Probabilistic Methods in PDE

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Lecture 47

Functional Stochastic Differential Equations

Today we are going to see another class of differential equation that we call Functional Stochastic Differential Equation, okay. The main difference of this from the other one is that here the coefficients like b and sigma does not only depend on the present state of the process, but possibly the whole process, whole timeline.

So b and sigma therefore are functionals. They are not functions of time and space point, but the time and the whole path. So now we first look at some definitions those are mostly recollections and after that we are going to see the statement of the differential equation.

We are not going to state any particular theorem about existence or uniqueness of solutions to those functional of stochastic differential equations. However, we are going to talk about some properties. So, if solution exists then what are the things we can talk about there?

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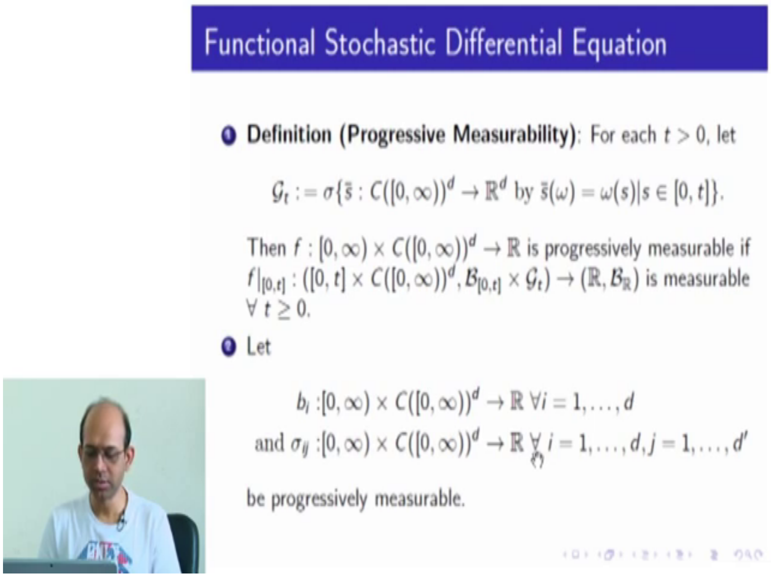

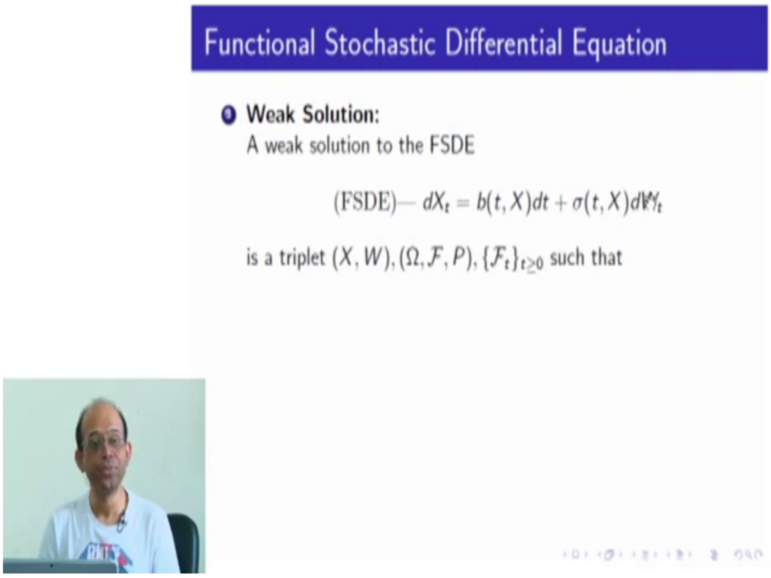

So we start here, so we recall the definition of progressive measurability, so for each time small t positive, let this sigma algebra G t is generated by all these maps s bar which is taking the path to the vector point R d in the following manner, that s bar of the whole path, the path is here written as omega s bar of omega is equal to omega of s for any s in 0 to t.

So here, let us start again here so, t was a fixed time positive, for this fixed time t positive we have the collection of time constants from 0 to t, this closed interval. For each and every s we consider the map s bar, which map so s bar takes the whole path to a point in R d and that is evaluation map, evaluation of the path at the point s. And hence this is a collection of maps which takes a whole path to a vector point.

And we are talking about the smallest sigma algebra under which these collection of maps are measurable, so each and every member in this collection is measurable, the smallest sigma algebra containing due to that is called the G t, okay. So here we have not introduced any particular probability.

Since we have not put any particular probability measure on this set and did not start with any particular sigma algebra, although we know that the natural sigma algebra would come from the Borel sigma algebra on this, but we have not done that. So, we do not also discuss about that whether G t is complete or not, so whether this measure on G t is complete or not? So, we just have this filtration.

And this gives me filtration, correct, because now I would change my t in increasing order and then this collection would grow and the sigma algebra would also become larger and larger so I would get a filtration, okay that we call script G filtration. Then function f from 0 to infinity to and Cartesian product with the space of paths this is space of path, correct, the space of all continuous maps from this non-negative interval to R d, correct?

Because if I write down just C 0 infinity that means set of all continuous path which is taking value in real number but to the power d that means I have Cartesian product d number of Cartesian product of such things so, I would get path in d dimensional space, so C 0 infinity R d.

So, f is a map you assume that it is taking time value and the whole path, the d domensional path and giving me a scalar, real scalar, so this is a function. We call these as a progressively measurable if the following is true that f this function if you restrict in the interval 0 to t. So what does it mean? Because this function f this functional is taking value t and also the whole path.

But if I look at the domain only not 0 to infinity but 0 to t, and on the second component I do not put any restriction. So, this is the restriction of the whole function in a smaller domain. And there that function is defined now on 0 to t across the whole path space exactly the same as before, but here it is smaller.

And then this with certain sigma algebra, which sigma algebra should be chosen natural one, the Borel sigma algebra on the interval 0 to t and on this, the sigma algebra what we have already introduced earlier, so for each and every since I have fixed t. So, I know for this t G t I have already defined. So, I take Cartesian product of these two sigma algebra.

I mean this is not, I mean Cartesian product of two sigma may not be sigma algebra. So, I mean that product sigma algebra, so product sigma algebra coming from these two sigma algebra, so this to R B R, this is measurable for each and every t, okay. So, if my functional f is such that this restriction on 0 to t is measurable for all t, then we are going to say f is progressively measurable map.

Now, here we introduce some notations. Now, since we are going to state stochastic differential equation which is functional in nature so functional stochastic differential equation. So, they are as I have mentioned earlier, the coefficients would looklitte different there. They would not depend only on the present state value but the whole path. So here b is a function of time and functional whole path.

So continuous functions on 0 infinity to R d and is a real valued, so I am here presenting all the one dimensional thing, and sigma i j is 0 to infinity times C 0 infinity, R d to R be progressively measurable so there is a typo. It should be R to the power d. So let me start from here, so there is a typo. So this b i is taking value time t and the whole path, d dimensional path and should give me a point in d dimension R d. And the Sigma i j is taking a time point t and the whole path.

And also should give me a proper matrix here. So that matrix multiplication this with Brownian motion gives me d dimensional point. However, I mean here we did not fix what should be the dimension of the Brownian motion, if we choose that it is 1, then sigma is just a d dimensional vector so I can write d, otherwise I should write down is a matrix d cross d prime.

So, here I have written that, so here for every i this is not a typo, it is correct, because here I am not writing b, I am writing b i. So, each and every components are scalar or real valued, Sigma i j is also real valued. So, these are progressively measurable that is the assumptions on b and Sigma.

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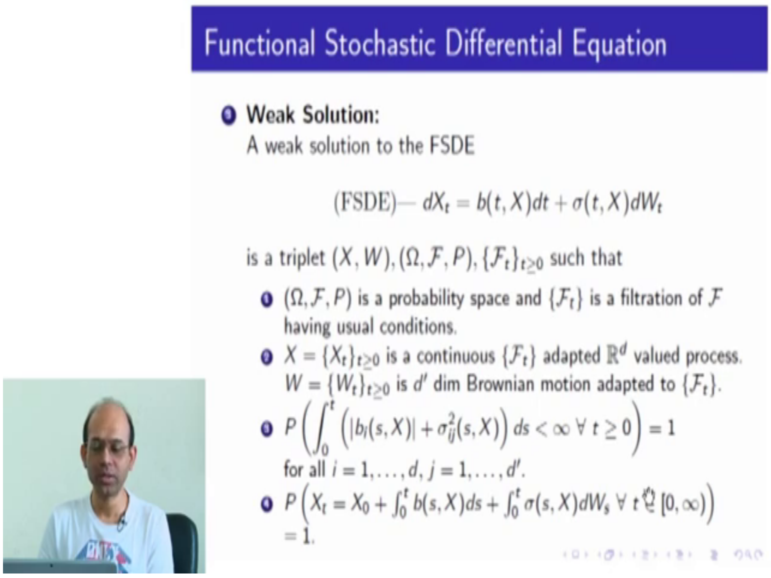

Now, a Weak Solution, we are defining, a weak solution to the following functional stochastic differential equations. So, this looks like this that d X t is equal to b of t comma X d t plus sigma of t comma X d W t. So here, earlier that SDE looked like d X t is equal to b of t comma X t dt plus Sigma of t comma X t d W t, but here b is functional in the whole path.

The whole path is present here, is not evaluated at time, also sigma depends on the whole path. And this FSDE we call in short FSDE because functional stochastic differential equation. So, for this FSDE we say that something is unique solution of FSDE is a triplet as we have also obtained a triplet for SDE so here also we are going to talk about this triplet.

This triplet is this X, W, omega F P and F t. So, this is the Brownian motion and the process solution and this is the probability space, this is a filtration, so the filtered probability space and then this solution. So, here the explanation of these symbols are given below, so omega, F, P is the probability space and F t is a filtration of F having usual conditions.

X is a continuous F t adapted R d valued process and W is d prime dimensional Brownian motion adapted to F t, and probability that mod of b plus sigma i j square, so this integration, this would be a random variable but this random variable is finite with probability 1 for all t. So, basically this random variable is finite for all t that happens with probability 1 for each and every choice of i and j.

So, this is also quite similar to before this condition, and this condition also we have seen earlier. Only thing is that earlier we have written in a different manner, it was b i s Xs, but here its subscript is missing. So, probability that X t is matching with this is 1. So, this says that choice of this W and this particular X if you put in this integral equation, then we would get equality for all t with probability 1 that is the definition of weak solution.

So, here we are not asserting any sufficient condition for existence or uniqueness of each solution, we are just stating what do we mean by weak solution of this particular FSDE.

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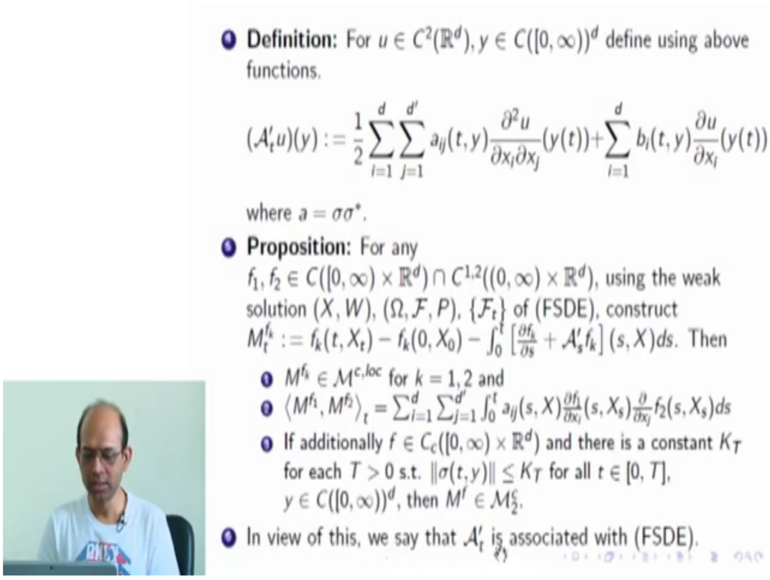

Now, here we are going to look at some other definitions. So, this is a generator and this is actually an operator we are going to introduce. This operator is introduced using the coefficients which appear in the functional stochastic differential equation. So let us just look at this operator, this A prime t, so that operator if it is evaluated at a function u which is twice continuously differentiable, this is a function of R d, it is not a functional, it is just a function.

It is taking R d value and gives just one real number and then y is a d dimensional path. So, this operator when it is operated on these real valued function, but evaluated on the whole path y, that is defined as half times this double summation sum over i and sum over j, a i j t y. So, here y is path and a is actually coming from sigma-sigma-star a, say sigma depends upon the whole path, so a also depends upon the whole path.

And it also depends on time and then double derivative, second order partial derivative of u with respect to x i and x j appear and that thing is evaluated at y t. Here I cannot write y because you know u is just a function of vector, u is just a function of the present location of the path. So, this double derivative this second order partial derivative of function that the path y t.

So, that is evaluated here and then this submission b i t y sum over all i. So, here also b depends on the whole path y and then del u del x i of y of t, so that is how we introduce this

A prime operator and this of course, depends on t because here t appears. And then this operator when we act on u, so we get a functional which depends on the whole path.

So, that path y is here appearing and that appears on the right hand side also. Now, we see a proposition which says that, if this FSDE what we have introduced in the earliest slide has a weak solution then how that solution is related to this operator? For any f 1 and f 2, which is continuous function of time and space and once continuously differentiable in time and twice continuously differentiable with respect to space f 1 and f 2.

So, here u was just a function of space variable, correct, twice differentiable, but here we are taking f 1 and f 2 which are all functions of time and space both. So when we take this f 1 and f 2 so using the weak solution, so that means we are assuming that equation FSDE is a weak solution so, whenever it has a weak solution so that we write down X W this triplet, correct.

We construct the following process, so 1 and 2, so k I am writing in general, so k is equal to 1 or 2. So from f 1 so like f k so k is equal to 1 is f 1. So from f 1 we get M f1 t, so this is a new process, this is defined, so, M f k t is defined as f k of t comma X t minus f k of 0 comma x 0 minus integration 0 to t del f k del s so this is saying that with respect to the first variable partial derivative plus this a A prime s f k.

So, that thing would be evaluated at is s X d s. Remember that when A prime is acting on a scalar valued function, we get a functional. So after acting, this is we get a function, so that is evaluated at X s so X is the whole power. On the other hand, this del f k del s, the first order partial derivative of that would be just a function of time and space, s and X.

And then the meaning of this would be just del f k del s, s X s, I mean it only take the value of X s. So, this is I mean this way we are writing, but it is understood that here we have also a composition of evaluation map. So, to be more precise I should have written actually composition of evaluation of second component at time at the time point which is given the first component.

So, s X s. so this is a function of t because 0 to t integration, so as t changes this changes, so this and it involves the random process X so, this is a stochastic process. Here also this involves X t so, we get on the right hand side is stochastic process running, with running variable t. And therefore, M f k t is also a stochastic process we get, and then what we obtain or what we assert in this proposition, ofcourse, I am not giving you the proof of this proposition, I am just going to state this proposition.

It asserts that this new process M f k is a continuous local martingale, okay, M f k is a continuous local Martingale for each k is equal to 1 and 2. Furthermore, we can talk about the quadratic co-variation of M f 1 and M f 2 is summation, sum over all possible i and j or integrations 0 to t and then see that for the time variable, this things do not appear here, only this thing appears, so here a i j is there.

So that appears here a i j s X, del f 1 del x i s X s del f del j s X s, okay d s, s is running from 0 to t. So, that process would be the quadratic co-variation of M f 1 and M f 2 in addition if f is a continuous function with compact support. So then and there exists a constant K T K subscript T for each T positive such that this sigma, so Sigma is here.

So, sigma actually gives rise to a, so sigma is bounded by K T for all small t which is between 0 to capital T, so uniformly bounded on this. So, and y is of course the path so d dimensional path. Then this M f, the way it is defined here would be square integrable continuous martingale.

As we have discussed in the last lecture that for just the, I mean general SDE that this somehow captures the dynamics therefore, because whatever the dynamics we get due to this X t after this subtraction of this operator acting on f, we get a continuous, we just get a continuous local martingale, correct?

So, and here for this specific case when f is a compact support etc, they are actually indeed that local martingale is a martingale. So you get a square integrable continuous martingale. So, this is the proposition which in a similar manner as we have done for standard classical SDE that this says that this operator is associated with the SDE, with FSDE so, this is associated with the solution of FSDE in this manner. So in view of this we say that this A prime t is associated with FSDE

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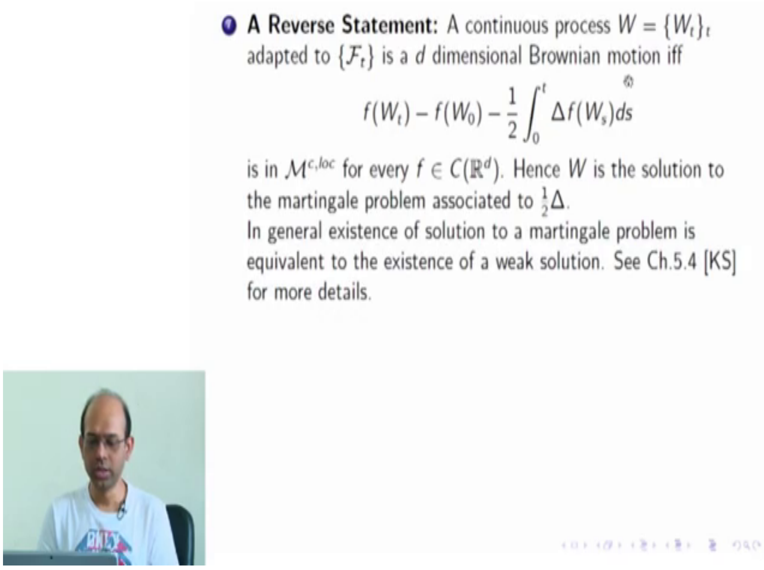

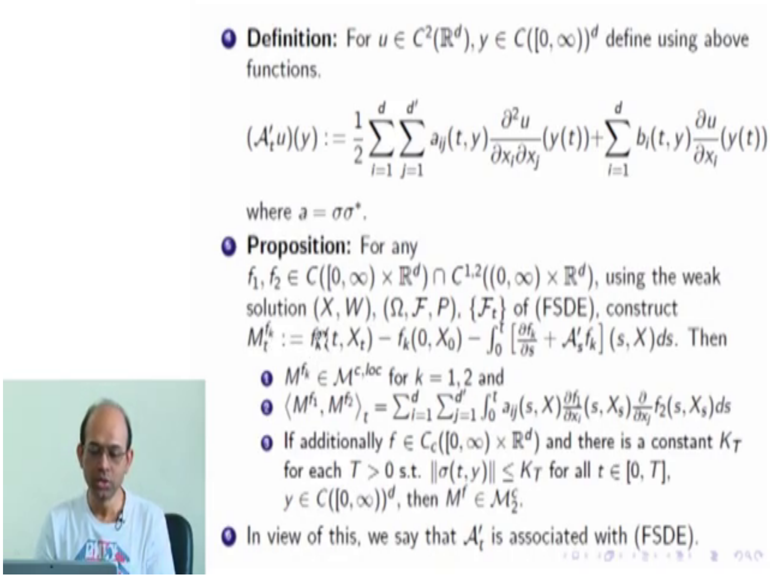

So now we look at a reverse statement, so the reverse statement is a continuous process W adapted to F t, so here, what type of reverse statement we are looking at? We are looking at a very special case, much special than this, this is FSDE but we are looking at only SDE case that we are going to say that the reverse thing is also true. That means here what we are saying?

We are saying that if we have a weak solution of the FSDE, and that we plug in here, and here, sorry that we plug in here and that FSDE has the coefficients, from that coefficients I come up with this operator, if I do all these things then I get a local martingale. But the reverse statement is that if I just take the coefficients of FSD and cook up this operator and ask for which process X the resulting thing would be a local martingale, no matter what f I choose, so that is a reverse statement.

So, this is the reverse statement for a very special case, only for Brownian motion, a continuous process W adapted to the filtration F t is d dimensional Brownian motion, if and only if, f of W t minus f of W 0 minus half integration 0 to t Laplacian f of W s d s is in M c loc continuous local martingale, for every f in CR d.

So, imagine that we have one particular process W such that, that Laplacian we just have minus half Laplacian, this half Laplacian we have. So, then this process f of W t minus f of W 0 minus half 0 to t Laplacian f of W s ds as a function of t. So, this process is a local martingale, continuous local martingale no matter what f I choose. So, here I have, there is a typo it should be C 2, twice continuously differentiable.

No matter what twice continuously differentiable f we choose, we would always get this as a martingale, continuous local martingale. So, such process we call that is a solution to the martingale problem associated to half Laplacian. So, if W is a Brownian motion of course this becomes a continuous local martingale that we have seen, but the reverse thing is also true.

Because a continuous process, this is a d dimensional Brownian motion if and only if that means if this becomes a martingale, I mean continuous local martingale then W is a Brownian motion okay so this is just you know if and only if condition. So in this case, what do we say? We say that the W is the solution to the martingale problem associated to half Laplacian. So, what is the martingale problem?

A martingale problem is basically if I have an operator here, and if I fix also the class of path, path space and then on the path space, we look at the coordinate process so, W is just the coordinate process and we do not know that what is the measure there but we are looking for the measure on the path space such that the resulting this process in T becomes a local continuous, local martingale for each and every choice of f.

And if that problem has a solution, the solution would be what? The solution would be measure on the path space, then that measure is called the solution of the martingale problem.

Here we actually say it in a very colloquial language that the W is the solution of the martingale problem, what does it mean?

That the law of Brownian motion is which is actually probability measure on the path space, which we also we call as Wiener measure, so that is the solution of the martingale problem associated with half Laplacian.

So, this is I mean we are not heading to go into more details about martingale problem. So, in general existence of solution to a martingale problem is equivalent to the existence of a weak solution, you can see the chapter 5.4 so, chapter 5 section 4 of Karatzas and Shreve for more details. So, that gives more details about this treatment, not only for only the Brownian motion, but for the SDE and FSDE also, okay so now I go to the next part.