Introduction to Probabilistic Methods in PDE
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Lecture 53
Growth property of C0 semigroup

In the last lecture what we have seen is that, for uniformly continuous semi group we have proved one particular theorem. So, let us look at that theorem and then we have proved that theorem completely.

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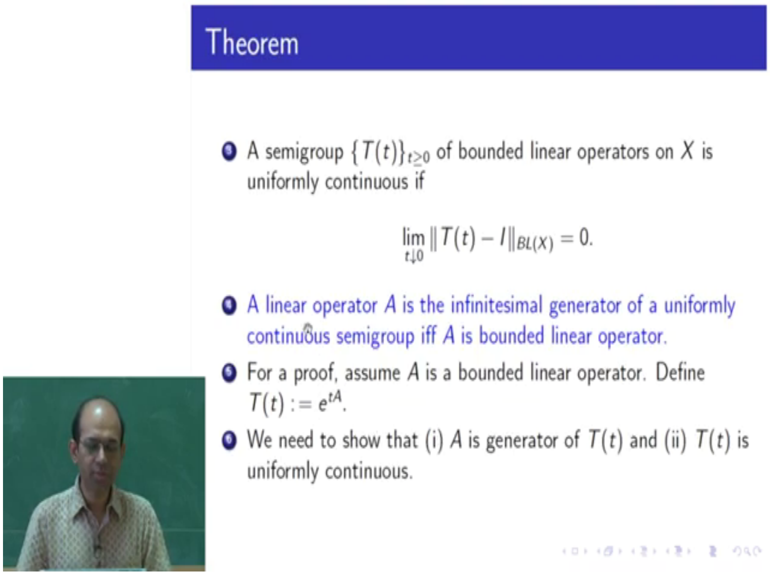

So, the theorem statement is that, a linear operator A is infinitesimal generator of a uniformly continuous semigroup if and only if A is bounded linear operator. So, what does it mean that if I have a bounded linear operator then I can view this as infinitesimal generator of a uniformly continuous semi group?

On the other hand, if A is identified as infinitesimal generator of a uniformly continuous semigroup then A is nothing but a bounded linear operator. However, this theorem does not necessarily says, how A and the generator and the semi group are associated. So, here if I have a bounded linear operator, I can always define what is e to the power t A as limit of the series expansion of the exponential series.

But this theorem does not say that the semigroup whatever we are going to get from e to the power of t A, whether there is any other semi group also for which A would be the infinitesimal generator. It does not say that the generator uniquely determines the uniformly continuous semigroup. I mean of course if you have one uniformly continuous semi, if you have a semigroup you get a generator that is okay, but given a generator whether you can get a unique semigroup so, that question is not addressed here.

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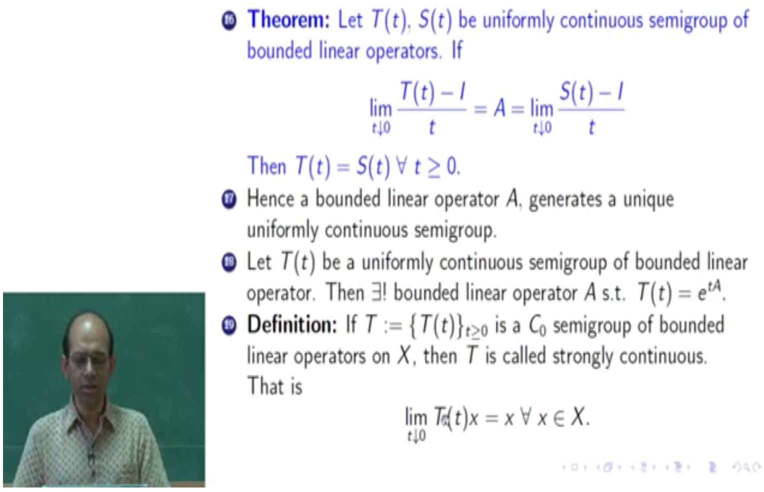

So, we are going to see one theorem today which really talks about that. So, this is the statement of theorem. So, let you have two different uniformly continuous semi groups of bounded linear operators T t and S t. Now, this is the generator T t minus I by t limit t tends to 0 and that you say A and S t minus I divided by t and limit t tends to 0 so that is A.

So, this limit exists because it is uniformly continuous, uniformly continuous semigroup for that we have seen that this limit exists, and that limit would give me some operator and that is bounded linear operator. So, imagine that you have two different uniformly continuous semigroup, for them this limit values are same. Now, question is that can I conclude that the semi groups are same, T and S both are same.

Basically, if you look at some any nonlinear functions or some functions, F and G and you just have that its derivative at 0 are matching, that does not necessarily mean that the function will be same. However, if they are linear functions and their slopes are matching both starting from 0 they are the same thing.

So, now here for semi group how to do this so that is the question. So, this is like you know derivative at 0 so the implication of this theorem is the following. So, it says that a bounded linear operator A generates a unique uniformly continuous semi group, because if A is generator of two different semi groups they are essentially the same so, A uniquely determines the semi group.

So, T t, let T t be a uniformly continuous semi group of bounded linear operator. So, then there exists a bounded linear operator A such that T t can be written as e to the power of t A this is also a consequence of this theorem. Why is it so? Because, I mean in earlier theorem, we just proved that if I have uniformly continuous semigroup then it has a, its infinitesimal generator is bonded linear operator.

And then from that bounded linear operator I could define e to the power of t A, but this e to the t A would be a semigroup, and this T t what we started with is also a semigroup, why should they be same? Because for both the infinitesimal generator is A is the same.

So, this is also a consequence of this theorem that every uniformly continuous semigroup of bounded linear operators can be written as e to the power of t A for some bounded linear A. So, this is one definition I mean this we have already seen the definition of C 0 semigroup, so, here this is just defining what is strongly continuous.

If T is a C 0 semi group of bounded linear operators on X then T is called strongly continuous because we have used the word uniformly continuous, there what did we get that limit t tends to 0 T of t minus identity that norm goes to 0. So, T t converges to identity in the operator norm.

However, here we have fixed some small x in the banach space and then the action of the operator on small x, then that would give me a vector in capital X, a member in the banach space and that member is a function of t and as t tends to 0 that member converges to x itself. So, this is like point wise convergence of T t to identity operator.

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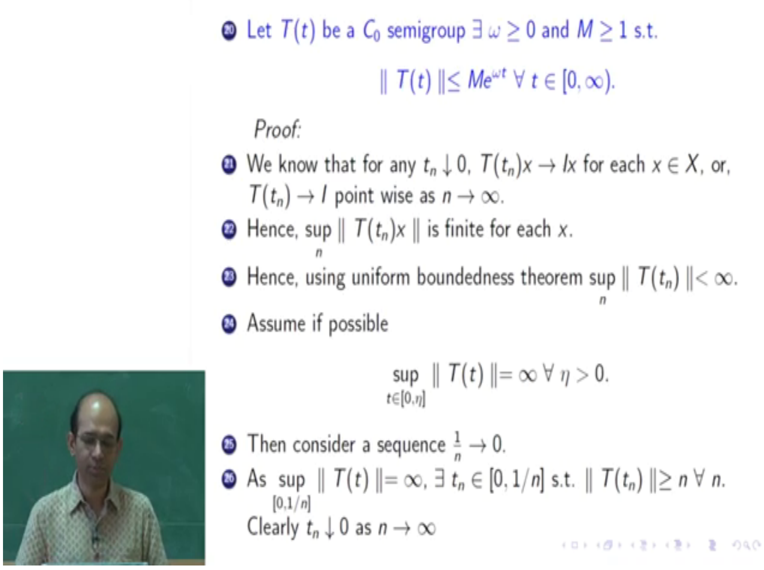

So, now we go to the next theorem that let T t be a C 0 semigroup. So, here we do not assume that capital T is a uniformly continuous semigroup. So, for uniformly continuous semigroup we have got a characterization, we have got very clearly that this is nothing but e to the power of t times A for some A and that A would be the infinitesimal generator and that A would also be a bounded linear operator.

However here, when we choose a T from a larger class, it is just a C 0 semigroup then can I say something similar, of course that particular statement is false here, but one can get something similar, what similar things that let T t be a C 0 semigroup. There exists some Omega non-negative and some capital M greater or equals to 1 such that norm of T t is less than or equal to M times e to the power of Omega t.

So, like in exponential expression the e to the power of t A kind of things of course, we cannot get for this general type of semi groups. However, the norm of that operator, for norm of the operator can be dominated by some exponential map, for many proofs actually, this is really-really helpful. So I have a semigroup so see semi groups definition clearly T of t composition T of s is equal to T of t plus S. So, like you know exponent e to the power of t times e to the s is e to the t plus s.

So, like you know that property itself is therefore exponential functions, so, it is very natural to expect that this growth as you know time goes to infinity its growth is like exponential, is very natural, but it is not obvious, it is natural to anticipate but it is not obvious. So, this theorem says that indeed 1 can do that so as t belongs to close 0 to infinity, so its growth can be upper bounded by an exponential map.

So, the proof is not very difficult, so it is, I mean it just uses one particular theorem of functional analysis that is uniform boundedness principle of family of bounded linear operators. So, we start this proof here. So, we know that since you know T t is from C 0 semigroup, we know that if I choose a sequence t n which converges to 0 then T t n of x converges to identity operator of x, T t n x converges to x because it is C 0 semigroup.

So, as t n goes to 0, T t n goes to T of 0 x, T t n of x goes to T of 0 x, T of 0 is identity. So, for each x we have this, so this is same as saying that we have a family of operators. So, that I mean for every n we have an operator so, this family of operators, they converge point wise to another operator identity operator so, point wise as n tends to infinity.

Now, from there we get that since this is converging so, this you know if I put x so, this is a converging sequence. So, a converging sequence is of course, a bounded sequence. So, if I take supremum of all possible n, T t n x is finite for each x, this is finite for each x. So, that we get so, that is also in the sufficient conditions for applying uniform boundedness principle for family of operators, we need that point wise boundedness. So, for every point x this family this application of this operator that is finite.

So, now, since this is true so we can apply uniform boundedness theorem from functional analysis course and then what we get is that from this point wise boundedness to the uniform boundedness. So, like supremum over all possible n then there we can take the operator norm. So, instead of the point wise here so I am taking operator norm here, so norm of T t n supremum of all possible n that is finite so, this is first thing you obtain here.

So, next what we do, we assume if possible actually this is wrong thing, but we are going to show this is wrong. So, prove by contradiction. Assume if possible that supremum of T of t is infinity for I mean when we take supremum small t from close 0 to Eta for any positive Eta.

So, let me put it in a different phrase, you fixed one Eta positive and then you get an interval 0 to Eta and then on that interval you take this function T of t norm, so norm of T of t that is a non-negative function so, and then we take supremum of this function.

The supremum of this function on this interval 0 to Eta, assume if possible that is infinity, no matter what Eta you choose. So, assume that if possible that is true. So, then what we do is that using this fact we should get a contradiction. So, what we do is that because here supremum over all possible t it is not directly saying that we are getting contradiction, we have to do little more work, we have to actually consider a sequence decreasing to 0 because t n converge to 0.

So, we do this, this this is a very elementary argument, then consider a sequence 1 over n converges to 0 as supremum 0 to 1 over n T of t, this is infinity because of this assumption, because Eta could be any positive so, 1 over n is also positive number. So, 0 to 1 over n on that supremum if I take of T t that will also be infinity due to assumption.

So, I can always find one t n in this interval such that T t n would be more than or equals to n, I can always do this why because here in this supremum on this, this is infinity. So, that means, I mean for sum t it is very very large and arbitrarily large, whatever you know large number n is, I can always find one t n in this interval such that for which the norm of T t n would be more than equals to n.

So, if you cannot do this then actually this is wrong so, if we assume this then of course, we will be able to find out such t n. When we have fix t n then we take instead n, n plus 1, so 0 to 1 over n plus 1 even in a shorter interval, but then 1 over n plus 1 you consider that is Eta, again use the same argument and you consider again another t n plus 1 if it is 0 to Eta or 0 to 1 over n plus 1 such that again you get it.

So, you generate such type of sequence of t n for which norm of T t n is increasingly larger more than n, etc. And then the choice of t n is such that it should go to 0 why because t n is always between 0 to 1 over n and as n tends to infinity this thing goes to 0, so t n is by construction converges to 0 as n tends to infinity. On the other hand T t n by its construction it is going to infinity. So, here we get a contradiction therefore.

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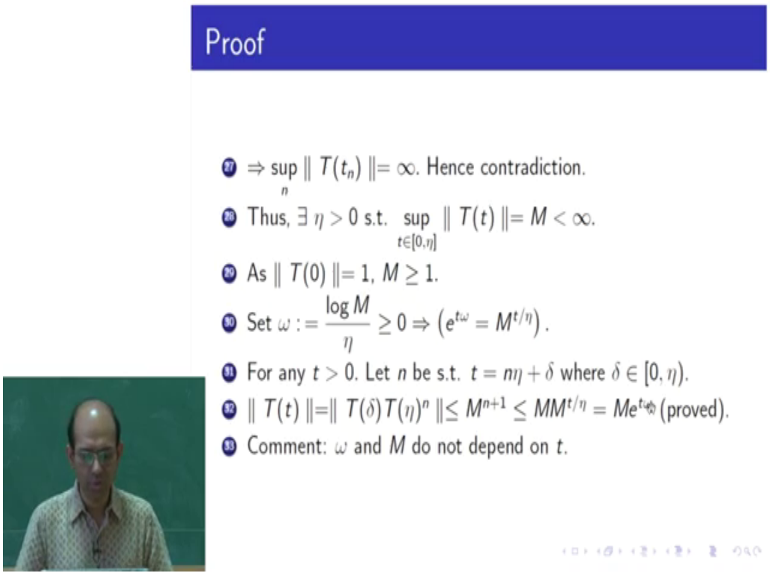

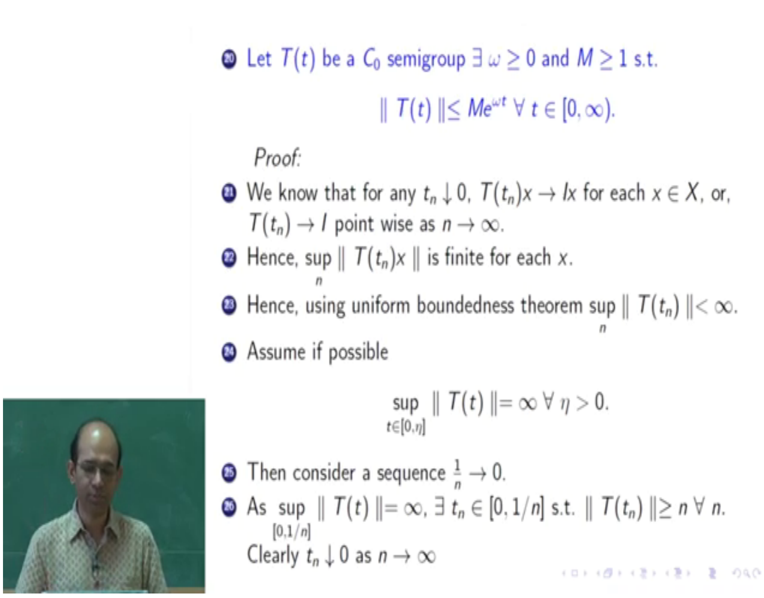

So, it implies that supremum of all T t n is infinity but here we have second supremum of all n T t n and is finite. So, this thing is contradicted due to this assumption, so this assumption is wrong, since this assumption is wrong so what we can assure is that at least there exists some Eta positive such that the supremum is finite, that we do.

Thus there exists some Eta positive such that supremum over 0 to Eta in this interval norm of T t is finite, we call that number as capital M, we call this number as capital M. Now, as T of 0 is equal to 1, and we are taking supremum, so norm of T 0 is 1, and this is a member and we are taking supremum over all possibilities so M would be at least 1, it will 1 or more than 1.

So, here we get existence of such M and also that it would be not less than 1. So, now, we have to cook up Omega, I mean recall the statement of the theorem, we have to come up with you know existing Omega in M. So, we got a candidate M there, now, we are going to cook up on Omega.

So, we define Omega to be log of capital M divided by Eta which Eta, this Eta, log of capital M divided by Eta. So, this is of course, greater or equals close to 0 because Eta is positive, M is 1 or more than 1, so, it cannot be negative so, Omega is non-negative. Now, if we define Omega this way, and we calculate e to the power of Omega so e to the power of Eta times Omega, e to the Eta times Omega is capital M. And then we take power of by t by Eta both sides.

So, e to the power of Eta Omega becomes e to the power of t Omega and right hand side will be M to the power of t by Eta. So, we get this nice you know relation between M, Eta and Omega. So Omega is defined this way so that means this thing.

So, now, we consider an arbitrary small t so here see I mean I have only, I could only assure that on the interval 0 to Eta, T of t is bounded so on that it is bounded, but for arbitrary t we need to get something is bounded by exponential growth I mean dominated by exponential growth that we need to come up with. And for that we of course would use the semigroup property of T t.

So, for any small t positive let n be such that t is equal to n times Eta plus Delta so like division algorithm so, like you know if I mean so if small t is more than Eta, if small t is less than Eta we do not need to worry much about this, for this we already got, if small t is say more than 2 times Eta but less than 3 times Eta, so we write down that t is equal to 2 times Eta plus Delta where Delta is between 0 and Eta.

Small t is any number, so we can actually divide small t by Eta, and then we take the floor function, floor function means the largest smaller integer that will be n because t by Eta so, that would be so, that is n. So, t is equal to n Eta plus Delta where Delta is between close 0 to open Eta, we would be able to always be able to find out such n.

So, now, norm of T t you should consider for this general t. So, this small t can be written as n Eta plus Delta. So, capital T of n Eta plus Delta is capital T of n Eta times I mean composition with capital T of Delta. So, capital T of Delta would be there and capital T of n times Eta. But capital T times t of n times Eta can be written as capital T of Eta plus Eta plus Eta n times and there again we use semigroup proper property, so you get T of Eta to the power of n.

So, we write down norm of T of t is equal to norm of T Delta times t of Eta to the power of n. Since Delta is between 0 and Eta, so, we know that T of Delta is less than or equals to n. So, T of Delta so this norm of the product operator is less than or equals products of the norms. So, here for norm of T Delta we are going to get one capital M and from T of Eta also we are going to get capital M to the power of n here. So, we are going to get M to the power of n plus 1.

However, as I told that this one if I take and then n, this n is what, n is the floor function of t by Eta, so I can write down that this is less than or equals to M times M to the power of t by Eta. So, now this t by Eta is here we see that M to the power t by Eta is e to the power of t Omega. So, we substitute that thing here, so we get is equal to M times e to the power of t Omega. So, norm of T of t can be written as less than or equals to M times e to the power of T Omega. So, that is the proof of this theorem.

So, one comment that here when you have obtained this Omega and capital M, so they are independent of the choice of t because here I mean what we have obtained that there exists some Eta such that, this supremum of whole thing is equal to M. So, this M is (())(23:57) choice of small t, this small running t and Omega is constructed using Eta and M.

So, here I mean this M and Omega are not function of t so here as a function of t so this is a constant and then e to the power of t. So, this is actually upper bounded by one exponential function. So, that is the proof of the theorem, thank you.