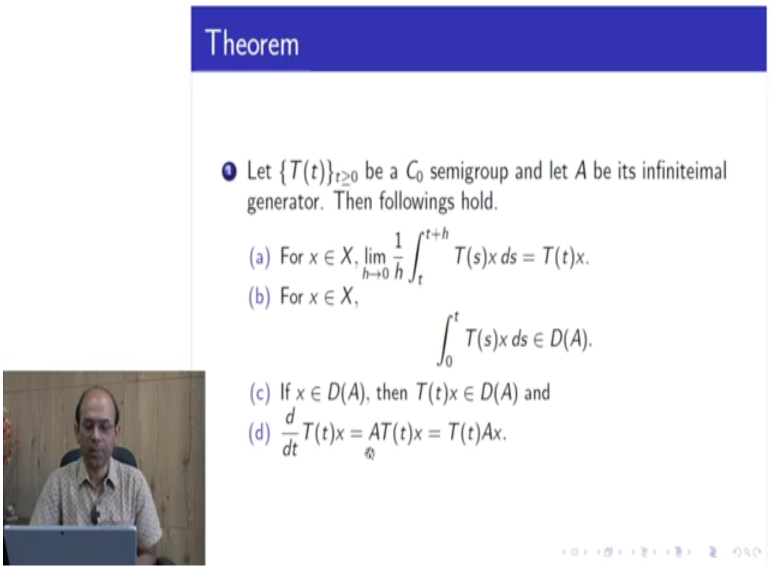

So, let us look at the theorem again. So, this is the end of the theorem. And this, the importance of the theorem would be now clear, because here what we have done is that we have obtained one Cauchy problem actually. So imagine that I do not know what is here it is unknown I have just a Cauchy problem d dt of something is equal to A times of same thing with T t x is here imagine that T t x is unknown.

So, then it is a Cauchy problem, given the operator A. Now, the goal is that find out the infinitesimal generator I mean identify capital A as an infinitesimal generator of a C 0 semi group, if you can then that semigroup acting on x would be the solution of that problem.

And then if your initial condition is also here that that T 0 of x, T 0 of x is x exactly and that is in the domain of A. So, if in this Cauchy problem the initial condition is also that initial vector is not anywhere in the banach space but in the domain of A then this would help us to solve one abstract Cauchy problem, so that is the importance of this theorem.

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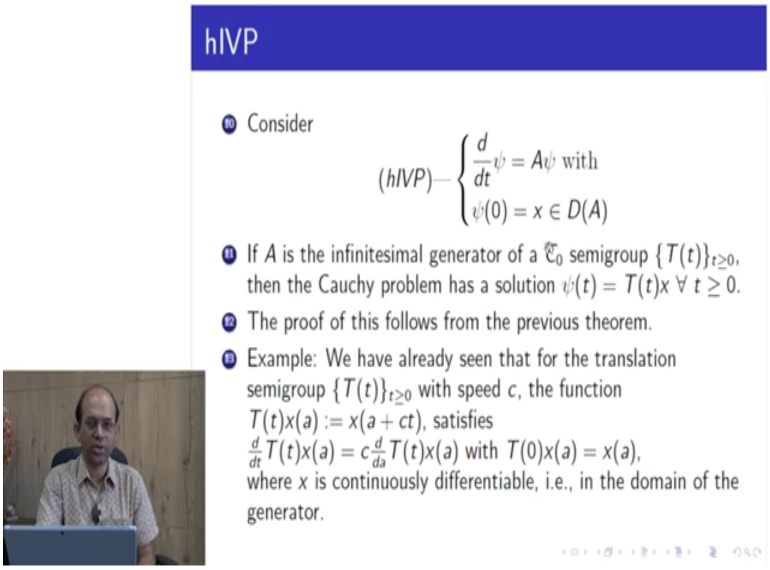

So, we summarize that thing here in this slide. So, consider a homogeneous initial value problem. We often call this as a Cauchy problem, sometimes we call this initial value problem, but it is important that we say that is homogeneous, why homogeneous because the unknown appears here and there is so it is a linear function, it is a linear equation and there is no other source term. So, this is the main reason that why are we consider this semigroup theory approach. Why, I mean, because you know, the in the whole course, we were looking at some partial differential equations and we are trying to solve that using probabilistic method.

In this part of semigroup theory we are not using any probability theory. However, we are using, I mean the semi groups and in the probability theory part what we have seen that we have used the semigroup generated by some Markov process or Brownian motion. So, using those we have solved some partial differential equations.

So, here in this part, we are considering the abstract Cauchy problem where the generator A need not be half times Laplacian because if it was half times Laplacian, then it was like the parabolic heat equation, probably PDE which you have solved, and for the semigroup I should look at the semigroup which is coming from the Brownian motion.

But in this section we are considering any arbitrary type of operator. So, this is abstraction of the things what we have already discussed for more general setting. So, this is really important for such applications because when we encounter new research problems, it is not necessarily the Cauchy problem, what we will be dealing with that the operator that is just you know, one of the half times Laplacian or some very standard operator, but it could be very complicated, but then the approach is that, can we identify that operator as you know, infinitesimal generator of some C 0 semigroup.

So, here this is a statement that if you consider this homogeneous initial value problem, so, if A is the infinitesimal generator of a C 0 semi group T then the Cauchy problem has a solution and the solution Psi of t is capital T of t of x what is the x, x is the initial condition. So, given a starting the x so this is the solution.

So, solution is also very you know easily written here however, the problem happens that most of the time the initial condition is not so good that means that it may not be in the domain of the operator. So, that is the kind of discussion we would be dealing with in the rest of the part of the course.

And here to understand this thing, let us recall another example, which we have seen just a few lectures back. I am not to giving the proof of this, because essentially I have proved this part actually in the earlier theorem is just the rewritten part of this or you can say is a very trivial corollary that follows from the previous theorem.

So, we look at this example that we have already seen that for the translation semigroup capital T with speed c, the function where T t of x a was defined as x of a plus c t. So, this, if this is the semi group, so, here what is x, x is a function of real number. And so, the banach space what you will choose is here, possibly we can choose continuous functions, but however, we do not do that we take, you can first take that is the continuous functions, but then the generator the infinitesimal generator of this semigroup turns out to be a differential operator. So, that is not defined on the whole banach space.

So, there we need to take a smaller subset here. So, here what we have already seen that the derivative d dt of T t of x a is equal to c times d da T t of x a. So, here this derivative is with respect to this variable small a, small a is real number and x is what, x is the function. So, here with T of 0, T of 0 is identity operator T of 0 x a is equal to x a itself.

So, this is a function, x function, so where x is now if we choose x in the domain of the generator so the generator is this differential operator. So, if x is differentiable, if x is continuously differentiable, that is in the domain of the generator. So, then already you know that this d dt of this T t x a, which is unknown here, we can imagine that way and then that equation is satisfied by this solution this x or a plus c t.

So, that is the solution of this Cauchy problem, where this is actually a partial differential equation of first order linear partial differential equation. So, here this function is a function of t and a variable two variables. So, here we have partial derivative with respect to t, here I did not write down partial I mean like del del operator notation, why, because this was understood as d dt of x, so, in a space of vectors and then evaluated that at a so that way otherwise if I would like to and here d dt x a, so if I now write down this as a function of two variables t and a together, then I should write down this as a partial derivative with respect to t and partial derivative with respect to a here.

So, that is anyways translation PDE and for the solution is that x of a plus c t is a solution of this equation, where I mean t is running from 0 to infinity. So, there is a discussion I wanted to make, so that with this example, we are recollecting that already we have seen some glimpse of such equations and we have seen solution of this equation.

And so here, only for a general case we have proved this theorem that whenever we have an operator A which we can identify as infinitesimal generator of a C 0 semigroup and if in that case the initial vector is also in the domain of this operator A, then the solution exists and the solution is written as T t of x, so let me conclude this lecture here. Thank you very much.