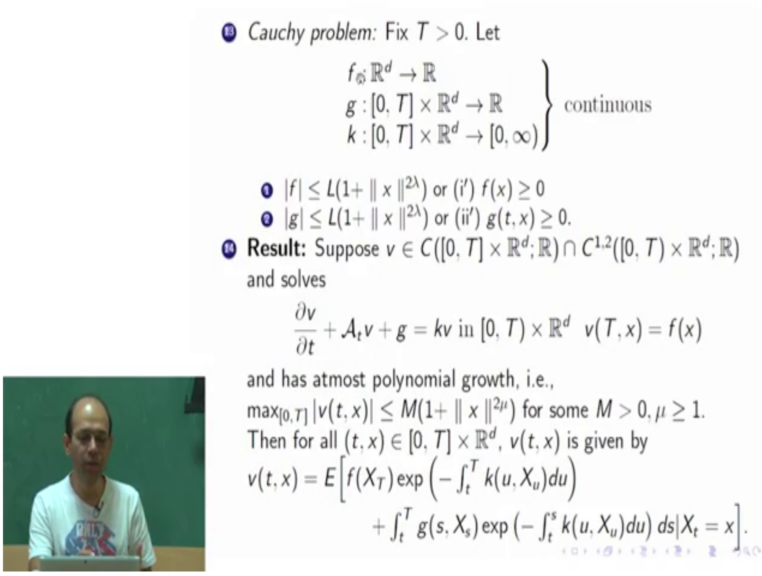
Probabilistic Methods in PDE
Professor Dr. Anindya Goswami
Department of Mathematics
Indian Institute of Science Education and Research, Pune
Lecture 49
Cauchy Problem with variable coefficients Feynman-Kac Formula Part 1

(Refer Slide Time: 00:22)



We discuss Cauchy problem for general elliptic operator here, I mean variable coefficients second order operator. So, for that what we do is that we as before we consider terminal data which is taking I mean multidimensional R d to R, and g is the function which is taking time and the space value and giving me a real value.

And k the potential function is non-negative and this is taking time t and space value to give me a non-negative value and this f g k are assumed to be continuous. In addition to this I also have further two more conditions that mod of f is less than or equal to L times 1 plus norm of x to the 2 lambda for some lambda.

So, this is I mean is similar, is equivalent of saying that, f has at most polynomial growth, so here the degree of polynomial will be 2 lambda I mean at most 2 lambda polynomial degree of growth.

So, here I am not writing explicitly that is a function of x, but is understood that I want to mean that and here also this is a function of t and x. So, modulus of these things is less than or equals to L times 1 plus norm of x to the power 2 lambda. So that means that f and g since they are functions of two variables correct, time and space and space is unbounded.

So, on that space I need to define that what type of function it is because just continuous functions you know all continuous function on unbounded space R d, they are not bounded and then sup norm would not work so. So, here we are clarifying the class of functions what we are considering here.

So, we are considering that f and g are having at most polynomial growth in the space variable. But on t we do not need to put growth condition, because this is compact set, correct, it is bounded. So for every other x so here g as a function of time it is bounded actually. So, these are the conditions on f and g, then this result says that if v is a continuous function on this 0 to t R d, so on this time and space it is continuous real valued function.

And also once continuously differentiable in first variable that is time variable and twice continuously differentiable with respect to the space variable. And also this v solves this equation, this equation is del v del t plus A t v plus g is equal to k v here.

So, here this A is the operator which comes which is associated with the SDE, which SDE? That d X t is equal to b of t X t d t plus sigma of t X t d W t, I am not talking about here the functional SDE, I am just talking about the SDE here, and is equal to k v. So, these equations should be satisfied by this function v on these closed 0 to open Tcross R d. So here and at the terminal time the unknown v should satisfy the terminal condition that it should match with f, okay, good?

And then in addition to that we also want to look for solution in a particular class, the class of functions with at most polynomial growth. Now thing is that, if you assume that we have a solution to this problem in this class, then for every time t and x in this set 0 to capital T this thing across R d, the solution of this problem has a particular expression, it has a representation.

So, this can be written as using conditional expectation, where that I would be able to use solution of stochastic differential equation which stochastic differential equation? With the same, I mean which involves exactly the same parameters which appear in definition of the operator script A.

So, then for every t x in this time and space domain, v t x is given by the expectation of terminal time f of capital X capital T e to the power of minus small t to capital T k u X u d u plus integration small t to capital T g s X s into e to the power of minus small t to s k u X u d u d s given X t is equal to small x.

So, here we must connect this result with the result what we have previously obtained in the Feynman-Kac theorem, we have not considered a variable coefficient general second order elliptic, second order differential operator we have taken half times Laplacian. So, then there we have obtained exactly similar expression but instead of X Brownian motion appeared, but here we have a general this operator.

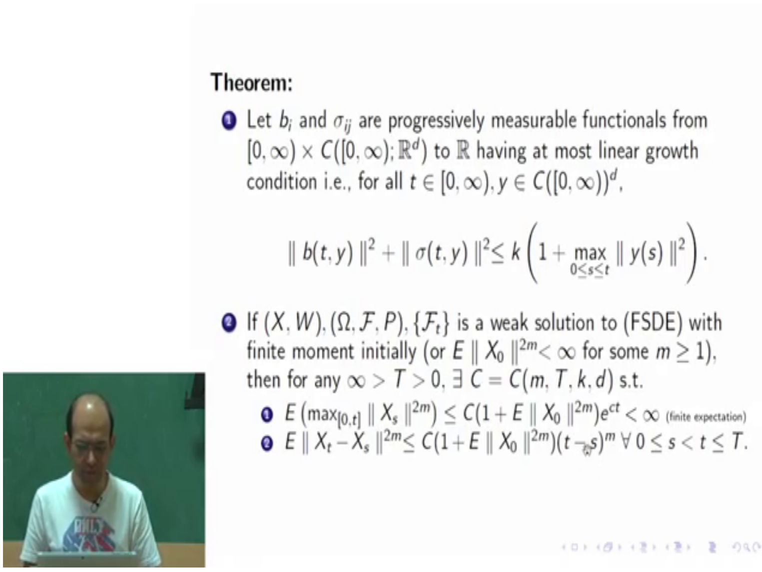
So, this involves the coefficients of which appears in SDE and therefore the solution of this problem is written as conditional expectation I mean functional of the stochastic process which is solution of the SDE. So, here trivially one can see that if at small t is equal to capital T, this satisfies this terminal condition.

So, let us check it easily so if you put small t is equal to capital T, then this part disappears. So, e to the power 0 is 1 and here this integration also disappears okay, because integral small t to capital T. So, what you would be left with an expectation of f of X capital T given X capital T is equal to small x because small t is equal to capital T.

So, given X capital T is equal to small x, expectation f of, X capital T is exactly f of x, so you would get f of x here, but inside the domain the interior why should be the expression of the solution of this equation, needs the detail proof. So, we are going to see the proof now, but before seeing the proof, we need to establish one more result that we are going to do first and then we would come back to this result and we prove this thing.

It is apparently clear that what should be the methodology because for special case where A was just half times Laplacian, there we have seen the proof that was just nothing but application of Ito's formula. Here also we are going to do that only thing is that we have to be careful for passing to the limit, appropriate manner so that things make sense.

(Refer Slide Time 09:01)



So, let bi and sigma i j are progressively measurable functionals from time domain and paths. So, here I am taking the general setting, where FSDE is considered b and sigma R functions of paths are having at most linear growth condition. So, here we are having a linear growth condition, so this is space of all continuous functions from this non-negative time to R d that means R d valued paths.

So, b is a function of time and R d valued paths, earlier b sigma were functions of t and position of the process, now b and sigma are the function of time and the whole realization of the process, so that is the kind of thing. So, having at most linear growth condition so, this is the condition we keep here. So, y is a particular member in this path space, space of path.

If this triplet is a weak solution to FSDE this triplet X W is a solution and this is the featured probability space weak solution with finite moment initially that means the starting X 0. So, there is some m, so expectation of this is finite for some m, then for any capital T finite and positive capital T there exists some C such that, so C ofcourse depends on the choice of m, T, k, d, etc such that expectation of maximum of modulus X s to the power 2 m.

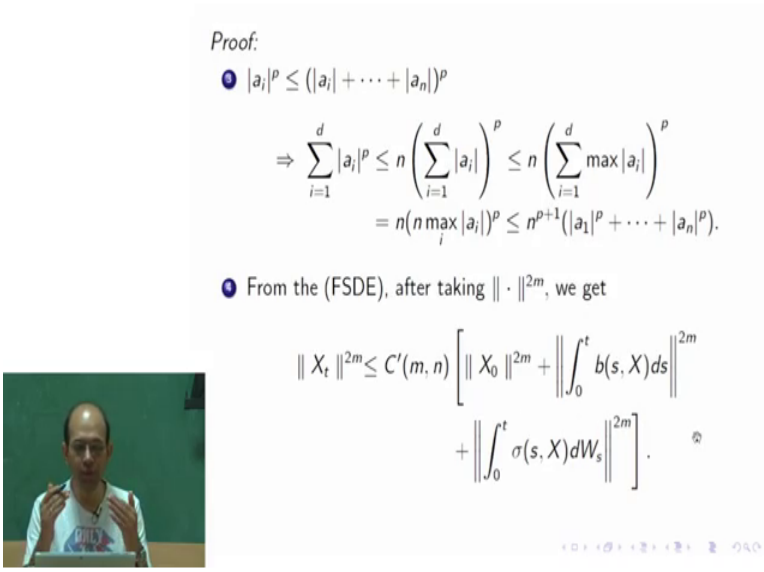
So here you see that this is 2 mth order moment. So I am writing that for not only 0, I am to saying here say s, s is running from 0 to t and then the 2 mth power of X s, I mean norm of X s and maximum of all these okay, so highest point during the path of 0 to t.

And expectation of that okay, expectation of that one can still get that less than or equal to that constant times this number, expectation of X naught to the 2 m times e to the power of C T that the C appears there. I think here by mistake is small c appear yeah but, okay and here this t is this same t as here it appears.

And expectation of small t is any number which is less than equals to capital T and then this C can be kept intact, this C does not depend on T that is very important. So, here the second result is that expectation of X t minus X s that is the increment so 2 mth order moment of increment, this is the thing is less than or equal to C times 1 plus expectation of X naught to the power 2 m which appears here times t minus s to the power of m.

So, with respect to increment also we can get some estimate here. So, these are very important things so, what we have obtained due to these properties of the FSDE. So here we are going to prove these theorems first these results and then we are going to use it. We would use actually this result extensively.

(Refer Slide Time 13:23)



So, proof starts here, for this proof we would first use this inequality. So, this is the inequality what we are going to use. So, let us derive this inequality. So, I have a 1, a 2, a n some you know real constants, the some scalars and then mod of a i to the power p is less than or equals to the pth power of the sum because this is only one particular i so, this type is a 1, a 2, etc, a n, so this for one particular thing.

For every i which is between 1 to n this is true of course because this right hand side has more number of non-negative elements. So, from here now we can sum both sides for I is equal to 1 to d. So, here we get the a i to the power p i is equal to 1 to d by summing this side, and this side what we do that this inside is sum over mod a i, i is equal to 1 to d, and then to the power p is there, and then whole thing does not depend on i if I sum over i is 1 to d, I just get n number of these things.

Actually I should have written, typo, so here it should be d, here should be d. So because there are only d number of components now, so here we would like to discuss everything in d dimensional space only because here d dimension etc. So here I should have written d here. So this is less than or equal to d times now this whole to the power of p is there.

So, what I do is that I change each and every mod a i by maximum of mod a i, so this term is independent of i now. So, sum maximum of all possible i so that I replace here, so n is as it is and to the power p is there, so, since that is a case, so I would get maximum of a i and then this sum would give me another n, but p is out outside to the power p.

So I would be able to write down then n into n to the p that means n to p plus 1 times max of mod a i to the power of p, but that max of mod a i to the power p is less than or equals to mod a i to the sum of all these because this is after for sum i naught this is achieved. So, here I have written all possible these sums, so this is a dominating this part.

So, in other words what I have obtained is that i is equal to so, here I should write d here, d here, i is equal to 1 to d mod a i to the power p is less than or equals to d to the power of p plus 1 so, that is a constant and then mod of a i to the p etc plus etc mod of a n the power of p, of course this is less than, this is trivial, but what is important is that in between this part is lower bounded by this and upper bounded by this.

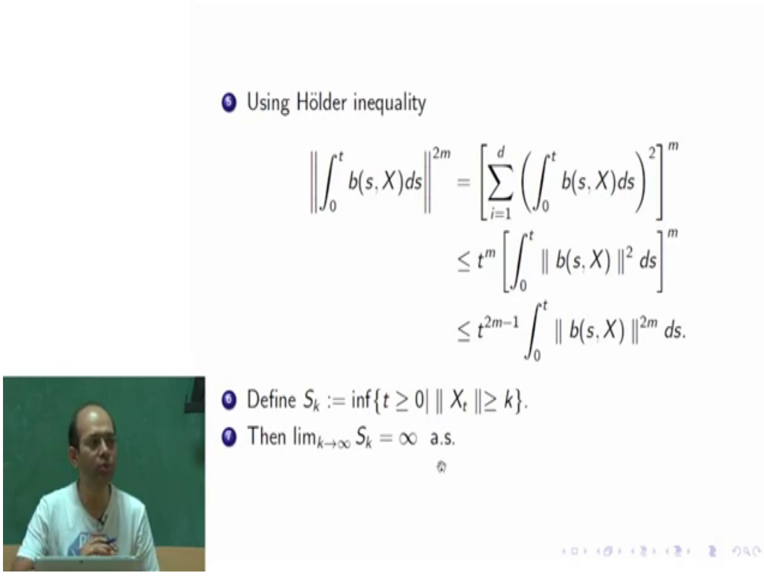
So, because this part so this would be used. So from the functional stochastic differential equation after taking the norm I mean 2 mth power of both sides of the functions stochastic differential equation. So, then we get that there are how many terms? 1, 2, 3 terms, correct? So, one term was 0^{th} term and the integration b and then another integration of sigma with respect to Brownian motion.

And then we are taking norm, etc so, we are using the triangular inequality. So, after doing all these you know triangular inequalities and these things, what we are going to get left hand side mod of norm of X t to power 2 m, and right hand side is less than or equals to C prime, this is just a constant. So, here norm of mod of X naught to the power of 2 m plus norm of integration 0 to t, b s X d s.

Here nom to the power 2 m, here plus integration 0 to t sigma of s X d W s norm to the power of 2 m. So, we are going to get this inequality which we need to analyse next. So, why are we looking at this to the power 2 m because we recall the statement I mean here we are going to prove these things, we are going to prove. So, we take this norm here for this reason.

So, now, I need to upper bound these quantities correct. So, that we do now, first I consider this term.

(Refer Slide Time: 18:52)



So we here use Holder’s inequality as expected. So here, 0 to t b s X d s here norm to the power 2 m, this is equal to summation i is equal to 1 to d, 0 to t. So, here what we are doing is that writing the Euclidean norm here, so Euclidean norm is so, yes, I can take m and 2 separately this square and then m.

So, this square is the component square and then some and then so, that is why we get this i is equal to 1 to d square of this integration and then of course, outside there is a power of m. Now, here at this stage we use Holder's inequality here, in square of this integration is less than or equals to t times integration of the square so that we use but then we take the summation inside to get square of the norm of b here inside.

So less than or equals to t to the m this norm square d s this to the power of m. So this t comes out so power m is this, so we get t to the power m here. So now here we get this norm square integration 0 to t d s and then we write down that thing, but here again we use here Holder's inequality. So, I would get here norm of b s X to the power 2 m, so I can either use Holder's inequality or I can use Jensen's inequality.

Because this power mth power is a convex function. So only thing is that integration 0 to t is not an interval whose measure is 1, so one has to you know reweight it, so by 1 over t, multiply and divide and then one applies that, one can apply Jensen's inequality there to get that t to the power of 2 m minus 1 that mean to the m minus 1 term would come here, so we would get this times integration b to the power 2 m d s. So, this thing is upper bounded by this quantity what we have obtained.

Now we go to the (second term), the third term, but before that we take this sequence of stopping times. So what is this, this is nothing but the exit time of the ball of radius k. So whenever first time, the process, the solution of SDE leaves the ball of radius k, that time is denoted by S k here.

So, here we also know that since they need large time to exit because if I write k is larger and larger, it takes larger amount of time to cross that with probability 1. So, therefore, limit k tends to infinity S k goes to infinity almost surely, so this stopping times are non-decreasing and goes to infinity with probability 1.

So, if I write down S k minimum capital T and then S k tends to infinity that would converge to capital T with probability 1.

(Refer Slide Time: 22:53)

- By taking maximum over $s \in [0, t \wedge S_k]$ on the both sides of (4) we get

$$\max_{[0,t\wedge S_k]} \| X_s \|^{2m} \leq C'(m,n)\left[\| X_0 \|^{2m} + t^{2m-1}\int_0^{t\wedge S_k} \| b \|^{2m} du + \max_{[0,t]}\left\|\left(\int_0^{s\wedge S_k} \sigma(u,X)dW_u\right)\right\|^{2m}\right] \text{ a.s.}$$

Proof:

- $|a_i|^p \leq (|a_i| + \cdots + |a_n|)^p$

$$\Rightarrow \sum_{i=1}^{d} |a_i|^p \leq n\left(\sum_{i=1}^{d} |a_i|\right)^p \leq n\left(\sum_{i=1}^{d} \max |a_i|\right)^p$$
$$= n(n \max_i |a_i|)^p \leq n^{p+1}(|a_1|^p + \cdots + |a_n|^p).$$

- From the (FSDE), after taking $\| \cdot \|^{2m}$, we get

$$\| X_t \|^{2m} \leq C'(m,n)\left[\| X_0 \|^{2m} + \left\|\int_0^t b(s,X)ds\right\|^{2m} + \left\|\int_0^t \sigma(s,X)dW_s\right\|^{2m}\right].$$

- Using Hölder inequality

$$\left\|\int_0^t b(s,X)ds\right\|^{2m} = \left[\sum_{i=1}^{d}\left(\int_0^t b(s,X)ds\right)^2\right]^m$$
$$\leq t^m\left[\int_0^t \| b(s,X) \|^2 ds\right]^m$$
$$\leq t^{2m-1}\int_0^t \| b(s,X) \|^{2m} ds.$$

- Define $S_k := \inf\{t \geq 0 | \, \| X_t \| \geq k\}$.
- Then $\lim_{k\to\infty} S_k = \infty$ a.s.

Now, what we do is that, we assimilate all these term together from the earlier slide here. So this part I am going to use the new estimate, here I would keep as it is for the time being. So, so here I take s and then I also vary s from 0 to t minimum S k. So we will worry about it little later so let us see what is there on the right hand side.

Here as before X naught to the power 2 m and t to the power 2 m minus 1 integration 0 to t minimum S k, norm b. So, here I am not writing it in full details that it is also a function of u and X, so the dependencies are omitted here just to put everything in the same line. And d u this integration where u is between 0 to t minimum S k, so small t is fixed for us, k is also fixed for us. And then we are looking at the taking maximum over all possible s running from 0 to t minimum S k.

So what we are going to get is that that would be less than or equals to this t is fixed here. So I mean earlier for s I should have get s here but here I am taking in maximum between 0 to t minimum S k, and then so here we get maximum 0 to t integration 0 to s minimum because here this is a function of s. So, you remember that here for this we got these, but here we did not get any estimates for this. So, here if you have s, you would get s here, correct.

Exactly, so that thing, but here for this case we can do that because you know it is non-negative, it is to strictly increasing, so we can actually for s also on the left hand side we can dominate it by t, but there we do not know actually. So here we have 0 to s minimum S k here to the 2 m and then max over 0 to t, clear? So this is the inequality what we obtain using the earlier things.