Probabilistic Methods in PDE
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Lecture 50
Cauchy Problem with variable coefficients Feynman-Kac Formula Part 2

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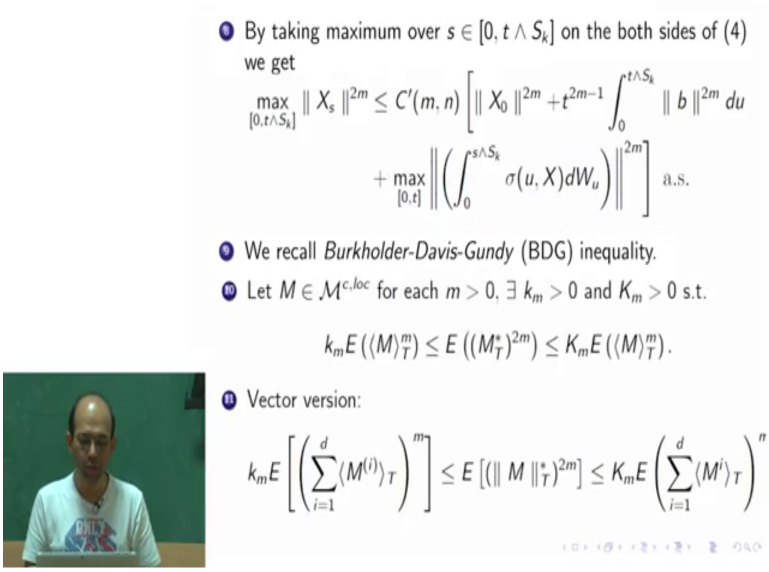

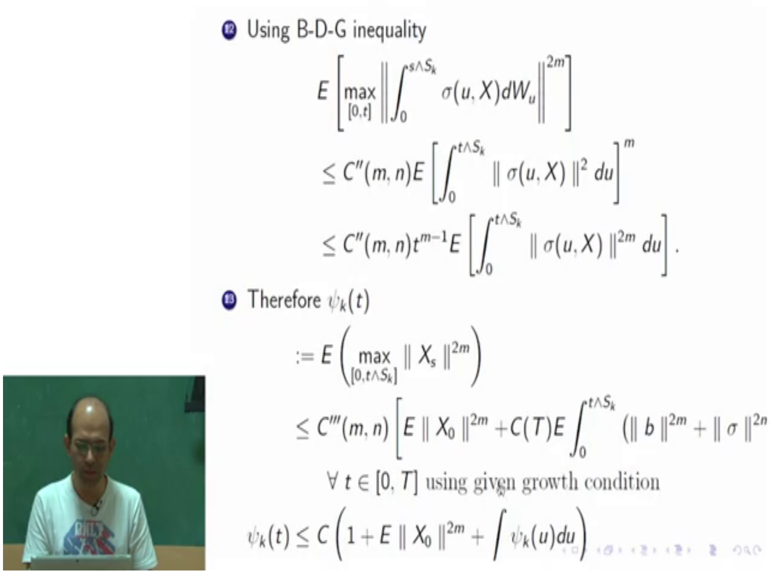

Okay so now, we are going to use the BDG Burkholder-Davis-Gundy inequality. So, we generally call this BDG inequality. So, what is that? So, this BDG inequality says that if we have a continuous local martingale M then for each small m positive there exist constant small k m and capital K m, okay non-negative, I mean positive constant such that this expectation can be upper bounded and lower bounded by some constant times expectation of this mth moment of the quadratic variation at capital T.

So, let us read this carefully, what is M star capital T? M star capital T is actually this thing, that M of small t mod and then take maximum, supremum over all small t between 0 to capital T. So, as capital T increases, M star t is non-decreasing, correct it increases. So, this is a very large number because 0 to t if you look at and in between the martingale can be very large also, okay.

There is always some probability that it would be a very large thing. And then finding out its 2 mth order moment. So that can be estimated using the mth moment of only the quadratic variation at capital T. So, this is the statement of BDG inequality for scalar valued local martingale, but there is also a vector version.

If M is the R d valued continuous local martingale then also one can write down that thing. So, here instead of this we would write down the norm here. So, norm M star T, capital T that is actually supremum of the process in the interval 0 to capital T. So, that to the power 2 m expectation that means the 2 mth order moment of that random variable. So, that is greater or equals to small k m times expectation of that i is equal 1 to d since there are d number of components i equal 1 to d.

And then quadratic variation of M i at capital T and then this whole sums is a random variable we look and this is the mth order moment of this random variable. And here similarly on the right hand side also we have this, you know this m is barely visible, but m is here, so it is also the same thing. So, this mth order moment of this random variable.

So, some constants multiple of this mth order moment of this random variable can bound this 2 mth power moment of this maximum achievers of this random variable. So we are going to use this thing, so here we understand that why is it so important for our case because we really came across that type of maximum, etc here, so we need to use this.

And for our case also we encounter the vector valued process, so here for example, Brownian motion is a vector value matrix so, this is a vector valued process so, we consider this term now. So, expectation of maximum 0 to t of these to the power 2 m. Why am I doing this because this is a as a function of s is a martingale, continuous Martingale because S k is defined such that this is bounded inside that.

So what we do is that we take this expectation of this 2 mth power of this quantity that is using BDGs inequality is less than or equal to some constant, so this k m I am writing down here as C double prime, why? Because C prime I think I have used somewhere, I have used here. So do not bother much about m here because most of the cases m is d.

So till now, I did not talk about any sequence, I actually want to write down every n as d because later I would really use n as a index for sequence, so this is d. So here C double prime m d actually, expectation of maximum 0 to t integration 0 to s minimum S k, sigma s u, X d W to the power of 2 m is less than or equal to C double prime m n expectation of 0 to t minimum s k and sigma u X norm square d u to the power of m.

How did they get it? Because this is the quadratic variation term. So, this term if I look at this term, instead of capital T if I replace that by s minimum S k I would get this term correct, this is the quadratic variation of this term of this local martingales. So here now, when we get this term here, Brownian motion does not appear here.

So, we are in a better shape here and now, again as earlier we are going to use I mean either appropriate Holder’s inequality or Jensen's inequality, so like to the power m, this m is 1 or more so, this is convex function, I mean so this is convex function and then this part just is not unit interval but we can actually divide and multiply to make it that way.

And then we would get a t minimum S k to the power m minus 1 but that is less than or equal to just if I write down t to the power m minus so, that is so, that we do. So, here we are going to get this m also would come here inside 0 to t minimum S k norm of sigma u X whole to the power of 2 m d u, so here this is the important thing what we were looking for here, we wanted to get one upper bound, we have obtained here, so which does not involve Brownian motion.

Now we write down this whole thing as psi k. So, this thing see this maximum of 0 to t minimum S k X s to the power of 2 m. So, this 0 to t minimum S k X s to the 2 m so what appears there on the top, so I have written that but I have taken expectation.

This is a random variable here in this side, and now what I do I take expectation of that. So, expectation, so, I need to do expectation because with expectation early we have obtained that upper bound here, so we need to do that. So, we call this psi k just writing these so that we can use these same thing this notation later to obtain one inequality involving psi k, so this inequality.

So in, and this depends on k correct, so we write down psi k, also depends on t so psi k of t, also it depends on m but we are suppressing that involvement because we are fixing m correct because that was an assumption that there exists some m for that X naught has finite moment of order m.

So, that m is fixed anyway from the beginning so, I mean here you confirm that these are all d, d. So now less than or equals to C triple prime, so another constant. Why am I writing another constant because I am going to sum these terms because this is the expectation of integration, you remember.

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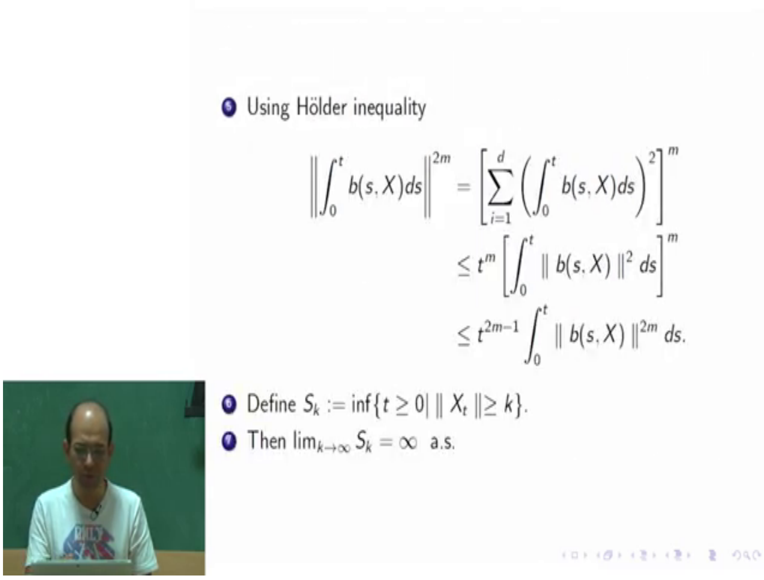

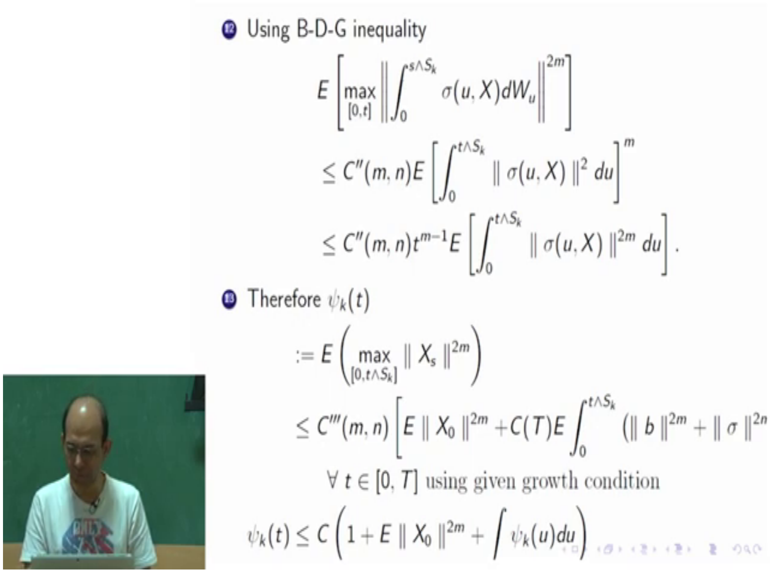

And then for b also we have obtained that some integration 0 to t d s. So, all these integrations are there, so we want to club those things together. So, here knows if we want to do that then all this you know constant should also be readjusted. So, imagine that we have done that, we have done that C (double prime) triple prime m n is readjusted with this thing and small t, etc we do not bother much about.

Because here that finite time interval only we are looking at. So, small t is always less than equals to capital T. So, we are going to get that, so expectation of norm of X naught to the power 2 m plus some another constant times expectation of 0 to t minimum S k norm of b to the power 2 m and then from here sigma norm of to the power 2 m and then d u.

So, this part is not appearing but this is just d u is written here. So that we obtain that is true for all small t. So, this psi k t is a function which you know satisfies this thing here. So now we use the growth condition, so what is given for b and sigma like the at most linear growth condition. So, when you do the growth condition what we are you going to get? We are going to get that constant L something there and then I think, what was the constant here?

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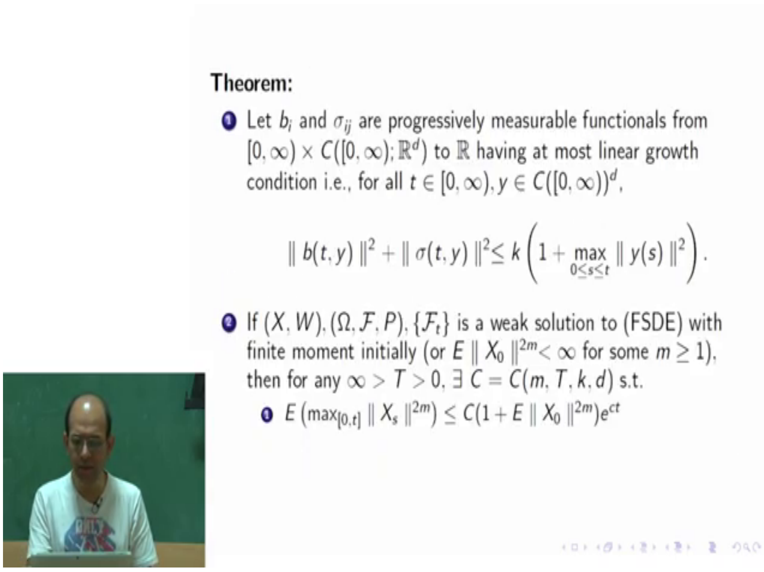

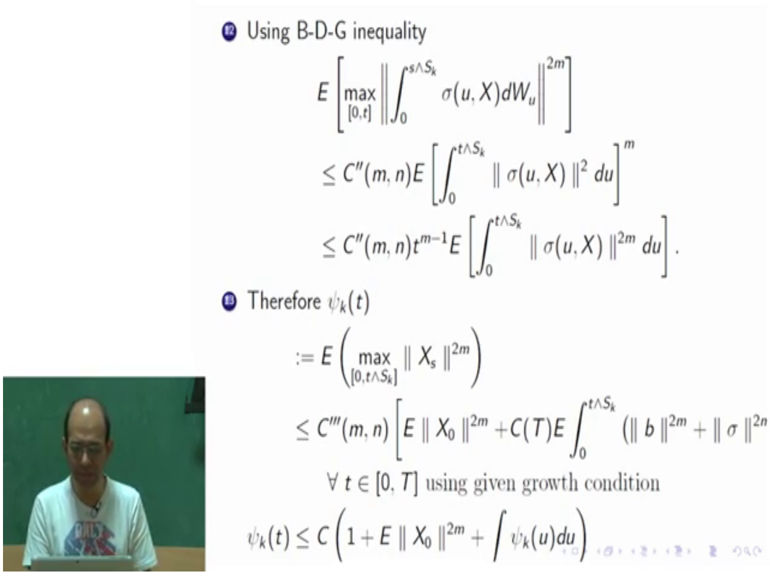

Yeah, some small k times this to the power this square, so this inside this thing would appear here, so when you do that, so here we get that inside this X to the power of 2 m would appear. And by the way, maximum of that thing or expectation is psi k. So if I replace that by maximum of these things and take expectation inside because non-negativity I can take, expectation inside.

So I would psi k here therefore psi k would satisfy some constant so I am writing here just C here, psi k of t is less than or equal to C times 1 plus expectation of norm X 0 to the power of 2 m plus integration psi k u d u. So all these constants, etc whatever was there in my hand, I can put everything in C and then we get this inequality.

This inequality reminds us that we can now apply Gronwall's inequality to get to one estimate of the growth of psi k.

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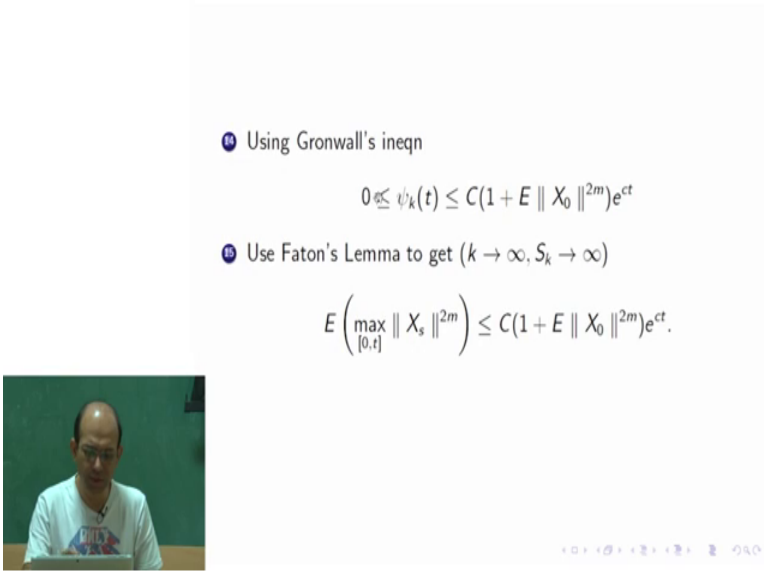

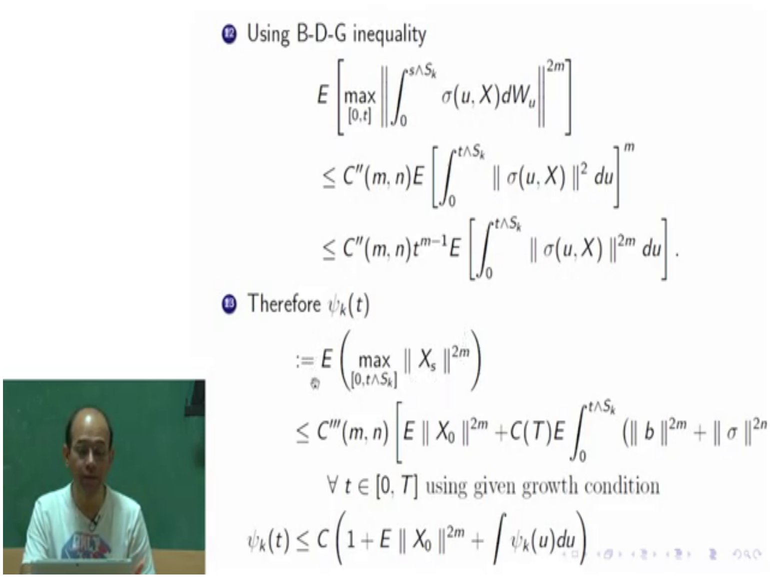

So we do Gronwall's also inequality. So that would give me that this thing is there. So, C times 1 plus expectation norm of X 0 to the power 2 m, e to the power of ct. So, this c is coming from all these constant whatever we have obtained till now so this thing. So this is psi k t.

So, what is the intuitive understanding of that that these expectation, this 2 mth power expectation as t increases you know that the solution that can be very large and that the probability that it is large is also becoming larger, etc. However, as a function of t, it is dominated by some exponential function of t that is a good thing that is a thing we just obtained here.

It is not like that to get c t square or something that only with t we can get bound or you can think log if I take then that can be bounded by some kind of linear growth function. So, that is the inequality we obtain and this is non-negative Of course, because it is norm expectation of norm of something.

Now, we are going to use these things, I mean in our result. So what we are going to do is that we are going to use Fatou's Lemma to get k tends to I mean we are taking k tends to infinity, as k tends to infinity, S k goes to infinity almost surely that already you have seen earlier.

And now, when psi k so, psi k is this thing, correct? If I now take k tends to infinity here, I will justify why that limit can go inside expectation. Thing is that since this is a non-negative random variable, so I can take use of Fatou's Lemma, so Fatou's Lemma says that limit of integration of non-negative random variable. So that thing is greater or equal to integration of the limit of the integrant.

So, say expectation is our integration here correct so, here this limit was taken so expectation of limit of integrant is less than or equals to the limit of the expectation but the limit of expectation so this is an expectation, its limit is less than or equals to this term anyway and that does not involves k so, this is directly written as it is.

So, what we have obtained is that now we are free from that localization that S k that localization was used just to ensure so that we can do the analysis whatever we wanted to do. Now, we are free from this S k so we have obtained that this expectation of this thing is less than equals to this.

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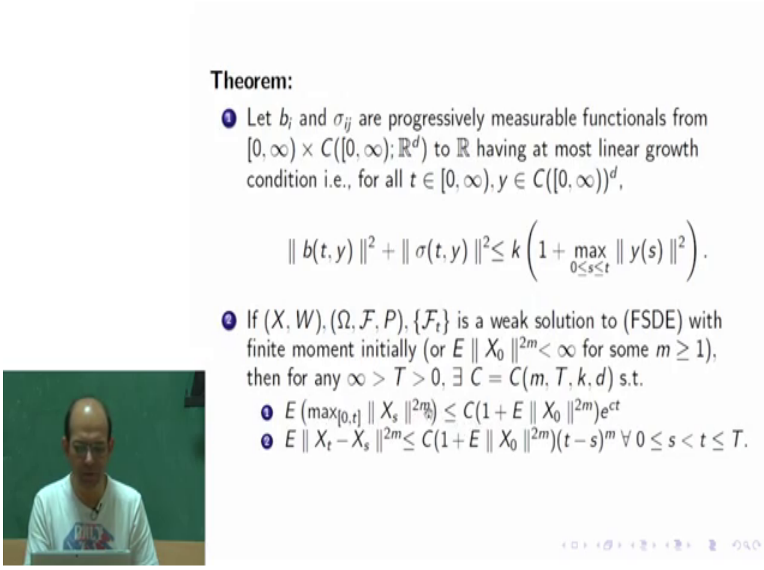

So, let us see that this is the property what we have aimed to prove expectation of this thing is maximum of X s to the 2 m less than equals to this thing that we have proved now. This thing I am not doing in here because this would be easy consequence from, it is not far different from this part but anyway, I am not doing it here because only this part would be required to prove the general Feynman-Kac formula.

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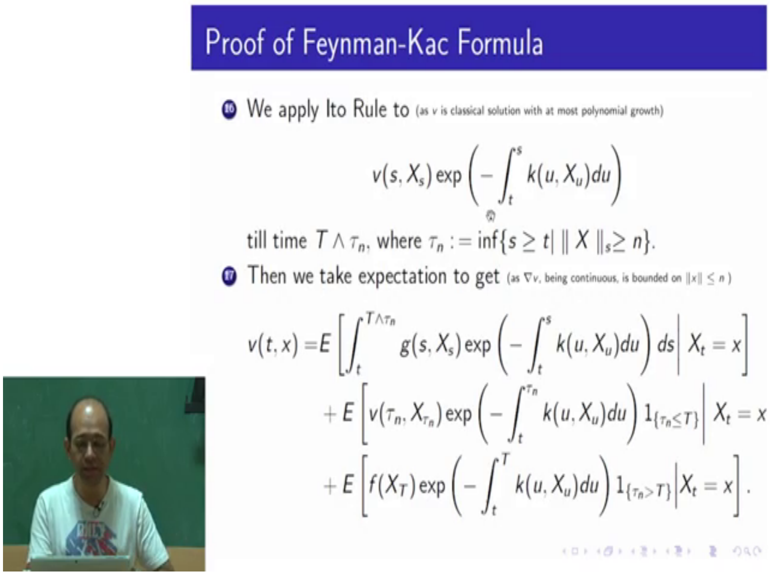

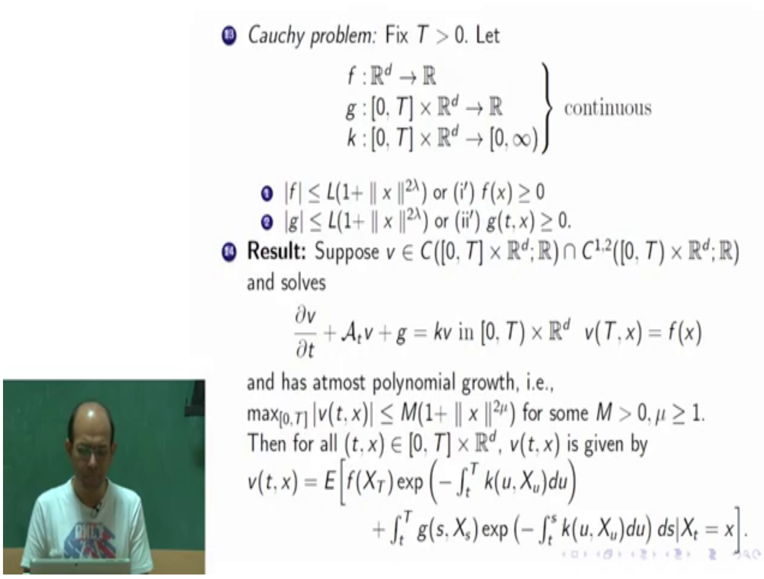

So now start proving Feynman-Kac formula. So, what do we do is that so let us since we have seen that formula before, so let us recall what was the problem. That this was the Cauchy problem, okay. So in this Cauchy problem we have variable coefficient second order partial differential operator.

And if this has a classical solution having at most polynomial growth then that classical solution has this stochastic presentation, okay so this is the thing we want to prove. And now as we have mentioned earlier that anyway we are going to use Ito's formula for this. So, but what function we are going to do, we are going to take for applying Ito's formula.

So we take this v e to the power of minus t to s this thing, we consider this function for taking Ito’s formula. So whenever one applies Ito's formula one should be careful that whether this is applicable here. So, first thing is that whether this is sufficiently smooth so that you can operate all the operators.

Second thing is that what the local martingale you are going to get, so is it a martingale? Otherwise you would not be able to take expectations 0, etc. So, these are the things we need to be careful, for smoothness we do not need to worry because v is already assumed to be the classical solution of the PDE, so it already has the desired smoothness, it is already in the domain of the operator.

Since the partial derivative of v is continuous and continuous function on compact set is bounded. So, the till the stopping time the domain is in the ball of radius n, so, del, the integrant of the stochastic integral would be bounded so, the stochastic integral would be a martingale.

So, with respect to time, so s is the time variable, correct, t is fixed here. So, so till time capital T minimum Tau n we are going to do that means s varies from small t to capital T minimum Tau n, what is Tau n? Tau n is again a sequence of stopping times, it is a localization sequence. So, these are like the first exit time for nth ball. So, this n is, so this is correct, so this is the sequence we are starting. So all other earlier inside actually d, okay so tau n.

So, when we do this, we take Ito’s formula we just do along this, on this then what are we going to get? We get v of T minimum Tau n and here s is equal to T minimum Tau n minus v of small t X t because this is the initial point small t X t, and then here, small t to t that is 0, t to the 0 is 1 so this thing.

With this difference is equal to then the partial derivative of v with respect to time and then integration with respect to the time and then partial derivative of v with respect to the space and integration with respect to the process and then quadratic variation and then higher order partial derivative of v and the quadratic variation of X, everything would appear. And from that, we already we know that what are those terms and those terms are already arise in the expression of the operator script A.

And we know that v is a solution of this equation, already this del v del t term and A t v term would appear in the Ito's formula, and this will be evaluated at s comma X s integrated from small t to s minimum capital T minimum Tau n. But this is v solve this equation, so this sum would be equals to k v minus g. And since I have taken k here also when I would take partial derivative with this time variable, I would also get 1 minus k there.

So, this term would be g plus n then minus k and this k would cancel also so, what would remain is just g. So, this is so, what would remain just g so, I am not writing the whole details it is an easy exercise one can do. So, this g and as I have mentioned that terminal time T minimum Tau n that would appear here.

So, the t minimum Tau n is written in two terms. So, these are the things which we have written earlier in details for the actual Feynman-Kac formula proof and many other proofs so, here we are skipping those details, writing the terms which comes up after the manipulations. So, g appears as g s X s e to the power of minus small t to s k u X u d u.

So this whole thing is a function of s and that is integrated d s. And then the left hand side term, there we had so, remember that here, I mean this is minus g, but I am writing g here so, left hand side is also a multiplied with negative sign. So, left hand side should we be v Tau n X Tau n minus v t x so minus sign is omitted here, so v t x.

And then minus v Tau n that is taken on this side so, you get plus v Tau n on this side, but I should have what capital T minimum Tau n, but here I am taking integrator functional of Tau n less than equals to T. So, when Tau n is equal or less than T, then I can ignore T minimum, I can write down Tau n here, but that is not the whole thing, the complement thing also I should write down.

So, that is the case when Tau n is more than capital T, then Tau n minimum T is actually capital T so, this is the way. And then v the solution satisfies the terminal condition so at capital T terminal condition v capital T minimum T comma X capital T is exactly f of X capital T, so that is the reason that we get f here. And here say for this case I get small t to T Tau and here I would get small to capital T, clear?

So, this is obtained using the Ito's rule on these function, you can take this as an exercise to actually get this thing all this manipulation together, you have to write down 2-3 steps to get it.

So now for I mean g is there, for g we need to look at the growth condition, g has a growth condition correct because here, for here this g. So, this g has, so here this is the result, this g has this growth condition. So, here when I stated actually I stated two different theorems compactly together, actually I followed this Karatzas and Shreves book they do that so I have also kept it, but it is not a very good idea to state two different theorems in the same place.

It is saying that, if these two follows then also you get this result otherwise, if these two follows then also you get this result, but if these two follows then you use some different techniques in this, then you get different techniques, but those different techniques are not too very different. They can be managed together if that no confusion arises.

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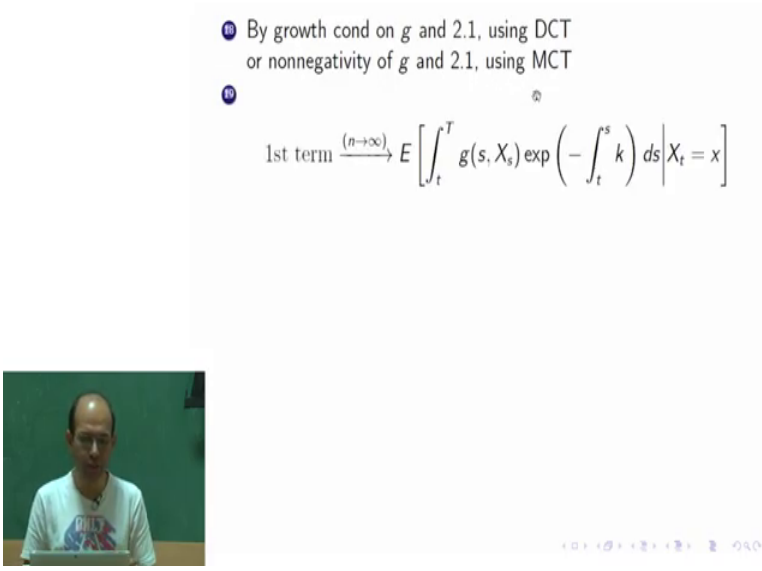

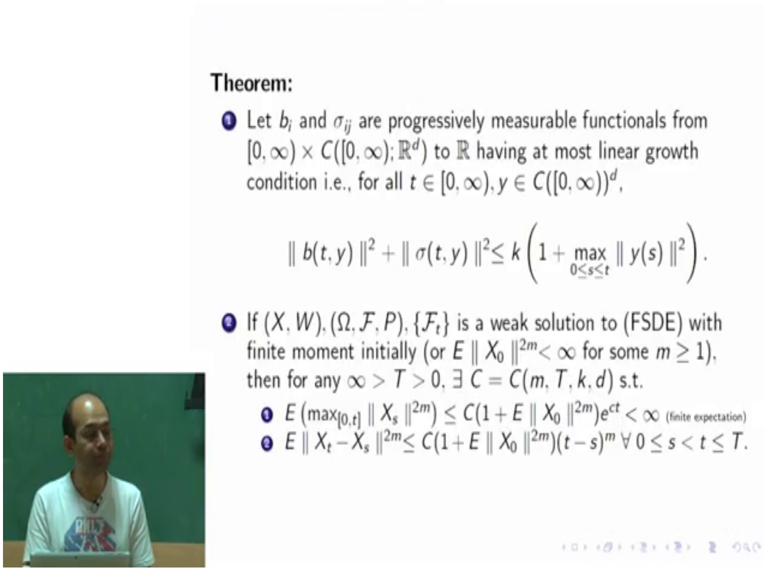

Okay, so here using the growth condition of g and 2.1 that means this thing which you have proved for the first half of the lecture. So, these two things we are going to use now, and we use a Dominated Convergence Theorem. Here I am writing that thing that if the other alternative condition holds that we do not put a growth condition on g, but non-negativity, then we can use monotone convergence theorem. So, how do you do that?

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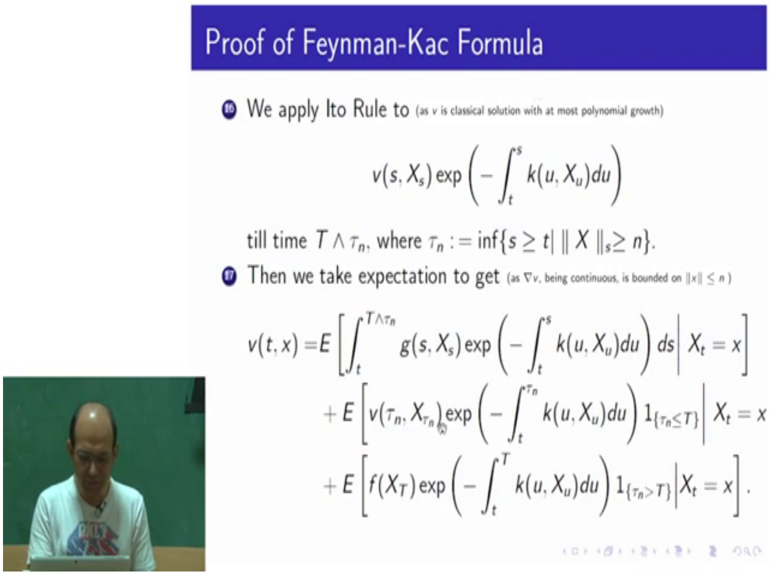

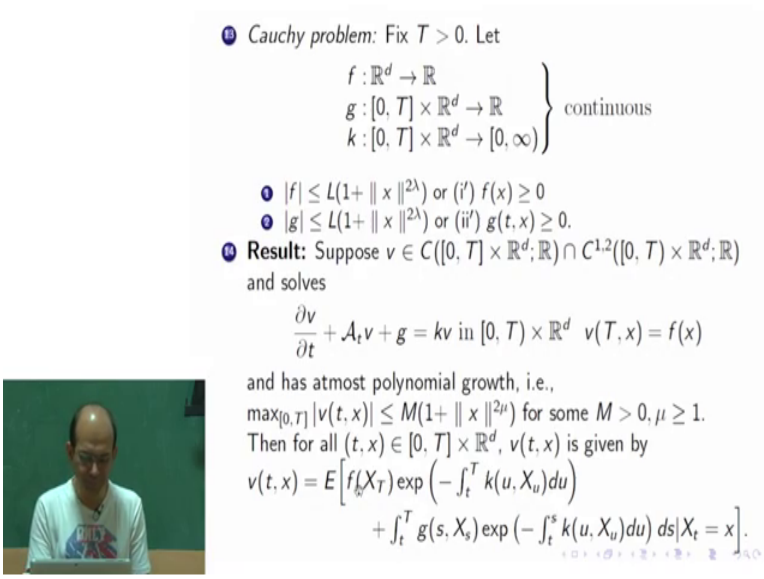

So, term by term so, first term, second term, third time. So, you remember that, we need to prove v t x is equal to this thing, f of XT times e to the power of this thing, you know that already appears here, only thing is Tau n greater than capital T appears here so that we need to just get rid of.

And this integration, this thing appears so, here also I have exactly same integrant see g s X s e to the power of minus small t s k, okay, so e to the power minus small t s k only thing is that here I have capital T minimum Tau n here I have capital, so I need to get rid of this Tau n, this Tau n from here, then I will leave it.

And this term I need to send it to 0, I have to argue that this can go, this should go to 0, because the basic intuition is that as n tends infinity, Tau n increases to infinity, then with probability 1 that the Tau n is more than T. So, this random variable converges to 0 almost surely. But the limit is outside, how to take this limit inside that is only the challenge we are going to discuss.

And then when you justify then we can do this. So, we know that what are the things we need to do with all these three terms, first term we need to send to this term, second term we need to send to 0 and third we need to send to this thing.

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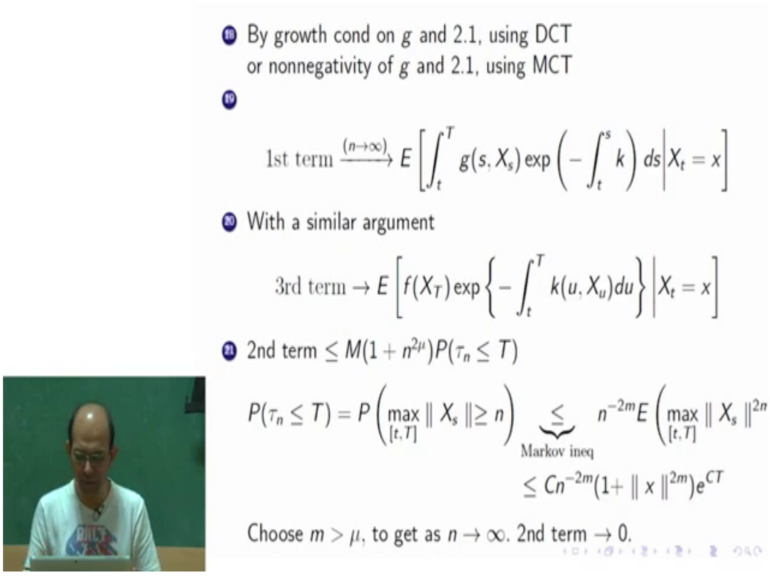

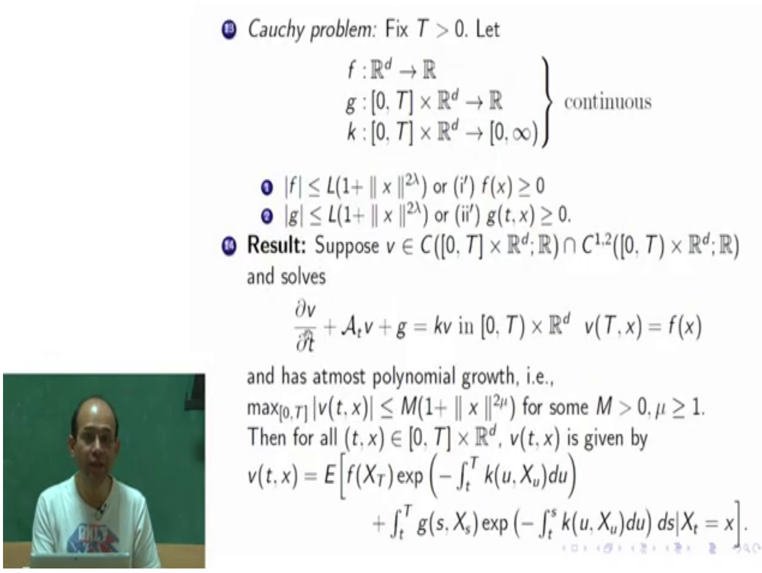

So, let us go back to this thing. So, first term as n tends to infinity we are pushing this limit inside, how do we do this because, we are using dominated convergence theorem, how can I do this? We can do this because g has at most this polynomial growth condition and then we are using this 2.1.

So, here this thing is saying that it is dominated by this term. So, first term so since g has at most polynomial growth, so, we can use that condition there so and this is a anyway bounded by 1 because k is non-negative. So, we can take the limit inside to small t to capital T this thing d s, where s is running from small t to capital T. So that would be the limit here and then in the same in similar manner for third term, so that I need to send there because that Tau n tends to infinity, so as n tends to infinity.

So, one indicator function of Tau n greater than capital T is converges to 1 almost surely. So, only thing is that I need to assure that this I can do but f has at most polynomial growth. So, I can use this I mean 2.1 here to conclude that this third term converges here, and for second term we need to show that it converges to 0.

So, here also what we do is that Tau is less than equals to capital T is same as probability that exists with the norm of X s for small t to capital T that is greater or equals to small n, otherwise it does not happen correct, because, sometimes norm of X is more than n between small t to capital T that is why the Tau n is smaller than capital T because Tau n is the exit time of the ball of radius 1, radius n.

So, now this probability we can use Markov's Inequality, Markov's Inequality says that expectation of a non negative random variable is less than or equals to I mean say a times probability that mod X is more than a. So, this probability is here and now, so this is more than n and then if you write down n here, but then this is same as to the power of 2 m to the power 2 m to the 2 m here.

So, n to the power 2 m times probability of mod x to the 2 m is greater than equals to n to the power 2 m is less than or equals to expectation of this to the power 2 m. So that is the thing we are using here and then we take this 2 m this side to get 1 over n to the power 2 m. So now for this we again use the inequality of what we have obtained earlier that the constant times e to the power of C T.

However, I have n to the power minus 2 m, as n tends to infinity this converges to 0. So, we can choose this m sufficiently large, say larger than mu, mu is the polynomial growth polynomial degree this thing. So if we choose m is more than this so, then we can there is some because here we are second term is less than equals to m times 1 plus n to the power 2 mu into probability of Tau n less than equals to capital T.

Now, because second term has this function, function v here so see, when we write down here the polynomial growth we have written the expression, yes, see, so this v, the classical solution of the Cauchy problem what we have considered satisfies at most polynomial growth of some particular order that order is taken as 2 mu, so I am talking about this mu here. mu is just greater or equals to 1.

So this mu we are talking about and then if we hear take m larger than this mu, then as n tends to infinity the second term goes to 0. So here it is actually n, so norm of X to the power 2 m. So here you have n to the power of 2 mu correct. So this grows and these decays, if m is larger than mu, then together multiplication goes to 0. Okay so this is the end of the proof of the Feynman-Kac formula. Thank you.