Introduction to Probabilistic Methods in PDE
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Lecture 45
Stochastic Differential Equations: Existence

In the last lecture, we have seen the existence and uniqueness theorem of strong solution of a given stochastic differential equation under some conditions on the drift and the volatility parameter.

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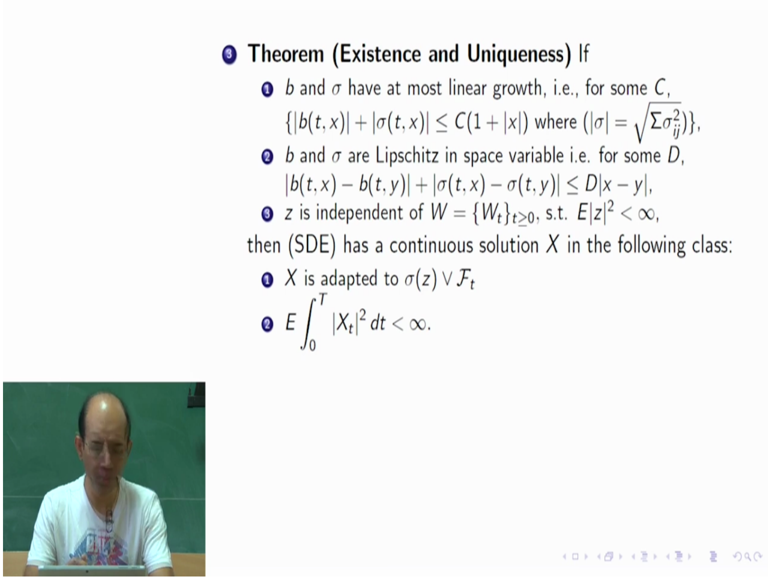

So, here in that proof, we have showed that given these 1, 2, 3 conditions. The first condition let us recall, this is nothing but the at most linear growth condition, second condition is Lipschitz continuity condition on second variable of the parameters. So, here this is drift parameter, this is diffusion parameter and then the third condition is saying that z initial point is independent to the Brownian motion. So, given all these conditions, we have asserted or this theorem asserts that the solution exists to the equation.

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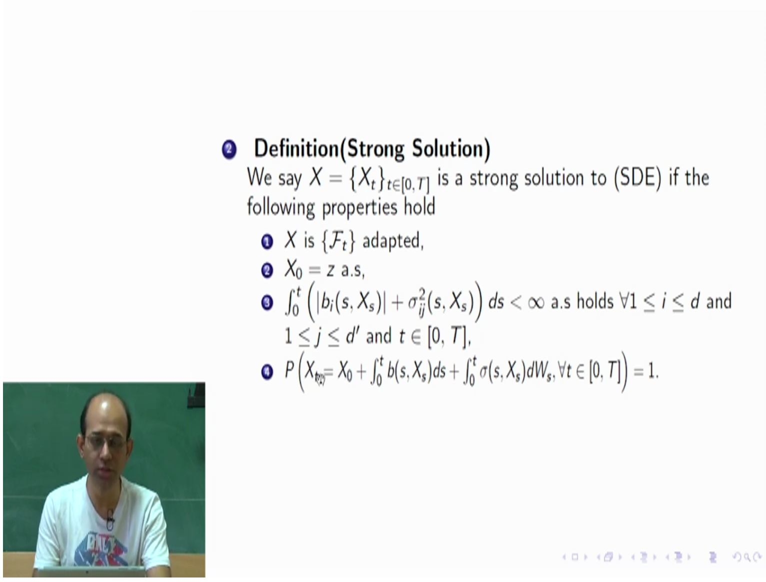

So, in this sense, strong solution, that there is a solution of this equation. So, x is unknown, this is integral equation actually, but it equals to a differential equation, this is actually stochastic integral equation. So, this is the statement what we have seen in the last lecture and also for the proof what we have obtained is the existence of SDE such that this third property is true. This fourth property is true that this solves, X is continuous and this solves. You remember that there we have denoted X by X tilde.

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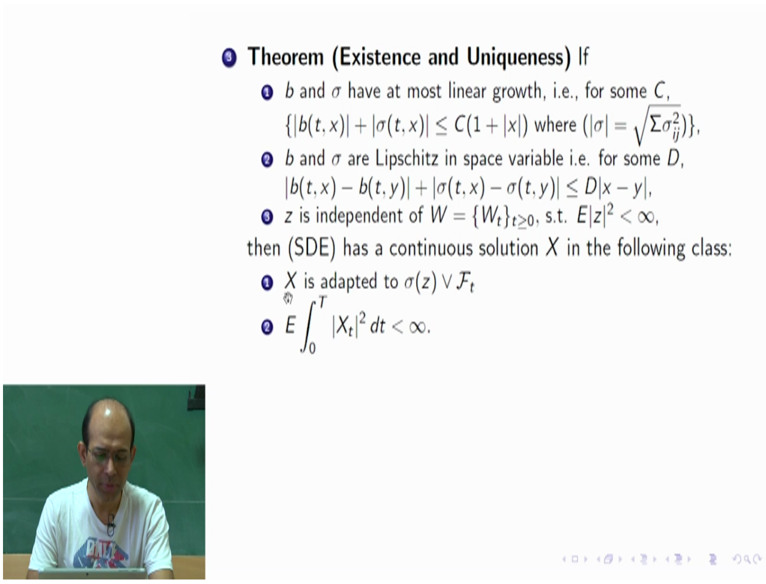

However, some more things to say about the solution that X is also adapted to the filtration this. What does it mean that X subscript t, that means at the t'th time, X that we call the Xt, that is the random variable that random variable is measurable with respect to the sigma algebra, this sigma algebra, for each and every t, that have not shown. And then this is of course, obvious from the proof as we have done in the last lecture.

So, we are going to see today this thing. Since we have already established that the equation has a solution, which is having continuous path with probability 1, and that solves the equation almost surely. So, we can use those property to derive this one.

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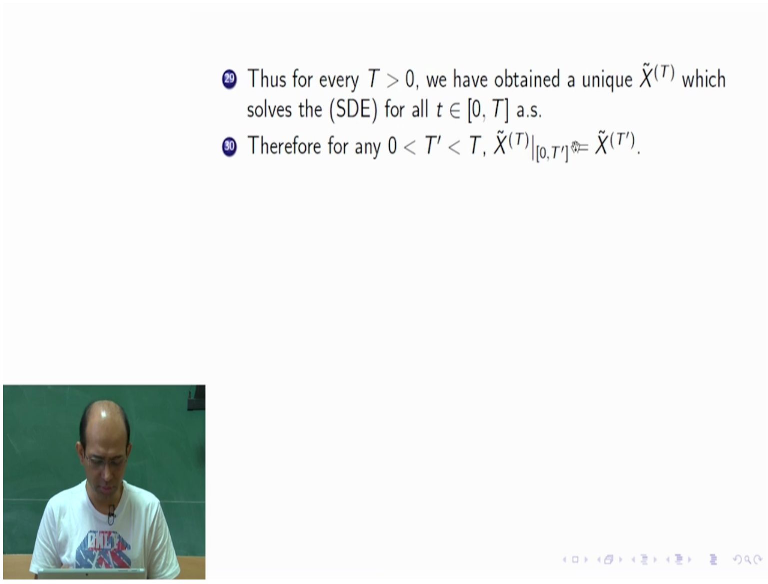

So, thus for every fixed capital T positive, we have obtained a unique X tilde t, as I was telling that we have denoted X tilde, there of course, we have not denoted X superscript this in the parenthesis capital T, but here we are doing. Why are we doing to emphasize that X tilde is obtained by fixing one time real time capital T. Just to retain the dependency of this thing on T we are writing this way.

However, this dependency is very, I mean trivial manner, so we are going to see that in the next bullet. So, which solves the SDE for all small t belongs to 0 to capital T almost surely, that we have already seen. Now, we consider another T prime which is less than capital T. So for this T prime, now consider this process. So, this is the process from 0 to capital T, X tilde.

So, this process if it restricts to 0 to capital T prime, which is smaller than capital T, so this process restrict to this, what do I get? Because this is obtained as I mean solution of integral equation. So, there on the top you have 0 to T and small t you replace by capital T to get the full thing, but instead of capital T you write capital T prime. So, that means, you just consider only the domain 0 to T prime capital.

So, the Brownian motion also you consider only 0 to T prime everything there. So, if you do that restriction that is same as, as we started only by considering 0 to T prime only. So, getting this first and then restricting 0 to T prime. So, this process is same as just the process what you obtained by solving the SDE between the interval 0 to T prime. So, this is true that we understand easily because that is the very definition here for this thing.

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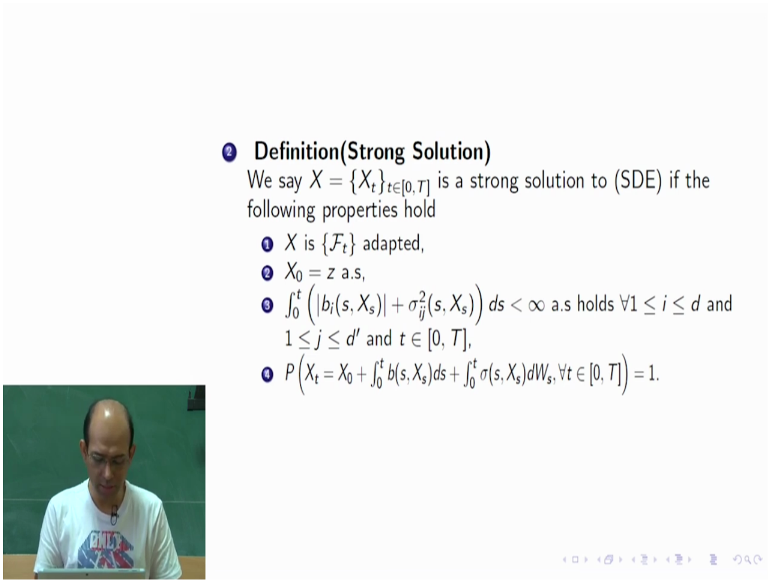

So I am talking about this equation, in this you just write down so this is true for all T. So you write down not capital T but T prime. So there that probability would also be 1 because for all small t belonging to 0 to capital T this is true, so this is true also for 0 to capital T prime. So that much only I am talking here.

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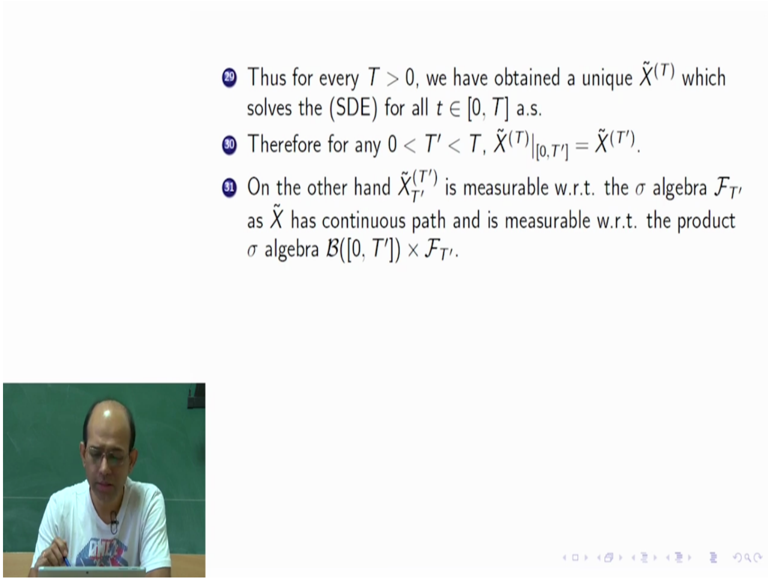

Now, here, why do we have obtained this thing? Now we consider this particular process which is defined from 0 to capital T prime. So this process evaluated at any given point t, we would like to ask about its measurability whether that process is measurable with respect to the sigma algebra, which sigma algebra, so here we see that this we have obtained as measurable with the product sigma algebra because this is a function of t and omega.

On small t you have the Borel sigma algebra on the close interval 0 to T capital T prime, on omega you have the sigma algebra fFT prime. So, since this is a function of two variables and it is a measurable function two variables, so these functions we would get that at T prime if you take T prime section, that is also measurable. I mean in general it is not true but here we are going to get it is measurable with sigma algebra fFT prime.

As this process is continuous, as this has continuous path and it is measurable with respect to the product sigma algebra this thing. So, this part let me elaborate in the following three bullets, because in general I cannot assure that at T prime, this prime section. So, I just know that this is measurable with respect to this product sigma algebra. From the how can I assure that its value at T prime is measurable with respect to this FT prime?

We do it in the following manner, that X tilde T prime evaluted at small t is F T prime measurable for almost every t is 0 to T prime. That we obtained from the argument of Fubini's theorem, there we do these things. That we get these you know, a particular section of the measurable function is measurable almost every for almost every t, so that we obtained.

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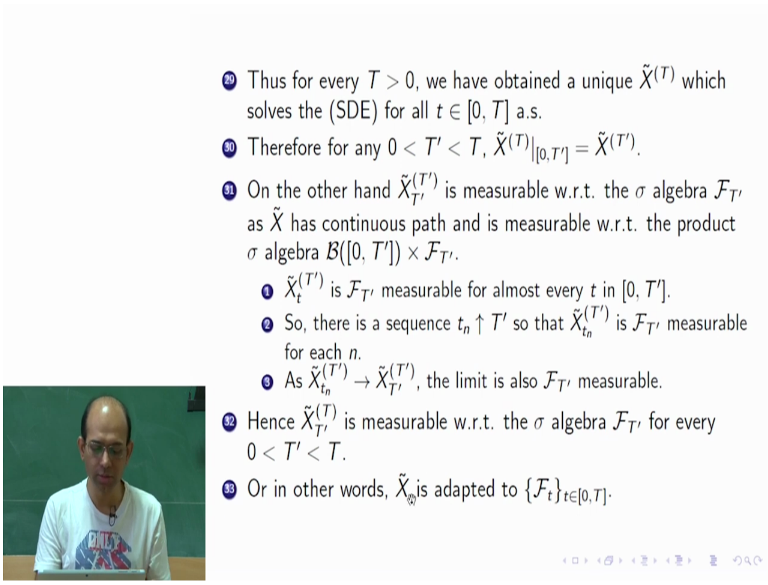

Now, since this is true for almost every t, now we take a neighborhood of T on the left hand side, so, that neighborhood would of course have a non empty intersection of the complement of the 0 measure set where this is not FT prime measurable. Or in other words, we would be able to get a sequence of tn which is in the complement of the zero measure, zero measure set where this may not be FT prime measurable.

So, this tn is a sequence which is increasing to T prime and at this tn, I can always find out this sequence of tn such that on at this tn if I evaluate, so this as a function of omega is measurable with is to FT prime, so this is true for each n. Now, we know that X tilde as a function of this small t is continuous. So, since it is continuous as tn goes to T prime X tilde T prime at tn converges to X tilde a T prime of T prime.

So, since tn converges to T prime and X tilde a is continuous with probability 1. So, this we are going to get, this converges to this almost surely. Now, this is a sequence of measurable functions that converges almost surely, so the limit is also measurable with respect to the same sigma algebra. So, this is measurable with respect to F T prime sigma algebra. So, this limit X tilde T prime at T prime is also FT prime measurable. So, that is the thing asserted here.

So these three bullets explain this claim. Now we go back to this scenario that, so X tilde capital T at 0 to T prime, this expression is same as this measurable function and from these we have obtained that T prime value is measurable in the F T prime. What does it mean, it means that this function's value at T prime is also measurable with respect to F T prime. But what is the value of this function T prime, that is nothing but X tilde T at T prime, because this is the just the restriction but at T prime its value is X of T, T prime.

So, what we have obtained is the following that X tilde a T at T prime is measurable with respect to the sigma algebra FT prime. But what is T prime for every T prime less than capital T. So this is saying the same thing that X tilde is adapted to the filtration Ft, small t belongs to 0 to capital T. For every time small t, the process is adapted to this Xt, X tilde t, small t is measurable with respect to Ft or in other words the full process is adapted to the filtration Ft.