Introduction to Probabilistic Methods in PDE
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Lecture 54
Unique semigroup generated by a bounded linear operator

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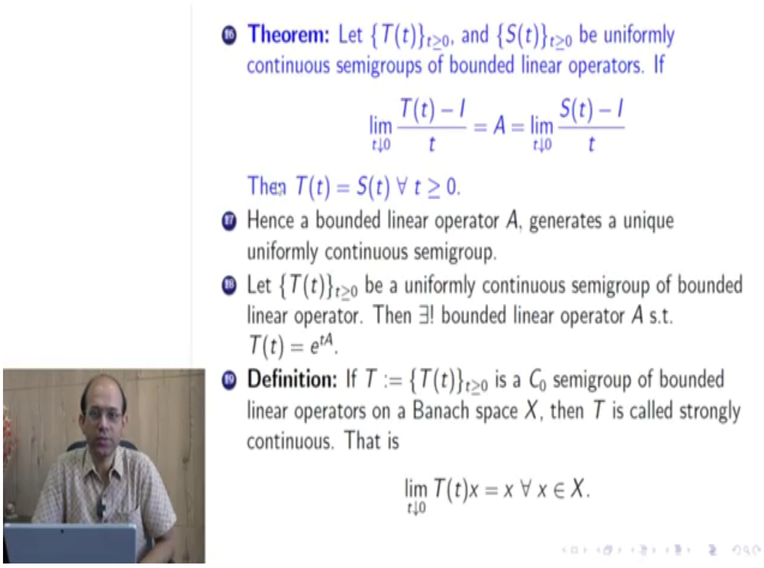

In the previous class, we have seen this theorem that when one has one bounded linear operator A then that operator A can generate only one single uniformly continuous semigroup. So, here this is the theorem which we have already seen in the last lecture. So, a bounded linear operator A generates a unique uniformly continuous semigroup. So, this is the statement of the theorem. So, let us prove this theorem, we have not proved this theorem earlier, we have just eated now, today we are going to prove this.

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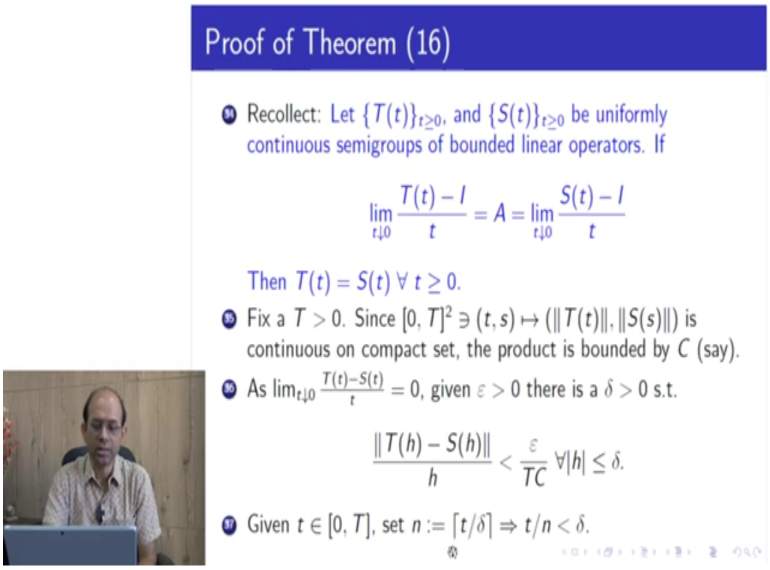

So, you recollect the statement of the theorem, let T t and S t be uniformly continuous semi group of bounded linear operators, if limit t tends to 0, T of t minus identity divided by t is equal to A and that is the same as this limit small t tends to 0 capital S of t minus identity divided by t and then we can conclude that T of t is equal to S of t for all t non-negative.

So, this is the statement of the theorem for proving this instead of considering the whole time horizon, we would consider first only a finite time horizon that means only from 0 to some capital T. So, we take finite time horizon, but this we have fixed arbitrarily and then at the end we would see that we can get rid of this finite capital T.

So, at present we fix one capital on positive capital T and then we take the unit, this closed square that is 0 to capital T square, the Cartesian product of the closed interval 0 to T. Now, given this closed square, we take a point small t and s in this and look at this map. So, this map is sending this norm of capital T of t and S to norm of capital S of s. So, this is order pair and we are getting another ordered pair.

Since, capital T and S are uniformly continuous semigroup so, in the norm they are continuous. So, these norms are continuous so this functions small t, comma s to this thing so, this map is a continuous map. So, now what do we have? We have a continuous map defined on a compact set, this compact set is 0 to capital T whole square.

So, a continuous map on a compact set is bounded. So, what we can conclude that if you take product of these two coordinates, the norm of T t and norm of S s so, then that function of two variables is a bounded map on 0 to capital T whole square and we named that upper bound to be capital C.

So, the product is bounded by C say for example, so we call this upper bound as C. So, this is the main part where we really require finite capital T, so this capital T should not be confused with this capital T. So, this capital T is a semigroup and this capital T always comes with a parenthesis of small t, but this is just one scalar just a positive real number. Next, what we do is that we subtract both the sides.

Since we already know that this limit left limit and right limit, these two limits are having the same things, so A is this limit of both these things. So, since both the limit exists, so if we subtract that limit would also exists and that will be 0. So if I subtract I would get T t minus S t so, we do that.

So, T t minus S t divided by t that goes to 0 as T tends to 0. So, from these we can write down or rewrite this limits statement in terms of Epsilon delta definition. So, given Epsilon positive, there exists a Delta positive such that norm of T h minus S h divided by h that is less than Epsilon by T C for all mod h less than or equals delta. So, here just renaming the variables instead of small t I am writing h.

So, since this goes to 0, so, given epsilon positive so instead of Epsilon, I am writing on the right hand side Epsilon divided by capital T which is already fixed and by fixing capital T we have already fixed what is capital C so, capital T and capital C is known to us. So, given any Epsilon, Epsilon by capital T by C is also a positive real number.

So, for that we can find out one Delta positive such that this fraction norm of T h minus S h divided by h is less than Epsilon by T C, for all mod h less than equals to delta, this relation we are going to use towards the end of the proof. So, now for the time being we go back to the main goal. Here, we would like to prove that T of t is equal S of t, if you need to prove that these two are equal, what do we need to do, we need to show that their difference is 0 for every t.

So, we take any t between 0 to capital T. So, remember that we are not taking all positive number real time we are taking only finite time horizon because 0 to capital T in that horizon only. So, we take a time small t in 0 to capital T and after fixing this small t we can find out one n which is ceiling function of t by delta, which Delta? This delta, t by delta ceiling function t by delta.

So, this n is dependent on small t, small n depends on time t. So, this would also imply that t by n is less than delta, less than or equals to delta because this is the way it is defined that n is I mean ceiling of t by delta so, t by n is less than or equals to delta. So, I should have use equal, less than or equals to here.

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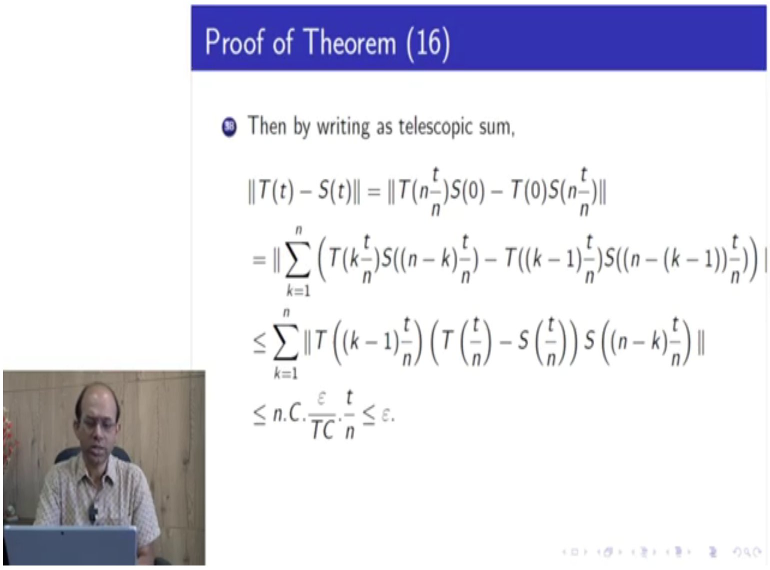

So now, what do, we do we look at this difference, norm of T t minus S t, we need to show that this is 0 indeed actually. So, now for showing that what we do is that we are going to rewrite this, rewrite using some kind of telescopic sum. So, what we do is that we rewrite capital T of t is equal to capital T of n multiply and divide by n, T of n t by n and then there is nothing so, I can think this is the identity operator so, S 0 I write where S 0 is also identity, S is a semigroup and then here also identity times S of t instead of identity I write T of 0 and S of t is rewritten as S of n t by n, just multiply and divided by n.

So, this thing is rewritten in this way and now here so, this difference I would like to write down as telescopic sum of many small differences. So, what we do is that we are going to vary this n, n minus 1, n minus 2, etc, till this would be like 0 and here I would also increase from 0 to up to n times t by n.

So, this is the main idea of this simplification so, here we write capital T of k t by n. So, imagine k is equal to n, when k is equal to n it is T of n t by n and n t by n is t so this thing, and when k is equal to n, n minus n is 0 so it is like this 0, S of 0 times t by n, this is 0 so this thing when k is equal to n so this term is like this.

And from this term, we subtract with the term here k is replaced by k minus 1. Observe here carefully, T of k t by n and here I have T of k minus 1 times t by n. Here I have S times n minus k times t by n, here S of n minus k minus 1 times t by n. So, here k is replaced by k minus 1 so, that is why I call this as a telescopic sum.

And then the last term here what would survive the discussion from here, where k is equal to 1 would occur because this is from the first term where k is equal to n occurs. So, this should appear actually when k is equal to 1 occurs, this term. So, let us cross verify here when k is equal to 1 happens then T of 1 minus 1 is 0, so T 0 and then S of n minus 1 minus 1, 1 minus 1 is 0 S of n.

So, it is clear that this difference is written as this telescopic sum of small differences. So, this norm is outside, here also there is a norm. So, now norm of sum is less than or equal to sum of norms. So, use this triangle inequality to write down that this summation k is equal to 1 to n norm of capital T of.

So, here what we do, this part we do little more simplification, what we do here that T of k t by n is rewritten as t of k minus 1 times t by n times capital T of t n. Why do we do this? Because we would like to take common. So, this part is already present here, and here this part this time is little more, so it is T k minus 1 by times t by n, here k times t by n. So, here we can use the semi group property of capital T and that is that T of k t by n is same as k minus 1 t by n plus I mean composition with capital T of only t by n.

So, we do that so, from here we take common of capital T k minus 1 t by n so what remain is without k only T by n so that I am writing here capital T of t by n. And from right hand side I have here S n minus k t by n, here I have S n minus k and minus minus is plus, plus 1. S n minus k plus 1 t by n.

So, if I take this common, S n minus k times t by n so then I can write down capital T of t by n here and from here since I have already taken common here this term, here I would get nothing identity however, here I have S of n minus k plus 1 but I have taken common S minus k. So, plus 1 that would remain here, so, this 1 t by n so that appears here, so capital T of t by n minus capital S of t by n.

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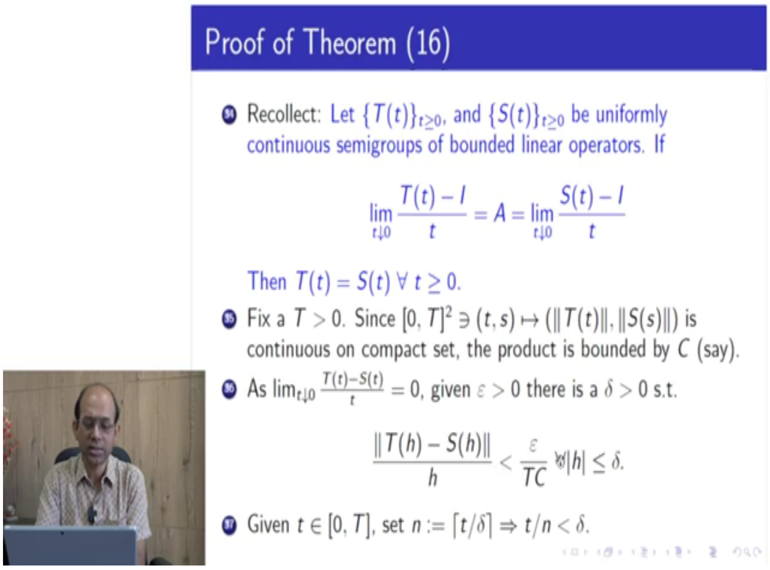

So, this simplified terms appear here. So now, I think you will be able to understand that why I am doing all this here because we got some difference, and the time interval is small enough less than or equals to delta. So, here say t by n is less than or equals to delta is there, and since that is the case, so instead of h if you have t by n, so you can apply this inequality capital T of now we read like this, capital T of t by n minus capital S of t by n divided by t by n is less than or equals to Epsilon by T times C. So, now this small t by n I can take on this right hand side because that is non-negative number, positive number here.

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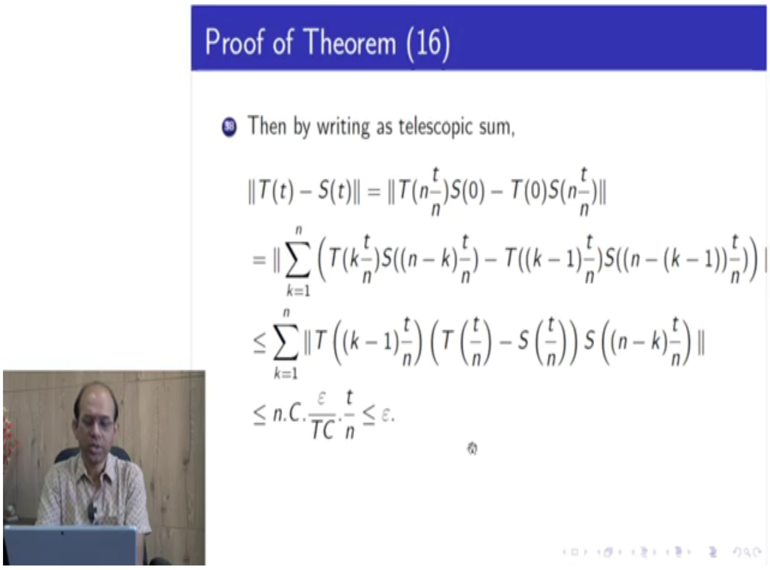

Here so we do this, so here we have n number of sums. So, for each and every term so n number of terms are there from n to n, and for each and every term I am now dominating this term this capital T of t by n minus S t by n because here this norm of the composition operators is less than or equals to products of the norms.

So, that we are using so this norm all these norms I mean there is the three different operators. So, you write down that as a product of norm of three different operators. And then these two things we are going to consider together and this separately for the norm of difference of this we are going to get Epsilon by T by C and then this t by n. So, this part we are going to get from this part.

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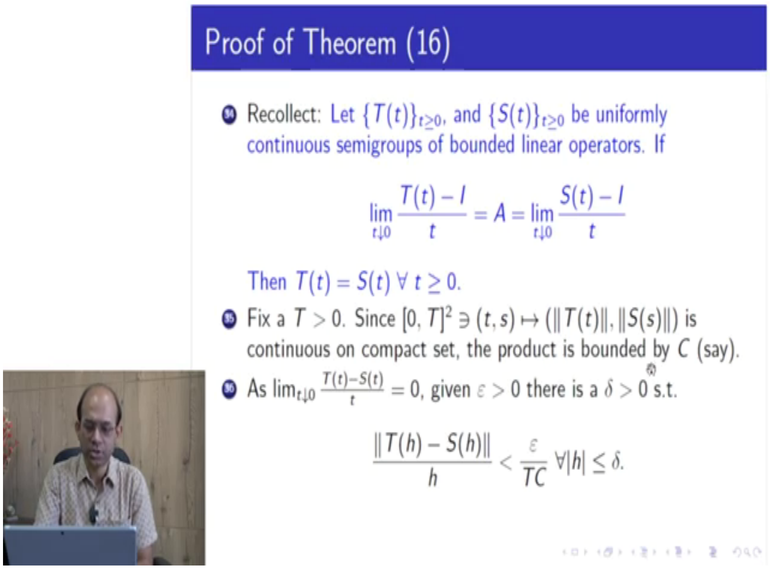

However, this capital T of k minus 1 t by n S n minus k t by n all this together we can bound by capital C, why because capital T and capital S the norm and product of the norms of capital T and capital S that is less than or equals to C that we have argued here, the product is bounded by C.

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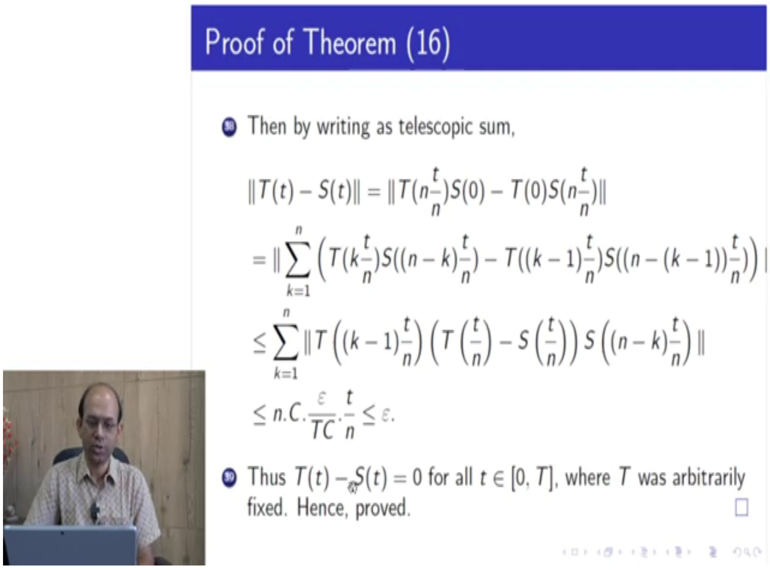

So, as long as the entries are in the interval 0 to capital T and that is true here because k is 1 to n, so that means this entry is at most t, this entry is also at most small t, but small t is less than or equals to capital T, so we can apply that inequality. So, n times capital C, capital C is coming from norm of these operators and then this Epsilon by T C times t by n is coming from this middle part.

So, now we have this upper bound, here C and this C cancels, this n and this n cancels, what remains is Epsilon times small t divided by capital T, but small t is less than or equals to capital T so this is less than or equals to 1 so we get only Epsilon. So, this is upper bounded by Epsilon.

So, what was Epsilon? Epsilon was arbitrarily chosen in the beginning. So, given any arbitrary choice of Epsilon, we have shown that T of t minus S of t is less than equals to Epsilon. So, what does it mean? It means that capital T of t is equal to S of t, this difference and this norm is 0 actually, this is less than or equals to any given arbitrary positive number so, that means this norm is 0. So, what we have achieved? We have achieved that capital T of t minus S of t is equal to 0 for all small t between 0 to capital T that we have proved.

However, this capital T was chosen arbitrarily, we have fixed capital T arbitrarily. So, whatever capital we choose, we are going to see this is true that capital T of t is equal to S of t during 0 to capital T. So, it does not matter how large your capital T is. So, since this is arbitrarily fixed so, what we conclude that capital T and capital S, this semigroup, T semigroup and S semi groups are identical for all finite time, so that is the end of the proof of this theorem.