Introduction to Probabilistic Methods in PDE
Professor Anindya Goswami
Department of Mathematics
Indian Institute of Science Education and Research, Pune
Lecture 46
Weak Solution Differential Equations: Weak Solution

Now, we come to the discussion of weak solution of the stochastic differential equation.

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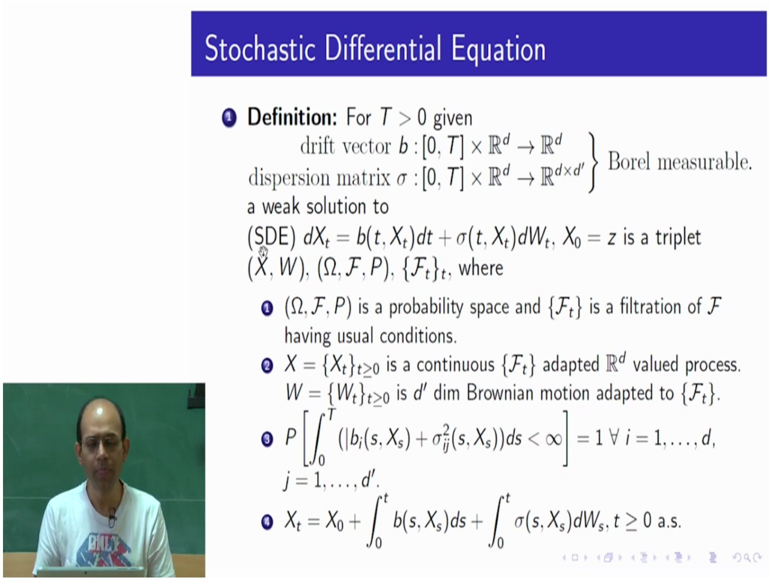

Here we again recall the stochastic differential equations under our consideration. So, here we take capital T positive drift vector is this b as before, a function of time and space and is the vector, drift vector and then dispersion matrix is sigma, that is also a function of time and space variable. And that gives me a matrix, a rectangular matrix d cross d prime and these maps are Borel measurable in appropriate sense. So Borel measurable, that means you take a Borel Sigma algebra here, so you endow this also with Borel sigma algebra, so inverse image of Borel set under this map is also Borel set here.

So that is the meaning of this. So, a weak solution to the stochastic differential equation dX t is equal to b of t Xt dt plus sigma of t Xt dwt. And initial condition is X0 is equal to z, this z is a triplet, this triplet. The first component is X, W, this pair, 2nd is the probability space and third thing is the filtration. Okay, so why are we doing this thing is that for a strong solution, we were given a probability space and on that we were given a particular Brownian motion.

We were just required to check whether there exists any stochastic process on the same probability space such that that solves that SDE. But here it is not the case. Here we are not specifying any probability space, we are not specifying any Brownian motion, we are just asking that, is there any probability space omega F P, any filtration F t and any Brownian motion on that, such that there exists some stochastic process X such that this is true. So, here the scope is quite broad.

So, here, it is very clear that a strong solution is also a weak solution, because in a strong solution everything is given. So, that means, those are there. So, that means given that SDE, there is everyone can get it. So strong solution is also weak solution. However, a weak solution need not be a strong solution. So, here if you want to understand the difference between the interpretation of weak and strong solution, it is as follows. That for weak solution, we are just looking at the law of the process.

So, what we have written is just the law. So we are just specifying the law, here Brownian motions law is important okay. It is not that we have specified a particular Brownian motion. So that a process X for which we are going to get this type of relation with Brownian motion. However, so for strong solution is concerned, there I mean you have a fixed Brownian motion and you are asking that whether that equation is satisfied with that Brownian motion.

So strong solution is relevant for constructing a stochastic process. Because that would give a stochastic process on that probability space. So, the meaning is here written explicitly here omega FP is a probability space, F t is a filtration of the Sigma algebra F having is well conditioned, that means that this filtration is drive continuous and complete. And here this X is a continuous process, which is adapted to this filtration and this W is a Brownian motion of dimension d prime, this d prime and adapted to the filtration Ft.

And also that when we have this X and W together, this B s X s sigma square ij s X, so this is a stochastic process. And then if you integrate that from 0 to t with probability 1, this integration will be finite. So this would be a finite valued random variable. I am not saying this is a bounded random variable, I am just saying a finite random variable. And this X should be the solution of this SDE on this thing.

So that is written here that X t solves this, that if you integrate both sides, so X t is equal to x 0 plus integration 0 to t, b of s X s d s plus 0 to t integration, Sigma s, Xs dWs for all t, 0 or more. Okay, so this is true for all t, of path wise, so this is true with probability 1, almost it is outside. So what does it mean that if this has a weak solution, this thing, then when you get the weak solution X and Ws, this triplet, then by fixing this filtered probability space and this W X is a strong solution on that.

So, remember, earlier what we were told, a strong solution is also a weak solution. However, if you get a weak solution, then from this weak solution, you would come up with probability space, suitable probability space and Brownian motion. And if you do that, then on that X is a strong solution. However, that does not necessarily prove that every weak solution is a strong solution. I mean, every SDE, which has a solution that has a strong solution, no it does not.

Why, because here to get a strong solution does not matter what probability space you start with, you are always assured to get one solution. But here it is not the case. If an SDE has a weak solution, then then you come up with a particular choice of Brownian motion and filter probability space and with this particular choice, you would get a strong solution. Is it clear? So, this idea would be clearer with the following example.

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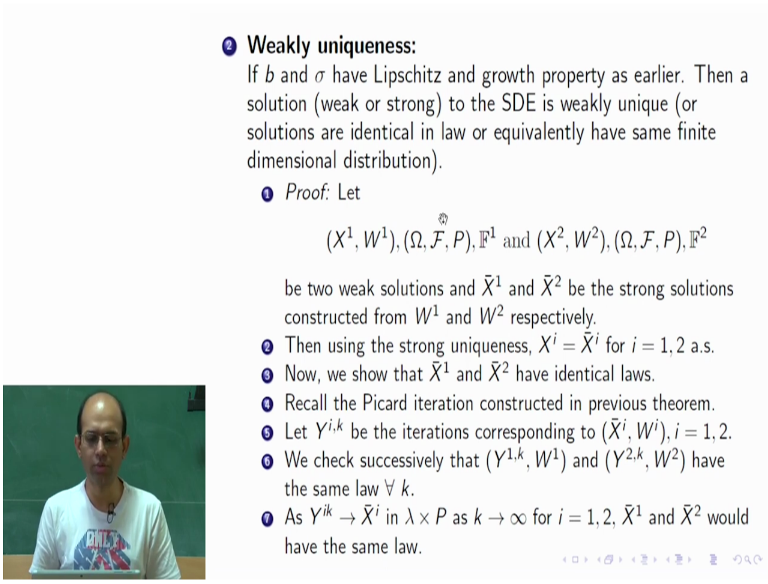

So that that is coming in the next slide. So, let us talk about weakly uniqueness. So, this is a theorem actually. So, it is saying that if b and sigma, the drift and dispersion matrices , these have Lipschitz and growth property as earlier, there is solution, does not matter whether weak or strong to the SDE is weakly unique. What does it mean that that the solutions are identical in law or equivalently has same finite dimensional distribution.

So, as you remember that when I have written that is SDE, I was talking that if no probability space is given, then this says only about its law, because Brownian motion's law is known. It is saying that small increment of X with respect to time t is same as this product of, I mean b of this the drift coefficient b t X t times dt. So, this is like a speed, velocity kind plus I mean this coefficient multiplied with the change in Brownian motion, small change in Brownian motion.

And small change of Brownian motion is known to us because this you know Brownian motions increments are normally distributed with mean 0 and variance of the time difference. So, here this precisely gives infinitesimal description of the law of the process X. So what it says is that does not matter what setting we are fixing, we are strong solution.

But so if we have SDE and b and sigma are in that particular class, then SDE admits uniquely weak solution, is weakly unique. So, here the proof goes. So, let X 1 W 1, so this is a triplet, this is another triplet. So, assume that there are two different triplets for a given SDE. So, both are weak solutions. What do we need to prove, we need to prove that X 1 and X 2 have the same law.

So, here for our convenience we have fixed same omega same probability space in both cases. So, let these be 2 different weak solutions. Now, since this is a weak solution, so, as I have mentioned earlier with this filter probability space and this particular Brownian motion X 1 behaves as a strong solution there of that SDE, similarly, X2 is also strong solution there. Now, okay between solutions and X 1 bar, X 2 bar be the strong solutions constructed from W 1 and W 2 respectively.

With this W 1 and this omega F P and W 2 omega F P, these are 2 strong solutions we consider. Then using the strong uniqueness okay how can I assure the existence of strong solution, because b and sigma satisfies the sufficient condition for existence strong solution, what we have already seen earlier. So then X1 bar, X2 bar we have obtained and now strong uniqueness we have seen also.

So this Xi is same as X bar i, that means X1 is same as the X 1 bar, X 2 is same as X 2 bar, with probability 1 almost surely they are equal. So, basically this justifies the comment I made that okay, this can be viewed as a strong solution of the equation under this particular choice of filter probability space and the Brownian motion. Next what we do is that we now concentrate on these strong solutions X bar 1 and X bar 2.

We show that this X bar 1 and X bar 2 have identical laws. Why do we expect that because after all both are coming from the same equation, same SDE, but with 2 different Brownian motions. But that needs a detailed proof that we are doing here that is just to know, what I have said is just is a intuition. So, recall the Picard's iteration, what do we have constructed in previous theorem to prove existence and uniqueness of strong solution of SDE. So, in the Picard iteration, we have constructed a sequence of processes Y 1, Y 2 etc.

So, we recall it that thing, so Y 1 k, k is 1, 2, 3, 4 etc. So, this process converge, we have shown that this process would converge to the strong solution. So, this will converge in probability. So, here we construct this Y k be the iterations corresponding to X bar i and Wi. So, we check successively now that this Y 1 k and this Y 2 k. This Y 1 k is equal to 1, 2, 3 etc are constructed using this W 1 and this is using W 2.

However, W1 and W 2 has the same law and, and this Y 1, Y 2 are constructed iteratively. So the beginning at zeroth step both are having the same law first level of it would have the same law because equation is exactly the same only W1 is replaced by W 2 here. So, both would have the same law for each and every k. So, this process has same law with this process. Whenever I say law I mean that I mean they would have same finite dimensional distributions, same law for all k.

And we also have seen there that this Y, this iterative processes converges to the strong solution in probability, in measure, so measure is on time domain it is Lebesgue measure, on omega domain it is probability measure. So here this converges to X 1 bar, this converges to X 2 bar in probability and both are having the same law. So X 1, bar X 2 bar the limits would also have the same law, because convergence in probability preserves the convergence in law, it is same as convergence in law.

So this is X1 bar and X 2 bar would also have the same law. So, what is the end result therefore, that X 1 bar and X 2 bar has the same law and X 1 bar, I mean they would have this identical laws. And that implies that X 1 and X 2 also have the same law, because both are X 1 is equal to X1 bar and X2 is equal to X2 bar. So, that establishes our anticipation that two different weak solution of same SDE would have same laws.

Or in other words, that SDE if I write down without mentioning the probability space and which particular Brownian motion, just the SDE, that uniquely determines a particular law of stochastic process, provided that equation has a weak solution.

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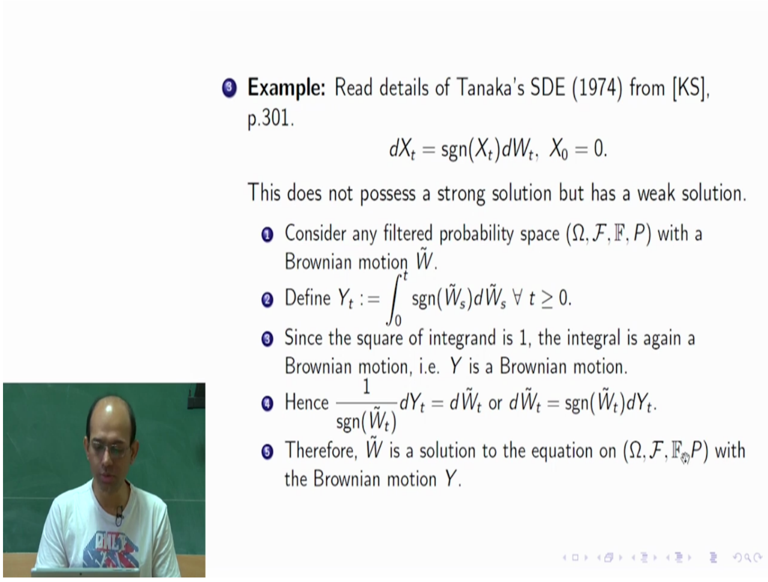

Now we come back to the example what I have promised that I would discuss. So this I am not giving the full details of this particular example, because here one has to show that for this example, this SDE does not admit strong solution in general. However, that part we are not presenting here, one can read the details from the book of Karatzas and Shreve's book, in page number 301. So, this is the book what I am referring to most of the time. So, it is there as a reference book in the course details.

So, this does not possess a strong solution, but has a weak solution. So, this is the thing we are going to show in details that it has a weak solution. So, the construction is as follows. So, when I say that this equation has a weak solution, I mean that would be able to come up with the triplet. So, consider any filter probability space omega FP, here F is a sigma algebra and this bold F is filtration of the Sigma algebra with a Brownian motion W tilde.

So, here you may think that okay why am I coming up with a fixing a probability space in the beginning, because after all there is no such hope I have promised. The thing is that this is not the Brownian motion you would work with, we are just for the convenience of discussion we are starting with something but we are going to change the Brownian motion, we are going to see actually this would not work here, we are going to see that.

So, now, with this W tilde we define a new process Yt that is integration from 0 to t sgn of W tilde s d W tilde s, so that for all small t greater or equal to 0. So this is a stochastic process we define as integration of this function of Brownian Motion and with respect to Brownian motion.

This integrand has modulus 1 all the time. When that thing happens, then that integral is also a Brownian motion, that one can get from the Levi's characterization of Brownian motion, one can find out the quadratic variation of Y that turns out to be t and that uniquely determines that okay that Y is also a Brownian motion, there is a characterization of Brownian motion.

So, we would get Y is also a Brownian motion. So, here we are writing this thing, this since the square of integrand is 1, the integral is again a Brownian motion, that is why it is Brownian motion. So, now how do we do that instead of this integral way of writing, we write down the differential manner. So, d Yt is equal to sgn W tilde t d Wt. So, we write down here d Yt is equal to dW t and Sgn W tilde t, but that we have to divided.

We can divide because this is nonzero anyway so one can say of course, W tilde, when this is positive sgn is 1, when it is negative sgn is minus 1, but it is 0 sometimes, so then sine of 0 is 0. One can do that okay, that okay, this is, I mean one can actually, when this is 0, we can write the sine of 0 is equal to 1, instead of making it 0. Otherwise, anyway, that would I mean does not matter what I have put value, because the time point when W tilde becomes 0, is of measure 0.

So on a measure 0 set of time, this becomes 0. So it does not really matter what I define sgn of 0, so I can define sgn of 0 is equal to 1 to prevent 1 over 0 here. So 1 over sgn W tilde t d Yt is equal to dWt. So this is written here. So 1 over sgn of a function is same as sgn of the function. So dW tilde t is equal to sgn W tilde t dYt, it is written. Now we recognize that here Y is like matching with W and then W tilde appears both sides like X appears here. So, what does it mean, it means that with this particular filter probability space, if I started with some Brownian motion and from there we have articulated one another Brownian motion Y.

So, if I take this Brownian motion on this filter probability space, then this W tilde becomes solution of this equation. So, that proves that this equation has a weak solution. But this does not actually prove that it does not have a strong solution, it is a little rather long proof. Not very long, actually, but that one can find from this book, that proof relies on the fact that from here I mean that proof is by contradiction actually.

So, if that exists one can, it has a strong solution and one can show that the filtration generated by X is strictly larger than the filtration generated by W that is a contradiction, but we are not giving the details here. So, we summarize that W tilde is a solution to the equation this, the weak solution with the Brownian motion Y.

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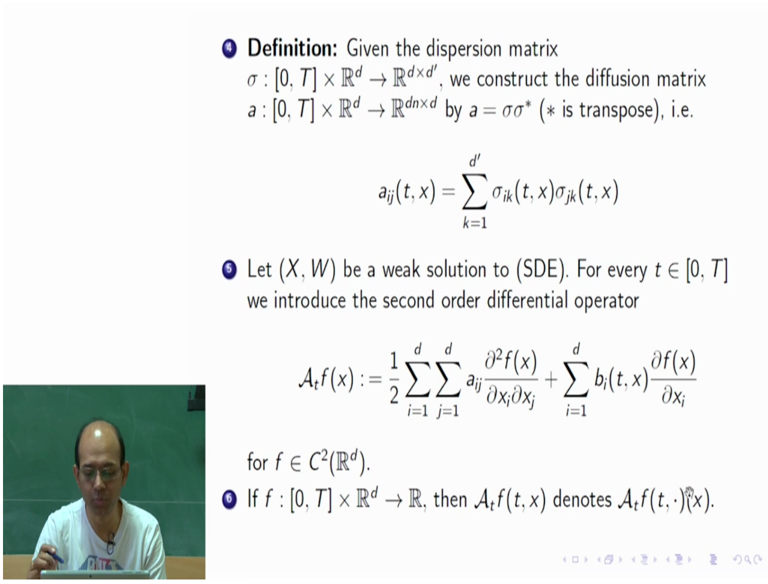

Now, we know defined few more definitions notations for a particular theorem, which we are going to see next, whose proof we are also going to see today. The definition, so let sigma be a function of time and space and then sigma is a matrix, rectangular matrix d cross d prime and a is the sigma sigma star is a typo. So, time and space to d cross d is a square matrix, star stands for transpose.

aij t x is defined as Sigma i k t x sigma ik tx okay k is running from 1 to d prime. So, this is called the diffusion matrix. So, this is called dispersion matrix, this is called diffusion matrix. Let X and W, X , W this pair be a weak solution to SDE. So, sometimes we do this way, sometimes for SDE, we do not talk about the whole triplet because the filtered probability space also should come up with and that should also be mentioned but for repeated, I mean for avoiding repeated mensioning of the filter probability space, we omit that.

Although you omit that but we always remember that that is also hidden here, that we just write down that just the pair, the solution and the Brownian motion. So when I say Brownian motion that means that this Brownian motion with a filtered probability space together. So that is like hidden thing, but we always should actually remember that is there.

So, let X , W be a quick solution to SDE for every small t in 0 to capital T to introduce the second order differential operator at At fx is half of this double summation over i and j, aij del 2 f del xi del xj, this is a second order partial derivative of f. So, for every f which is continuous , twice differentiable we can define this operator and it also has bi del f del xi. Basically, this is a dot product. And here you can also write down these in terms of gradiant, but Here I have not used the vector notation writing this way.

So, here if f is a function of t and x, real valued, then here is the function of x here I have f is a function of time and space, then if I write down At ft x, I mean the following, I mean that is f of t dot, that means as a function of this function of x variable. So, At is acting on that function and then what the new function we are going to get, we evaluate that at x, that is the convention of notation we are going to follow.

So, now, we are going to state a theorem and that result clarifies that there is a connection between this differential operator At and the stochastic differential equation the SDE or I should say the solution of the SDE. I mean that connection will be built and due to that connection we are going to say that okay At is associated with this is SDE.

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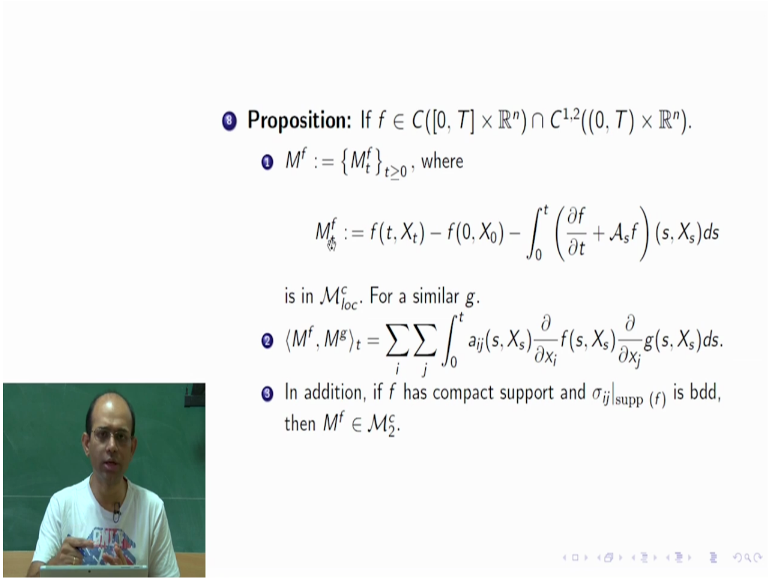

So, actually if these variables are not t dependent, like t independent, so we get SDE as time homogeneous diffusion and for time homogeneous diffusion this A also, you know time independent then that A is actually called generator of the diffusion which is solving that SDE. So, these are just terms that either you see associated or you say the generator. So, the result starts here. If f is a continuous function on 0 to capital T and Rn and it is twice continuously differentiable space variable, once continuously differentiable with respect to time variable.

Then with this f I define another process M superscript f, this is just a notation, which is defined as this process evaluated at time t is f of t , X t minus f of 0 ,X 0 minus integrations 0 to t del f del t plus A s f evaluated at s, Xs ds. Here I must explain what does it mean? This is partial derivative of f with respect to the first variable. But that would give me a function, that function should be valued at s, Xs not at t, something else, so this t stands for the partial derivative with respect to the time variable, the first variable, it is not a function of t.

Here of course, we get a function of t and Xt. So this thing is a function s and Xs and then you integrate this function over the interval 0 to t with respect to ds, with respect to s variable. So you get one stochastic process here, because as you change time it is changing the randomness is already here, here, so you get a random process. So, you get a stochastic process. And filtration is adapted etc. is very clear because X is adapted, so you are going to get adaptability here. So M ft would be a stochastic process.

But this proposition says that due to our very specific choice of A, this Mft becomes a continous local martingale. What does it mean, this is very interesting, I mean if I would have chosen any arbitrary operator A or anything, I would have got some kind of stochastic processes of course, but that might not become a martingale. So local martingale means what, so for the time being let us just think that okay f is like very nice so, that is a martingale.

So what is a martingale? Martingale means that on an average it is not moving. So, conditional expectation of Mt given the information till time 0 to till time s is same as the position of the process at time s. So, on an average in future if you find out the explanation, it is the present location. X t is a stochastic process, it has its drift, it has its motion and f is any arbitrary you know smooth function, so, you would get some stochastic process here.

But this subtraction nullifies its dynamics and leaves only a martingale, which on an average does not move. So, what does that mean? That means that these operators somehow captures the dynamics of the process X. So, that is the understanding for which we call this is associated with the process X. And that is not surprising because this process A is after all defined using the parameter which arises in the SDE of the process X. But appropriately chosen to, appropriately combined to get it.

The proof is actually very simple, the result is striking, I mean is very interesting but the proof is very simple. So for a similar kind of g like similar smooth these things, we would also be able to say that okay M gt is also like one can define, that would also be M c loc and then 2 different continuous local martingales quadratic variation I can find out and here this quadratic variation one can find out the formula, it formula also.

That formula turns out to be aij s, Xs del del xi f s, Xs and del del xj, g s, Xs. Because f is a function of time and x, so but here it is multivariable. So, in that multi variables you take partial derivatives with respect to i'th coordinate, you get another function which is also a function of time and multivariable Rd and in that function you evaluate at s, Xs. So, that is the meaning of this notation, similar meaning of this is also there.

So, then this sum. So, this integration would be the quadratic co-variation of Mf and Mg, this is quadratic co-variation. In addition, if f has compact support okay. So, like not on full Rn but a compact set it is nonzero, else it is 0. And on that set where f is nonzero, there Sigma ij is a bounded function, in that case Sigma ij if I consider that is continuous function, continuous process, then of course, but otherwise that is not assured. So Sigma ij is bounded, then this process Mf is square integral continuous martingale.

I mean not only local martingale, it is a martingle, square integrable martingale. So these are the things, so for this proof we are going to use Ito's formula. So here actually you can understand that if I directly use Ito's formula of f here, so what are you going to get. f of t Xt minus f 0, X 0 is equal to the partial derivative of f with respect to time t and dt that anyway appears here, del f del t this thing.

And then partial derivative of f with respect to space variables, that is actually a gradient vector, the dot product with, I mean the Brownian motion and then the remaining terms like second order, the quadratic variation term, the half times double derivative of f etc all these things would appear. So, those things if you compute, those things would actually appear here. So, by applying Ito's lemma it is very clear what should be the expression of Mft.

I know what is the expression of M ft. So that we are going to look at here. But that expression of Mft is integration with this Brownian motion, I just need to show that integration is continuous local martingale. We know that to show a stochastic integral as continuous local martingale, I just need to see the integrand is well behaved. Our integrand is in the class of L star W, so it is predictable and then this is square integrable etc.

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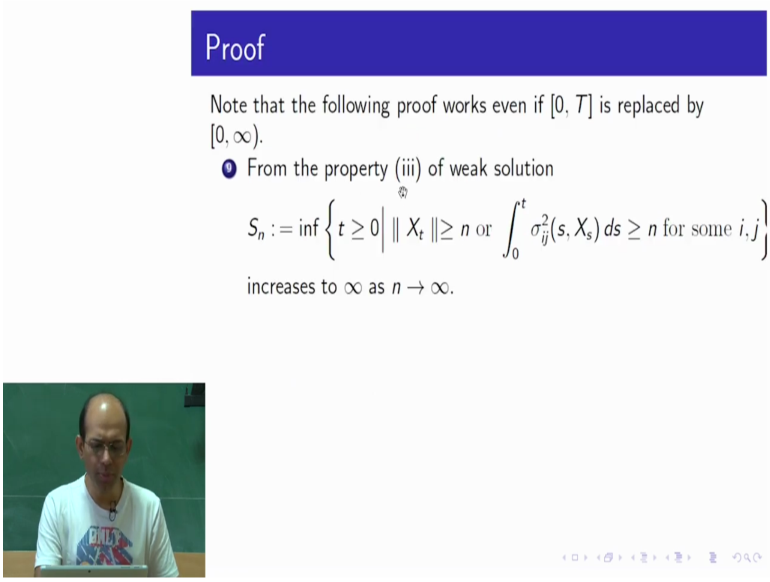

So, those are the things only you need to check okay. So, here we come here okay note that the following proof works even. So, here instead of 0 to capital T we take 0 to infinity. So, this thing whole time.

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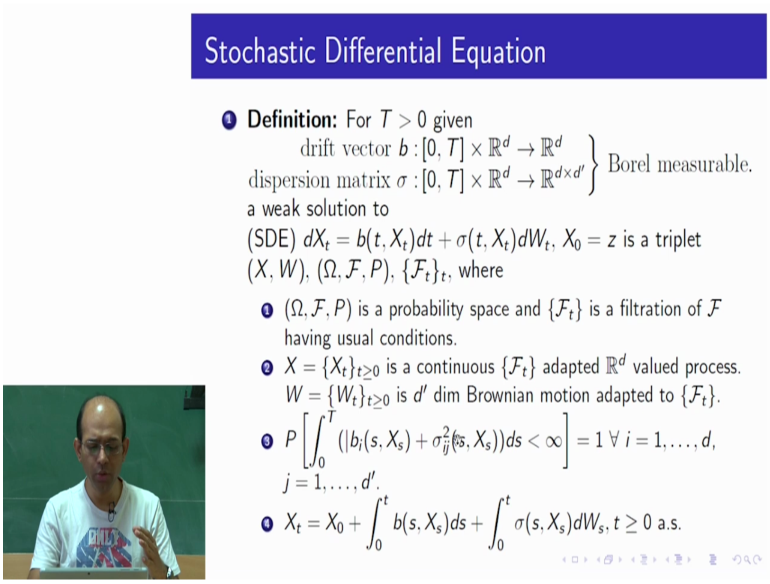

So, from the property 3 of weak solution, what is the property 3, the property 3 is talking about, this is the property 3 of weak solution that mod bi plus sigma square i j, if you integrate on 0 to capital T, you get a random variable that random variable is finite value with probability 1. So, for every T it is a finite valued random variable. So, now if I ask that the first Time this becomes large, larger than small n or something, we are going to get some time and that time should increase as n goes to infinity. Otherwise, you would not get it.

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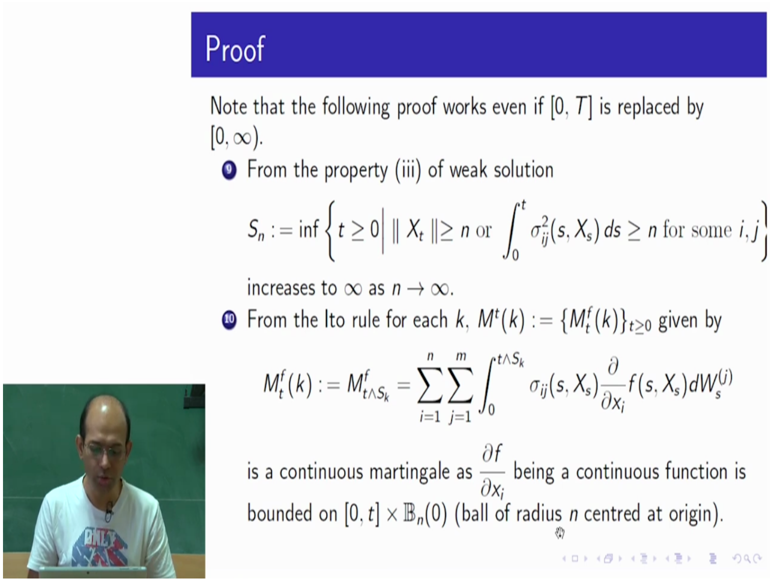

So this is the part what I am going to explain now. So, consider sigma square ij s, Xs ds integration 0 to t is greater or equal to n. So, for the first time, the time when this integral becomes more than n. So, this time is of course a random time because it depends on the random process X. Also mod of norm of X is more than n. If either of these happens okay for the first time, we get Sn and then we write n plus 1 and for n plus 1 we check when either of these happens, is when n plus 1.

And you get S 1, S 2, S 3, S n+1, etc., some sequence of stopping times. But this sequence increases to infinity as n tends to infinity. So why is it so, because if it is not, if it does not grow to infinity that means the sequence, increasing sequence has an upper bound and that upper bound is finite time. So on that finite time, the integration would be infinity then because it is more than all possible n. So your going to get a contradiction.

So this kind of, if it does not go to infinity that time would appear and this is infinity, but we have seen that from the definition that that would happen only with probability 0. So that means only with probability 0 Sn might not go to infinity. Or in other words, it increases to infinity as n tends to infinity almost surely, with probability 1. So from the Ito's rule, so as I have explained the Ito's rule if you do that, so here instead of 0 to t, we write down t minimum Sk.

So here we are doing this localization, so minimum of t and Sk. What is the good part of this, the good part of this is when s is between 0 to this interval, then X, norm of X is less than less than k. And because of this thing, less than k actually because Sk, less than k. And here, this integration, I mean this whole thing, if f is anyway continuous on the domain, and here X is between the 0 to n, so I mean mod of X. So that means we are talking about the bounded domain.

So that we can take closure of that, so with ball of radius k. On the ball of radius k, so this continuous function would be bounded. So, this part would be bounded. On the other hand here for this part we have already obtained this. So, the sigma square of ds which we might get using that, after using Ito's isometry this thing will square of that norm. So, that would be also less than n. So, here for this integration, we know that we are going to get this is a martingale.

So, it is a continuous martingale because this integrand is bounded here, I mean this integration and because this is square integrable and this is bounded here and as a function of t it is a martingale for each and every k. So, what we have done we have obtained a increasing sequence of stopping time such that the stopped process is martingale for each and every k. Or in other words, we have proved that Mft , Mf t is a local marketing.

So this part is written here elaborately, that ball of radius k I should write because k is here, ball of radius k, so instead of n I should have written k, so there it should be bounded. So that clarifies that Mf is a continuous local martingale.

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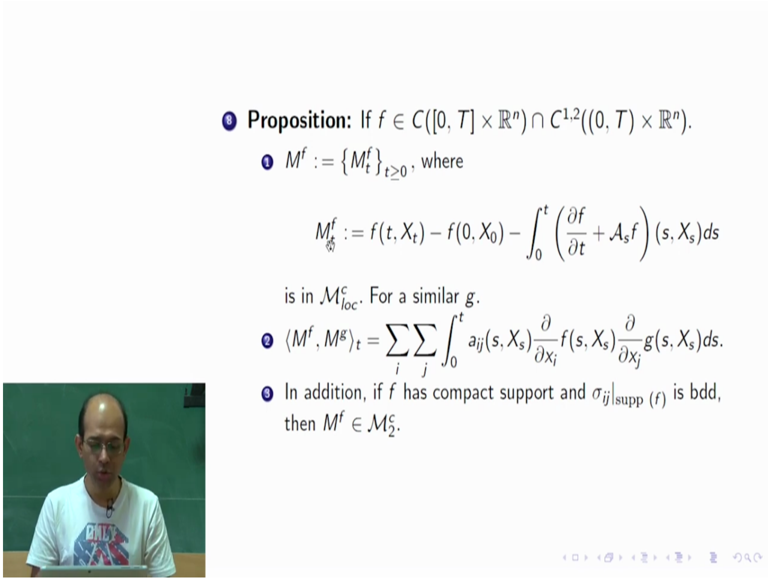

Now from here to here is actually trivial, it is just a formula you do not need any other arguments for this, because for quadratic variation, you know that already this, it has this expression the sigma del f del x dW and you take quadratic variation. So, it is a, you multiply Mf and Mg and quadratic covariation is the natural increasing process appears in the Doob-Meyer decomposition of the product of the process Mf Mg. And there if you do that product, you are going to get exactly this term del f del xi del g del xj.

So they are you are going to get. Of course you have to take actually the dot product there. So, this you are going to get and then third condition is trivial because here whatever we had to worry about we do not need to worry there because the support of f, there Sigma ij is bounded already and f has compact support. So, you do not need to consider the sequence is S1, S2, Sk there, without doing that you would be able to get that this would itself be a martingale, squared integrable continuous martingale. So, this third property also follows for that. Okay, thank you very much.