Probabilistic Methods in PDE
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Module 10
Lecture 52
Semigroup of bounded linear operators on Banach space, Part 2

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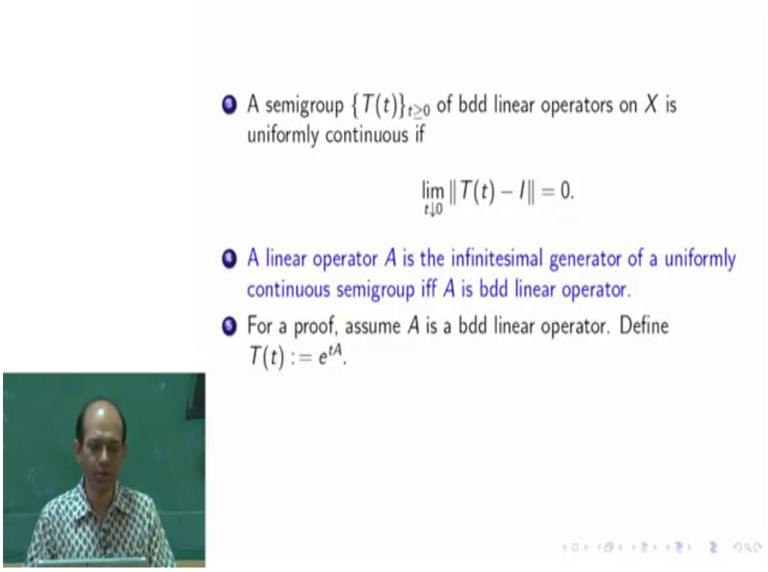

In the last lecture, we have seen this theorem. A linear operator A is infinitesimal generator of a uniformly continuous semi group, if and only if the operator A is a bounded linear operator. For this, what we have seen is only one side of the proof that when A is a bounded linear operator, then that A generates one semigroup or in other words A is infinitesimal generator of a uniformly continuous semigroup.

So, that we have done in the following manner, we have constructed one semigroup T t using e to the power t A and then we have shown that this is a candidate, but we have shown that this is indeed a semigroup whose generator is A, and not only that this semigroup is uniformly continuous. So, that is the thing what we have seen here so, this part was done.

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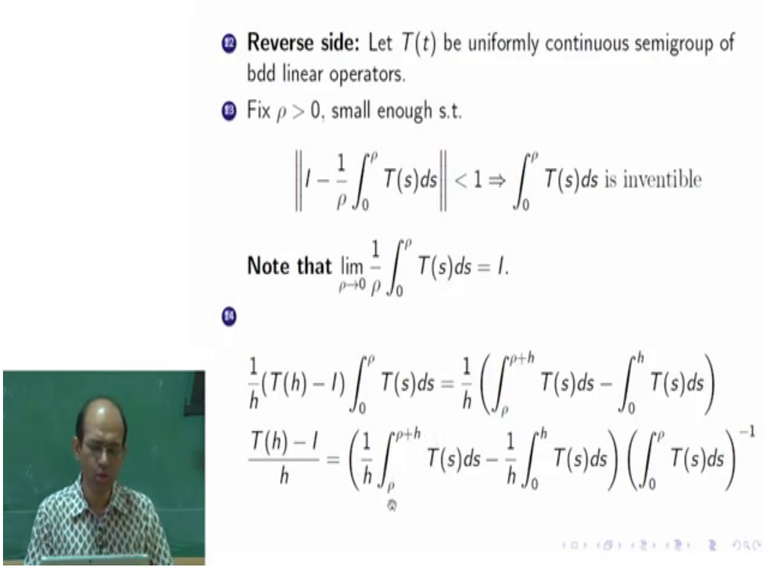

Next what we would do the reverse side, reverse side of the statement, which is not much complicated. So, what is this, this is that you have uniformly continuous semigroup T t. You just need to show that generator is bounded linear operator. So, for this uniformly continuous semigroup, this is uniformly continuity is a very nice property we can take you know, limits there.

So, that is the thing what we first need to show that when it is uniformly continuous semigroup then actually you know that we can take that norm limit correct. So, T t minus identity divided by small t and the norm of that and t tends to 0 so that converges. So, that is the thing what we need to show, we just know that, if it is uniformly continuous, we just know that T of t minus identity that norm goes to 0 as small t goes to 0.

So, that does not necessarily mean that T of t minus identity divided by t, that will also converge somewhere, it does not remain. However, that actually is true so that means, you know little justification that we are going to see in a standard manner. So, what we do is that we first observed that we are going to see in a standard manner. So what we do is that, we first observe that, so far a small positive constant Rho.

So, we consider this difference identity minus 1 over Rho integration 0 to Rho T of s ds. So, if you look at this fraction, this fraction has Rho, you know I mean, so, this is, what is this? This is basically as Rho tends to 0, due to the continuity of this, this would converge to T of 0 that is identity. So, this is coming from uniform continuity of this T that this whole thing would go to 0 as Rho goes to 0.

Now, I can therefore choose some positive Rho sufficiently small such that this norm is less than 1. Now, we know that whenever I have an operator B such that norm of I minus B is less than 1, then B is invertible. Correct. So here we apply that. So, since this I minus this operator's norm is less than 1 so, this operator would be invertible or in other words so, here it is the scalar multiple, so, we can say that integration 0 to Rho T of s ds so this would be an operator because the integration of operator correct. So, this is invertible.

I mean if you are seeing for the first time, integral of operator valued functions then I would motivate this way that ok after all these T's are operators in the Banach space of B L X, I mean bounded linear operators there. And whenever you have a Banach space valued functions, continuous function, so then one can define this integration like you know in the Riemann sense. In that one can take the sum basically the Riemann sum and that sum would converge. So, that is the thing we are going to know that there is a notion of this integration.

So, when we have invertibility of this so, this we are going to use for our this thing. So, here we are basically using this fact that limit Rho tends to 0, 1 over Rho integration 0 to Rho T s ds. So this is actually a limit 0 by 0 form, but we know that whenever we have continuous function and then we know that this limit is I.

Actually I mean one needs no so continuity at point 0 and I mean one does not need continuity everywhere, just continuity at 0 and locally L 1 for that also one can get the same thing, but here we have better thing we have a T s is continuous. So next what we do, we now take this following product, we take 1 over h times T of h minus identity.

So, this is the fraction which we must consider because we are after all trying to find out the infinitesimal generator of the semigroup T, and that is nothing but limit of this thing, correct. So, anyway this I anyway need to consider, but instead of considering only alone this, we consider this multiplied with this thing. And while multiplying this, I do not mind multiplying because anyway this is invertible, I can take this out also on the right hand side.

So, that is the idea. So, instead of just considering this fraction, we consider this multiplied with 0 to Rho T s ds. So, when you do this, then we do some more manipulations. So, T of h, this is a linear operator, so this is bounded linear operator, so T of h. So, this and this integration 0 to Rho, so this and when this comes inside, so this thing product with this. So, if I put it inside I would get difference of two integrals.

So, one would be T of h composition T of s ds 0 to Rho. And another one minus T of s, 0 to Rho ds and everything divided by h. So, if I do that, so what should I get? I should get here h here so, because T of h composition T of s ds, so I would get, I would use the semigroup property of the semi group property. So, here T of h composition T of s is T of s plus h, so then we would write down that with simple change of variable that will be h to Rho plus h T of s ds, and then minus, from here we are going to get 0 to Rho T s ds.

So first integral is h to Rho plus h, next is 0 to Rho. So now this second integral 0 to Rho and first is this. So in the first if I add 0 to h again that part and subtract, so this we subtract minus 0 to h T s ds and add with the first part, then the first integral would be 0 to Rho plus h T s ds, and second would be just minus 0 to Rho T s ds. Then first and second integral if we take difference, we would get Rho to Rho plus h. Is it clear?

So Rho to Rho plus h T s ds. Let me repeat again, so here from here and here together, I would get h to Rho plus h, but below h, I would like to make it 0. So I would write 0 to Rho plus h, since I have added, so I just have to subtract. So minus 0 to h T s ds so I subtract then I would get 0 to Rho plus h, but from here, I would get 0 to Rho. So 0 to Rho plus h minus 0 to Rho, this difference would give me Rho to Rho plus h. So 1 over h integration Rho to Rho plus h T of s ds minus integration 0 to h T s ds.

So now after getting this expression, so this expression we have obtained because of this term. Now, we take the inverse of this term, because this is invertible after all. So, left hand side would be T of h minus identity divided by h, on the right hand side I have as it is this term, and this term on the inverse of that would appear on the right hand side here. So, here it is 1 over h Rho to Rho plus h T of s ds minus 1 over h 0 to h T s ds and then this thing.

Now from here, it is evident that what would happen if you take limit h tends to 0, what would happen? So this does not depend on h, this remains as it is, but here what happens? So, here so, this is as h tends to 0 this whole thing would be 1 over h integration 0 to h T s ds would converge to T of 0, correct T of 0. And then this T of 0 is identity. So, I would get identity as the limit of this term. And from this time, what I am going to get? I would get Rho to Rho plus h T s ds and 1 over h. So this as h tends to 0, you should converge to T of Rho good so, we do that.

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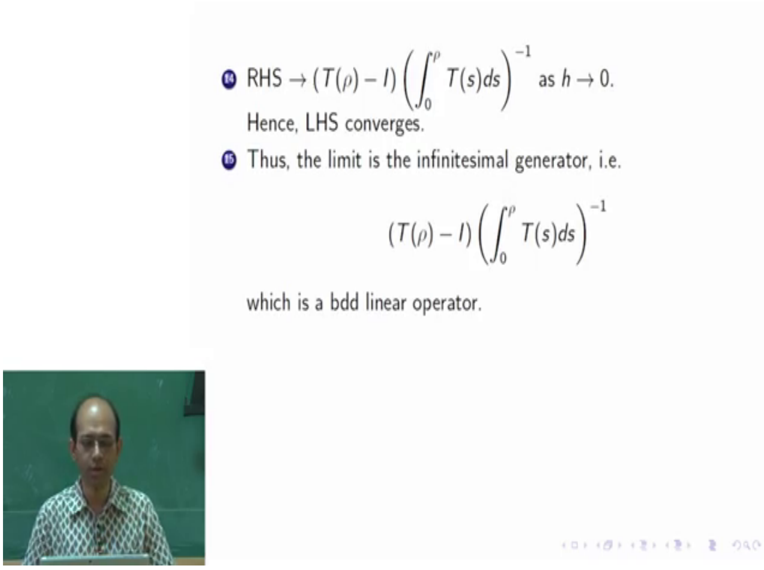

So, the right hand side converges to T of Rho minus identity times integration 0 to Rho T s ds inverse as h tends to 0. So, the right hand side converges as h tends to 0. So, initially I was not confident that whether this should also converge in norm as h tends to 0, but now yes we can say that because right hand side surely converges here, so left hand side should also converge. So, thus is the limit is the inverse.

So, whatever the limit that is infinitesimal if that is the infinitesimal generator by definition, so left hand side is the infinitesimal generator, and right hand side is this, so the infinitesimal generator is given by T of rho minus identity times integration 0 to rho T of s ds whole iniverse. So, this is a bounded linear operator because I mean this is of course T and this is bounded linear operator. So, here what we have proved that uniformly continuous semigroup has a bounded linear generator, so the generator is a bound linear operator.

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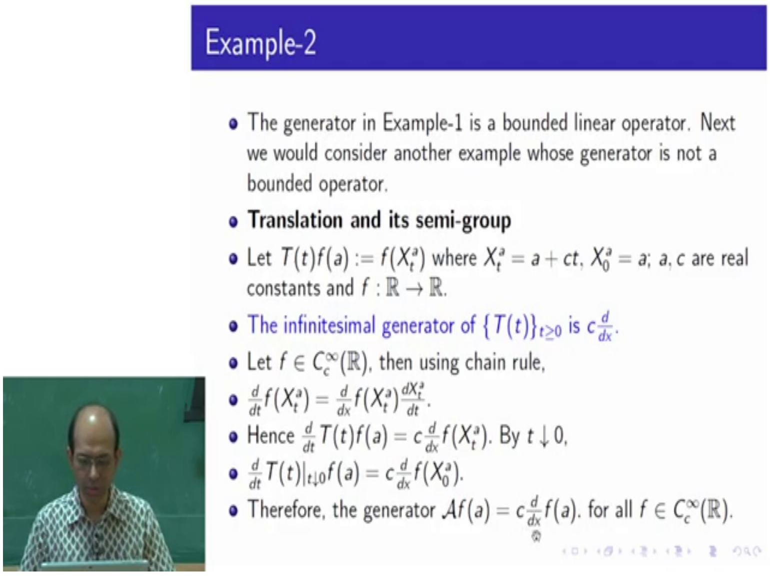

So, next we come to some example, remember that in the first example, what we have seen the Markov process, finite state Markov chain. And for that we have computed the infinitesimal generator that turns out to be the rate matrix, and the rate matrix is actually having finitely many elements, where k cross k matrix, where k is the number of steps. So that is a bounded linear operator.

Now here that this theorem is also talking about that. So only considering the case of bounded linear operator as the generator. But it is not necessary that every semigroup would have generators as bounded linear operator. So, we are going to see some example where that semigroup would not have a bounded linear operator as generator and is a very simple, this is a very simple semigroup.

I have chosen this very simple semigroup just because to emphasize the fact that most of the semigroup what we are going to discuss or what people encounters do not have you know the generator I mean what I mean to say that at the time of studying differential equations, etcetera or any dynamics continuous dynamics, so the generator what we obtained the semi group we obtained, their generators are not bounded linear operators.

So, here we would see that, so we take translation semigroup translation. So, what is translation? It is a very simple thing that at time t it is a plus ct, at time 0 it is a and any anytime t it is a plus ct, so that is the process deterministic process. Just linear translation this is, just starting from a and speed is c, c is the speed.

So far this translation dynamics, I come up with a semigroup here. So, we consider f is a function from R to R, a and c are real number. So, here T of t f that when T t is applied on f so, that is defined as a f of X t. Why is it a semigroup? It is a very easy exercise I expect that you would do that, just show that. So take that as an exercise that this T t is a semigroup indeed. Now for this semi group we are going to find out it is generator.

So, the infinitesimal generator of this semigroup is c times d dx. So, partial derivative operator, so that is the infinitesimal generator. So, let us see the proof of this. So, we consider f C c infinity, C infinity function with compact support, I mean for this prove actually if I take that differentiable functions bounded, with the bounded derivative that is sufficient, but anyway, so we better take just you know for some other discussions I need this.

So, we take this class of test functions, so, this C c infinity. So, f is there, so now we use chain rule, so derivative so here what is this? This is actually composition of two functions, X is anyway function of t and f of X of t. So, d d t of f of X of t is d dx, so its derivative and then dx dt. But X has this formula, so dx dt is just see the speed constant. So, we get that dx dt is c and d dx f is this and then this thing is same as T t f.

So, I am writing down now the expression of the semigroup, so d dt of capital T t of f evaluted at a is equal to c times d dx of f, you will get X t a. So now we take t tends to 0 from the right. So, when we do that so as we have defined the generator of the semigroup it is just the derivative at time 0 correct. So T h minus I divided by h limit h tends to 0.

So, we are going to get exactly the same thing here. So d dt T t, t tends to 0 here f of a is equal to c d dx f of X a 0, so t tends 0 so because of due to the continuity we can do this. Derivative is also continuous so all this is f is nice function. So, here what we have obtained is the generator A. A f of a is equal to c times d dx of f. So A is c d dx. So, this is the proof that this generator of the translation semigroup is c times d dx.

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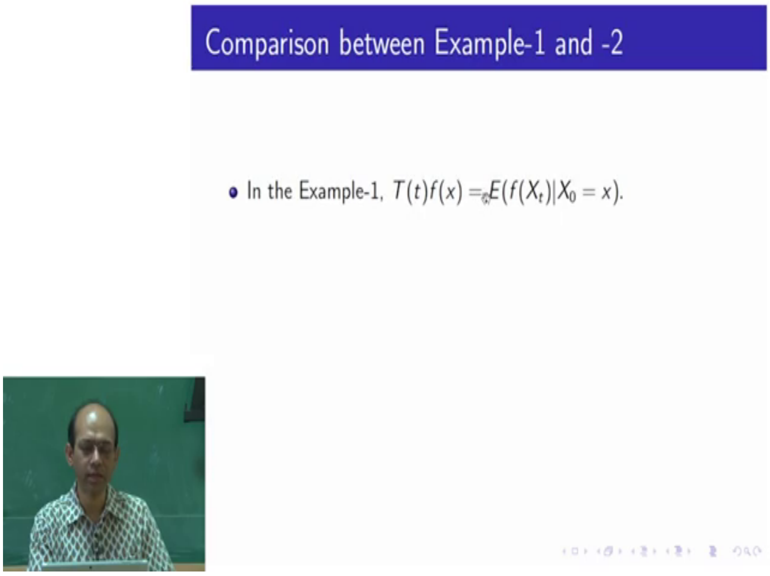

So now we just have some discussion. So in example one what we have taken? We have taken finite state Markov chain X t. And T t of f of x was taken as expectation of f of X t given X 0 is equal to x. So, let us go back there.

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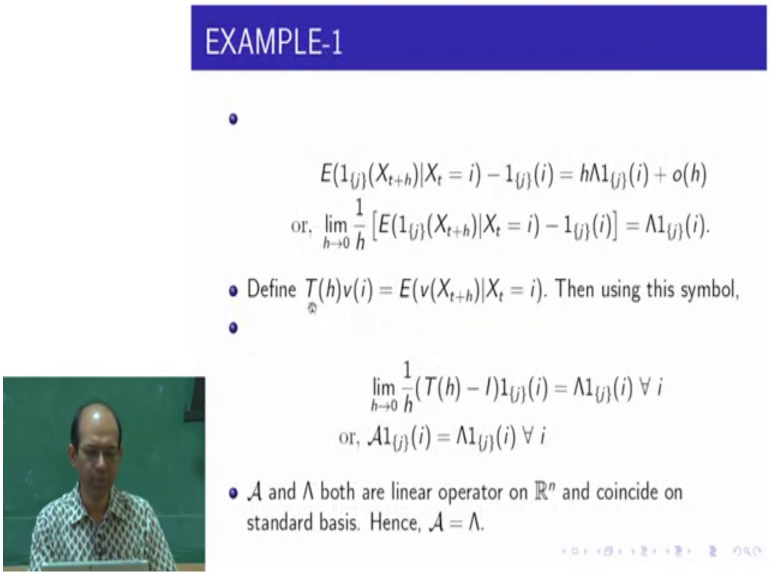

Here you see, in example one we have seen T of h v i is equal to v of X t plus h given X t is equal to i. This i does not depend on this T i parameter so, I can put 0 0 here. So, T h v i is equal to expectation of v of X h given X 0 is equal to i. So, v is a vector because state was only finitely many state, a function on a finite set can be viewed as a vector, so it was written this way. So, in general, we can use the notation f as a function here. And then what I was writing here.

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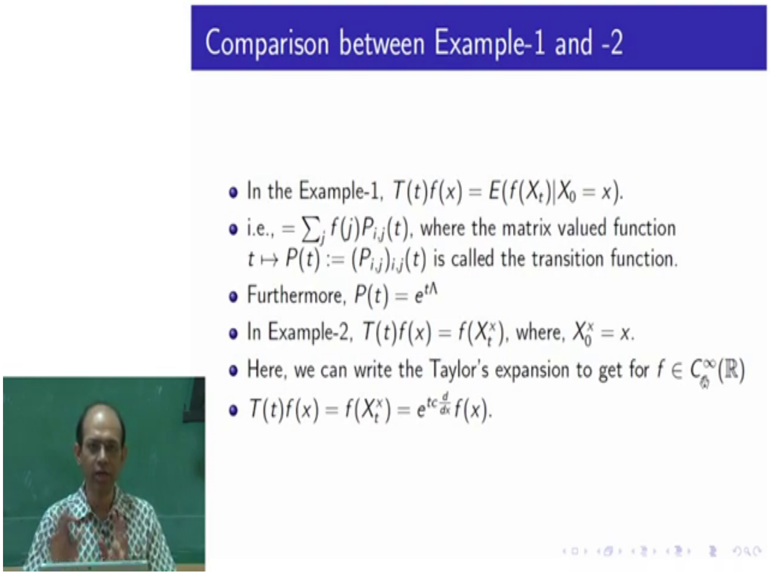

That T t of f of x is equal to expectation of f of X t given X 0 equals to x. Now, this right hand side, what is its formula? We can actually write down this, correct. I mean this is the expectation after all of X t. So, there are finite state, so this expectation is equal to summation over all possible f of j times P i j t sum of all possible j correct. So what is P i j? Where the metric value function T to P t where P t is that P i j.

So, this matrix is called transition function. So, this semi group can be written in terms of this matrices and then this matrix can also further be written as P of t is e to the power of t lambda. That we know because that is actually coming from the discussion what we already had. So, here next in example 2, we have considered T t of f x is equal to f of X t where, so here it looks a little different from there so the above because here X 0 is equal to x.

But it is not very different also because here f of X t where this process is such that it starts from x. So here also starts from x, here also starts from x. Here you do not need to take expectations because it is deterministic after all. So, it is quite similar as before and then here we can write the Taylor's expansion to get for every f C c infinity functions that T t f of x is equal to f of X t, but this if you write down Taylor's expansion of this, what are we going to get?

We are going to get f of X t is equal to f of X 0 plus derivative and then dX dt. And then one factorial 2 factorial then second order derivative all these things and then this T square, c square etcetera. So if you take here those operators together and can view that this operator is multiplying again and again, then you see that this is nothing but expansion of the exponential function of the operators.

So, from there one can write down e to the power t c d dx f of x. So, here also so as before here say for example, for this case the semi group I can write down as e to the power t times of the generator, here also for the special cases of functions in C c infinity, so that we can take all these submissions, all this makes sense so, there also you can write down this manner.

Otherwise, I mean if we do not take f form here, it does not make sense because this is not a bounded linear operator. However, when f is in Cc infinity, those summations makes sense because the power series converges, the Taylor series converges, so we can write down this way. Okay, thank you very much.