Probabilistic Methods in PDE
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Lecture 48
Statement of Dirichlet and Cauchy Problems with Variable Coefficients Elliptic Operators

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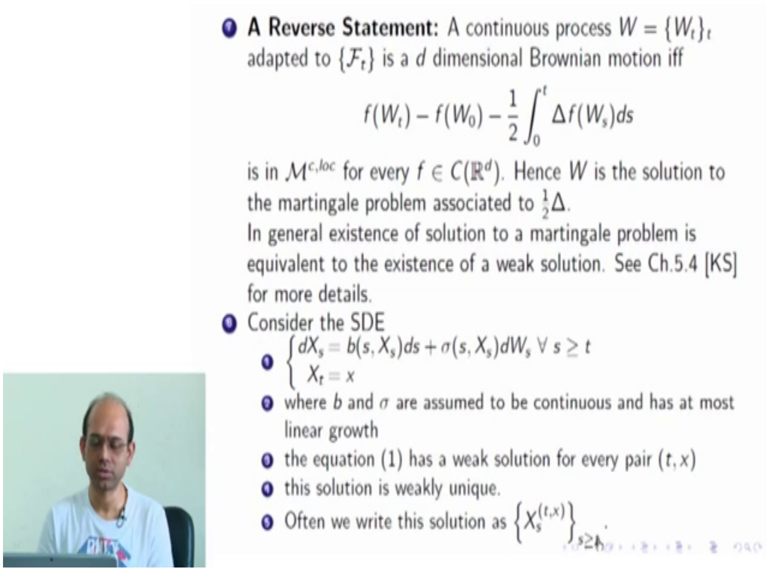

Now, we see some more definitions, we are not going to prove any theorem in this part but we are going to see more notations and definitions and some results also. The last result would be a general version of Feynman-Kac formula that we are going to prove in another lecture.

So, consider SDE, d X s equals to b s X s ds plus sigma s X s d W s for all s greater or equals to t and X t is equal to small x. This is not a new SDE, we have seen some this type of SDEs before but only thing is that every time when we wrote SDEs, we have written all the time s is greater than equal to 0 but here that we are writing s is greater than equal to t so, that means t is a particular positive time and a particular member in the space and then at that time at that position x we now start looking at the evolution of the SDE.

And if we consider such type of SDEs, which is not starting from origin not starting from 0 time, but initial time is a different time t. So, we would like to often denote the solution using this t comma x also so, we would like often like to carry this notation t, x there, here b and sigma has to be continuous and has at most linear growth all this condition as before.

And then this equation this one has a weak solution for every pair t, x. So the proof of this would be exactly as before, so weak solution and then this solution is weakly unique and we are going to denote this solution as X superscript t comma x s and s is greater than equals to t.

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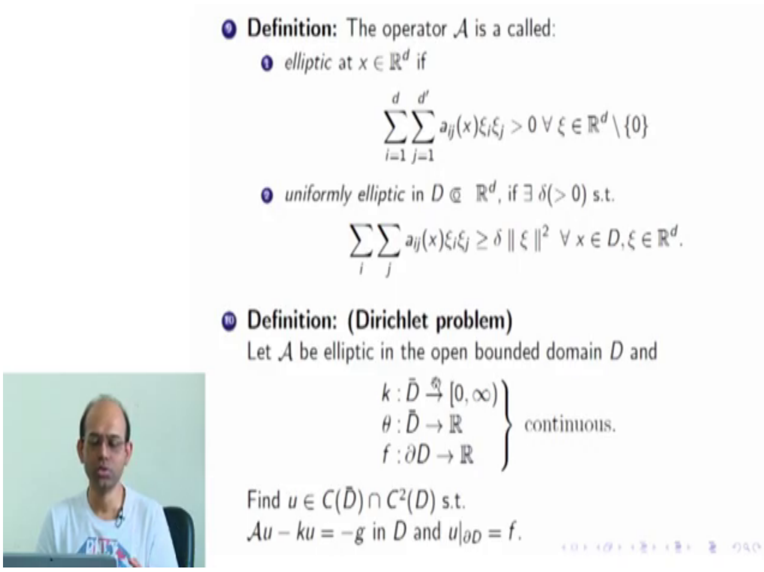

The operator A is called elliptic this is another definition, definition of some classes of operators, an operator A is called elliptic if at each and every x, a i j x sigma zi i zi j so, this is nothing but the quadratic form. So, zi transpose a zi that is written in terms of double summation.

So, here, yes, there is a typo it should be j is equal to 1 to d, a i j sigma zi i zi j greater than 0 for all zi in R d minus 0. Of course zi, when zi is origin ofcourse this is 0 and there is nothing big deal, but when zi is non-zero, then this is non-zero. This is a strong condition on the coefficient a, and if that condition is true, then the operator A is called elliptic.

We call a as uniformly elliptic in a domain D which a open domain D. So, this is like open subset of R d if there exists one positive delta such that the same quantity ai j x zi i zi j is greater or equal to delta times norm zi square, this is a stronger condition. It does not matter what x is in D, so for all x in D it is true. This implies this but the reverse is not true.

So, now, what is the additional problem in this general setting? Remember we have discussed Dirichlet problem earlier also, we have spent quite some lectures on that about the solution of the Dirichlet problem and then when can you assure existence of a solution. So, for the sufficient condition, we looked at the regularity property of the boundary of the domain.

So, here what we do is that again we are revisiting Dirichlet problem because there in that Dirichlet problem only Laplacian operator we have considered, since Laplacian operator is associated with half of Laplace, half times Laplacian is with Brownian motion so, only Brownian motion was used to study those problems.

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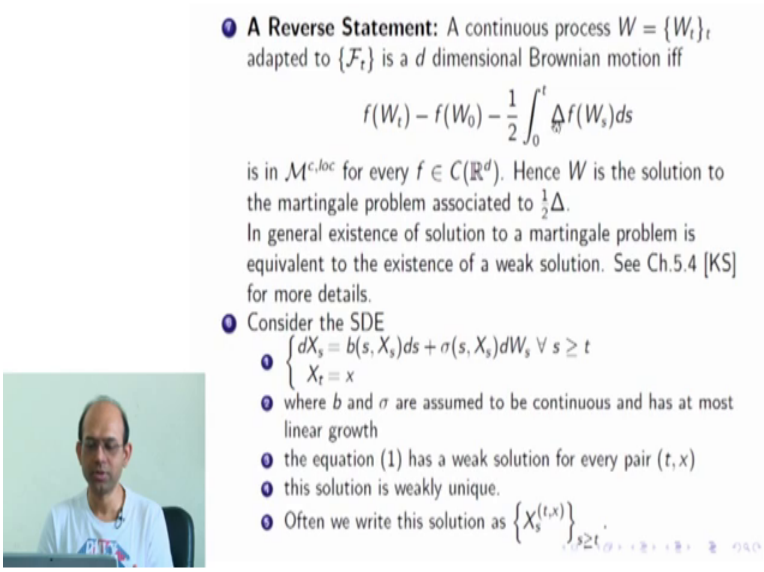

And that connection is clear from here because this Laplacian and Brownian motion are connected correct, because this is a solution of Brownian motion is the Martingale solution for the halftimes Laplacian operator.

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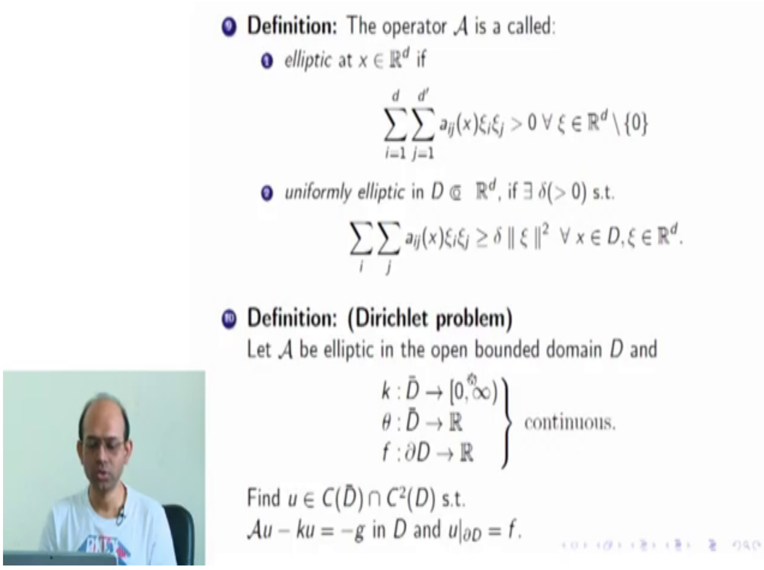

However, here what we are going to do is that we are going to look at a different more general equation where the operator A would appear, that would have maybe variable coefficients and that may also have first order term. So, A u minus k u is equal to minus g in the domain D and on the boundary, it is having some boundary data that u on the boundary of D is equal to f.

So, we ask that what is the solution of this equation, find u which is continents on the closer of the domain and twice differentiable inside the domain such that this is true, where this k, g are given this way that so here this is g, there is a typo, it is not theta.

So, k is a non-negative function defined on the whole closer and g is a real valued function defined on the whole closure of D, and f is defined on the boundary of the domain, this is the real function, and all these functions are assumed to be continuous. So, that is the settings what we are going to discuss so.

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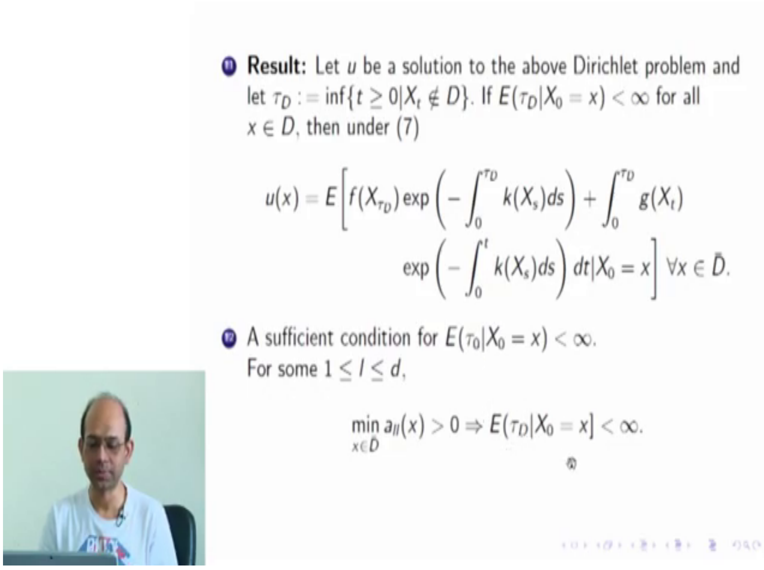

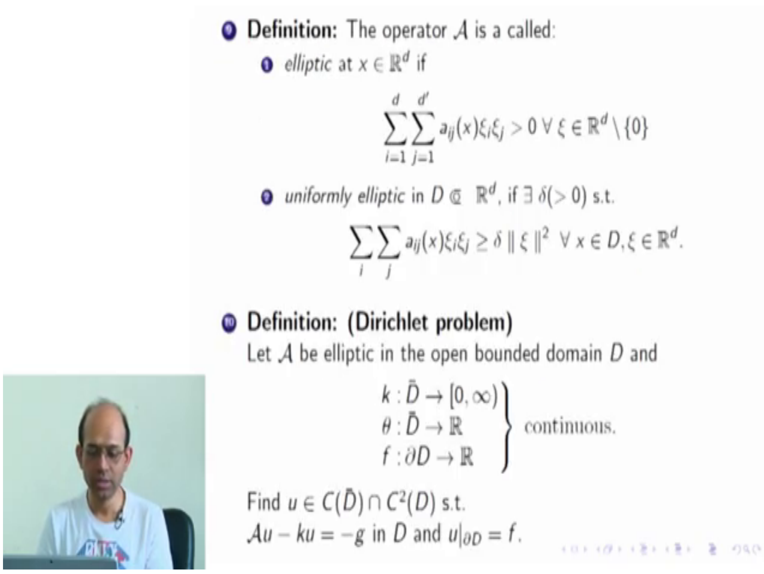

I am not going to give a proof of existence of this but just stating this result that let u be a solution. So, here we are not asserting anything about the existence, we are assuming that if it exists that let u be a solution to the above Dirichlet problem, then we are going to write down the solution in terms of conditional expectation of functionals of the process X which solves the SDE, here in this case maybe FSDE.

Here this A is coming from the SDE part. So where we do not have the parameters depending on the whole path functional, but only at the position of the path. So, that is my A, so to distinguish between this A and A (prime) use these two more notations. So, this X is coming from that SDE so, then the solution would be written in this manner.

Where tau D is first exit time from the domain D, so here it is written this way. And we also assume that conditional expectation of tau D is finite for any starting point x from domain D so, under those assumptions, the solution of Dirichlet the problem u of x is we can written as conditional expectation.

Expectation f of X tau D times e to the power of minus 0 to tau D, k of X s d s plus integration 0 to tau D g of X t times e to the power of minus 0 to t k X s d s d t. So, here t is running from 0 to tau D and here s is running from 0 to tau D and here s is running from 0 to t okay, given X 0 is equal to x for all x in the D closer so, that defines a function on the D closer.

So, these all these things are well defined because of this property I mean this is essential thing to assume, because if tau D is not finite value, expectation is not finite valued then we might not be able to do much. Actually tau D should be first finite valued and then this little extra condition than that, that expectation is finite. So if tau D is not finite valued, of course one cannot write down these things correct because then X infinity makes no sense.

Now we write down a sufficient condition for this thing, there is a typo, this is tau subscript D sufficient condition for expression of tau subscript D given X 0 is equal to x is less than infinity so, this expectation is finite. When can it happen? So, one would ask that can I find out any condition on the coefficients of the SDE under which this can be assured?

Yes, that can be but of course, the domain also plays an important role. So, here a l l x and you run over all possible x and take the minimum, okay. If the minimum of a l l x, for all x in D closer so, minimum of a l l x on the D closer set, if that minimum is positive and this is true for some l in 1 to d that would imply that expectation of tau D so, here it is correct so, expectation of tau D is finite.

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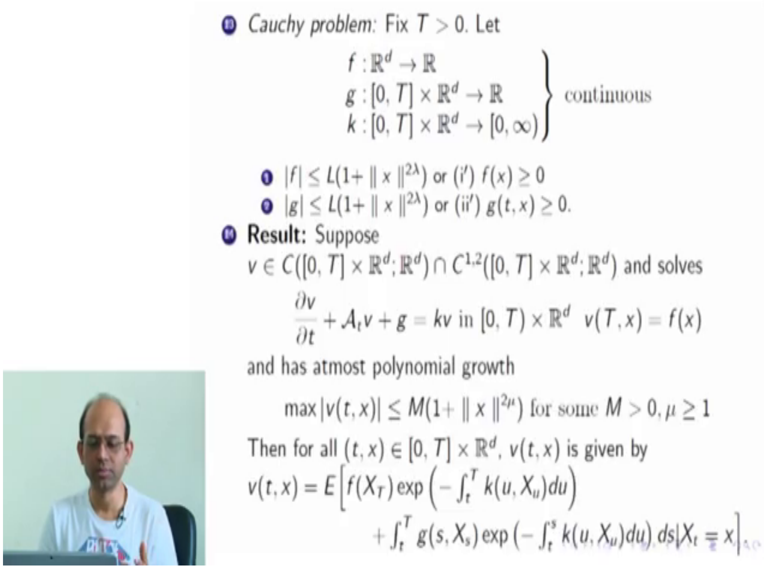

So, now I state another results, so here I have studied this result for Dirichlet problem we had a proof. Now here we are stating another problem the Cauchy problem in this more general setting where it is not only coming from the Laplacian, but the variable coefficient, general second order elliptic operator.

So, here let f, g, k are continuous these are exactly having the same meaning like g is Lagrangian, k is potential, and f is terminal condition and then these are at most polynomial growth, f has at most polynomial growth. So here 2 lambda, so polynomial degree 2 lambda, and the g is also at most polynomial growth. So, either this condition one can choose, alternatively one can choose that okay both are non-negative.

So, for the sake of statement let us just concentrate on that, I mean just let us just read this part. So, mod f and mod g are at most linear at most polynomial growth and then we can assert the following. What is this? This is saying that suppose v is a continuous function or the function of time and space and Rd valued, typo here, it should not be R d should be R only, there should not be d here.

Because this v is a real valued function. So v is a real valued function continuous function and once continuously differential with time and twice continuously differential with space and solves del v del t plus A t v plus g is equal to k v, this k, okay. So, see here I am not writing the details like g is also a function of t comma x k of t comma x, v of t comma x, but that is understood.

So in this is true in the domain that close 0 to open T Rd and the terminal time V T x is equal to f of x. And this has at most polynomial growth the solution. So, we here suppose that this Cauchy problem del v del t plus A t v plus g is equal to k v has one classical solution with at most polynomial growth, that mod of v is less than equal to m times 1 plus mod x to 2 mu so, this is the polynomial degree, for some capital M greater than 0 and mu greater than equals to 1.

Then, we can write down the solution v in terms of conditional expectation. Then for all t comma x in 0 to capital T cross R d, v t x is given by this formula, v of t x is equal to expectation of f of X capital T e to the power of minus small t to capital T k of u X u d u plus integration small t to capital T g of s X s into e to the power of minus small t to s k u X u d u and integration is with respect to s d s, s is running from small t to capital T here given X t is equal to small x.

Here, we are going to see its proof in the next lecture. So, few things are very pretty much clear here, when we write down say small t is close to capital T. So then say small t is equal to capital T if you put here, so e to the power of minus 0 because this is integration or this capital T to capital T so that would be 1 and then here this integration will vanish.

So it will be f of X capital T given X of capital T is equal to x, so that is f of x itself, so you would get this terminal condition to be true. If you remember that when you have stated Feynman-Kac formula, we did not have X as a solution of SDE, but we had Brownian motion itself here, why? Because there the operator was just half Laplacian here.

Since it was just half Laplacian, only we have taken the Brownian motion, but here it is a general variable coefficient, second order elliptic operator. So, we are looking at the solution of the SDE, so that solution we are picking up here to write down the solution of this equation. This equation is written as expectation of the function of the solution of the SDE, where the coefficient of the SDEs do appear in this operator, so this is the statement of general version of Feynman-Kac theorem, thank you very much.