Introduction to Probabilistic Methods in PDE
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Lecture 44
Stochastic Differential Equations: Existence (Part 02)

We have already seen in this sequence of stochastic processes Yk which are like Picard iterations.

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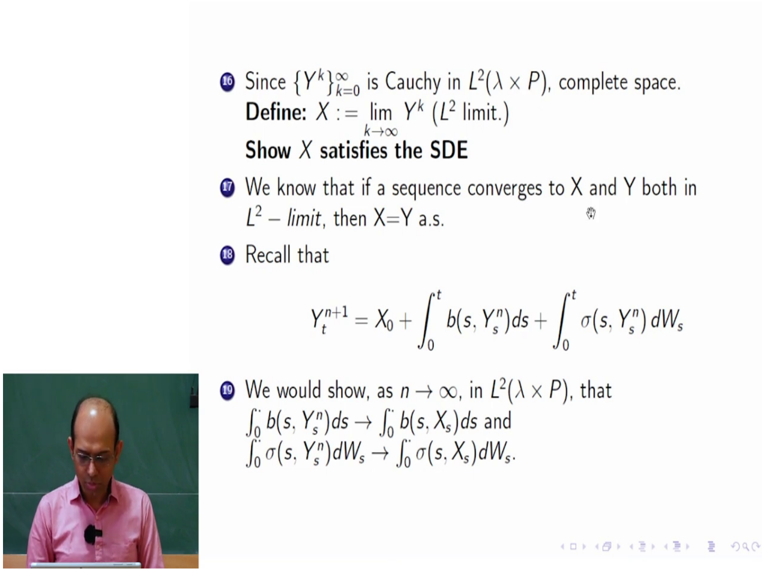

And then we have shown that this Yk the sequence is Cauchy in the L2 space with respect to the product measure and then that limit of this we denote by X and then this X is the candidate process which we need to show that this solves the Stochastic Differential Equation it is not quite right to say that we are going to prove that X itself solves that equation, but from X we are going to find out another which is very close to X a modification. So that would be this. So, here we recall that this iterations Yn plus 1 is defined using Yn here for all t. Now, we consider this right hand side.

So, the right hand side if you vary t. So, this is also a process, for every n we get a stochastic process here, for every n which gives us stochastic process here and the new goal is to show that this stochastic process 0 to dot, dot means there is t but t is running. So, general t, b s Y n s d s convert this to b s X s d s which X this X. So, that is the goal, we have our goal is that showing that this converges to this part, this converges to this part and this any way converges to in this L2 norm of the product measure in an appropriate manner we need to show this.

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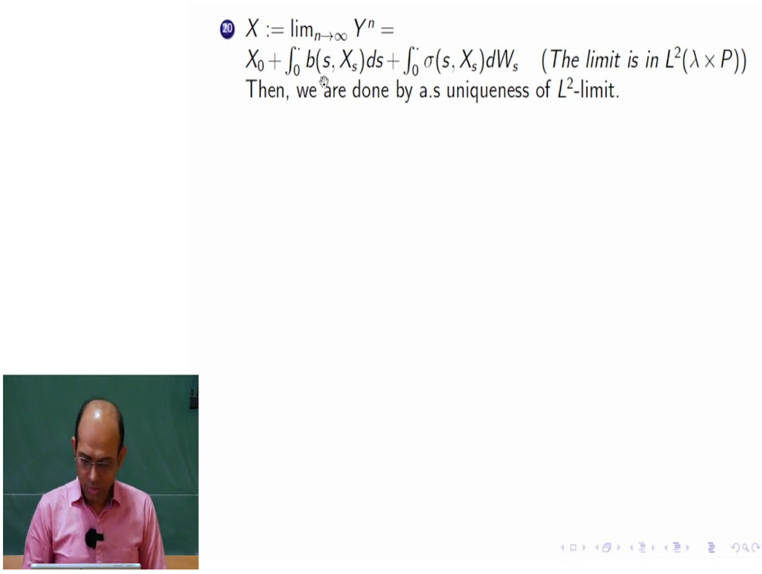

So first thing we do in the next slide. So, here, before doing this, let me emphasize the plan, here the plan is that X is the limit. And then Yn is anyway this thing.

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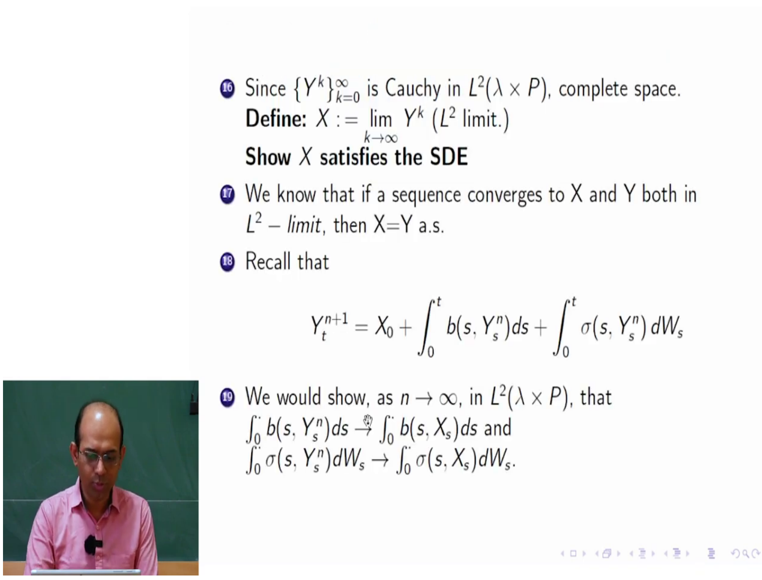

And then if we would be able to prove if we prove these two things.

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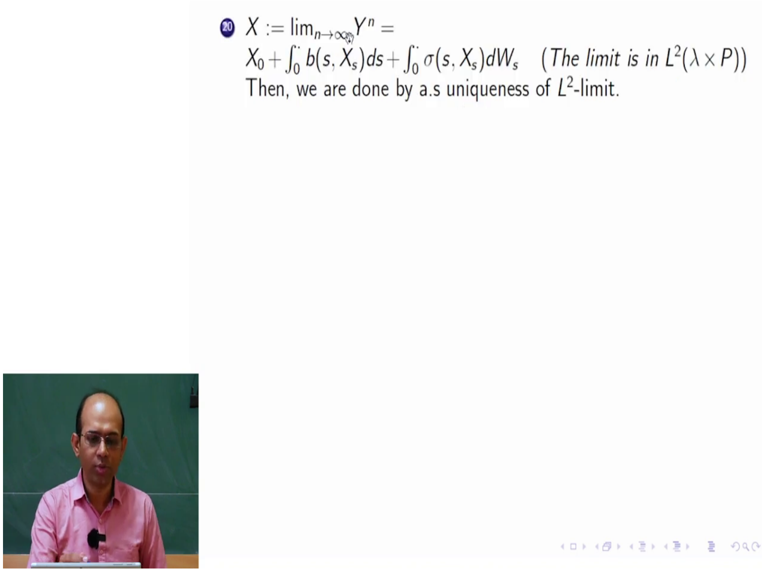

We will be able to prove that this limit will be this. And then that means X is same as this almost every where with respect to this measure lambda cross P. But that is not quite same as the statement but very close to the statement and with this, we are going to find out the final solution. So that is the plan.

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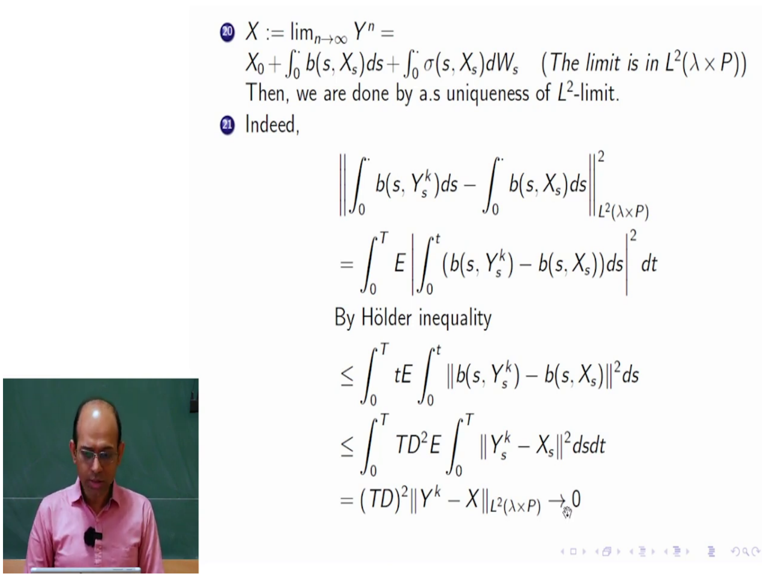

So, we now go to the first claim, the claim number one, so, this bsYks ds minus b s Xs ds. So this difference I need to show goes to 0 with respect to this norm L2 norm of lambda cross P? So what we do is that we write down the meaning of this, the meaning is 0 to T integration expectation inside or outside is because it is nonnegative term using Tonelli we can do and then here mod of 0 to t.

So, at t time whatever is value is bs Y ks minus b s Xs ds 0 to t that is the value, but then this t also you need to integrate because it is lambda is corresponding to the measure on the time domain P is measure on the Omega domain. So for that we are getting expectation. Now here we are using Holder's continuity for inside thing expectation of you know so here square of the integration.

So here what we are doing is that we are going to get that is less than or equal to t times that integration of the square. So that we are going to get anyway this is like a vector so, we will get norm square. So, here now, this thing, we can again, upper bound by small t you can replace the capital T and here small t can be replaced with capital T because anyway the integrand is non-negative.

So, we can have this upper bound and here norm of this is like is a Lipschitz property of the function b, this is the second variable, we get this list less than or equal to D square times this variable Yks minus Xs whole square. So, we are using that Lipschitz property of b here. So, here we have integration s running from zero to T, but here outside the whole inside is not independent small t. So, this TD square such as comes and this is capital T another time comes.

So, I would get capital T square, capital TD square and then this thing we need to identify, right? What is this? That expectation of 0 to capital T of this there is a stochastic process square and ds that is nothing but the L2 norm with respect to the product manager lambda cross P. So this inside part is this thing but we have already shown that Yk convergence X in this norm. So this part goes to 0 or in other words that this converges to 0, so here the first claim is proved now we go to the second claim.

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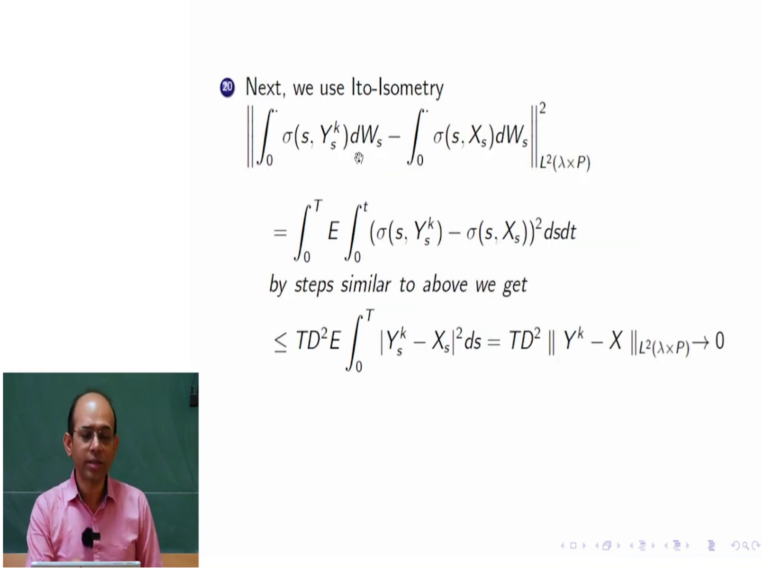

Second claim that involves that integration with respect to the Brownian motion. So next we use the Ito's isometry for this regard. So again 0 to dot, 0 to dot. So, these two processes we want to write down then the square of the L2 lambda cross P norm. Here we use that the definition of this norm, that this integration 0 to capital T expectation of 0 to small t at t value whatever his value is that is this thing, sigma square s square.

And here we have also use that Ito's isometry so everything together actually here I have jumped some steps I mean this whole square was there and then the integrand squared times ds because of Ito's isometry and then dt. So, basically I have used two things here the definition of the norm and then Ito's isometry. So, after getting this we do as we have done earlier that we use the Lipschitz continuity property of sigma with respect to the second variable here.

So, again we are going to get D here since square is this so I will get D square and Yk minus X whole square and then this small t I can have upper bound by capital T here also. So, here on this integration, outside integration will be capital T, So, I would get capital T from the outside integration because inside would be independent small t because I have replaced small t by capital T here. Now identify this term expectation of integration of the process square and ds. What is this? Again norm L2 norm under the product measure. So, that is Yk minus X norms here actually I should have written Square. Square is missing.

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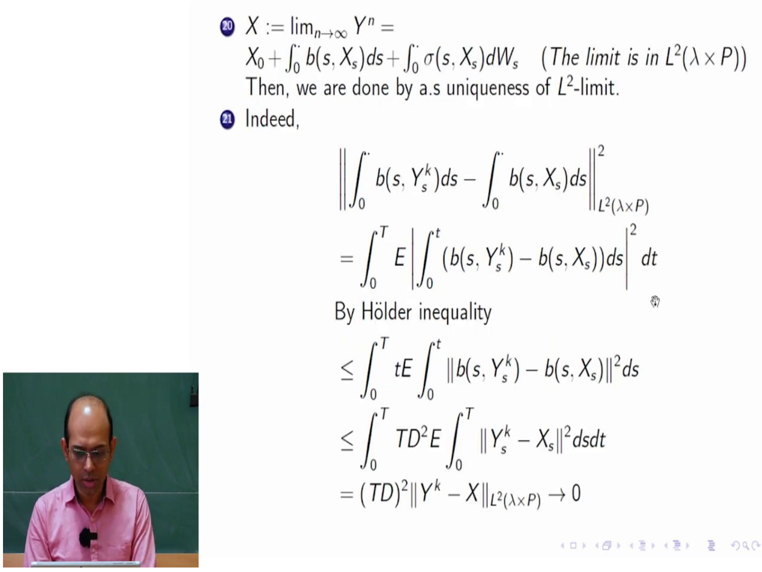

Or earlier also, here also I made the same mistake here I should have written square here

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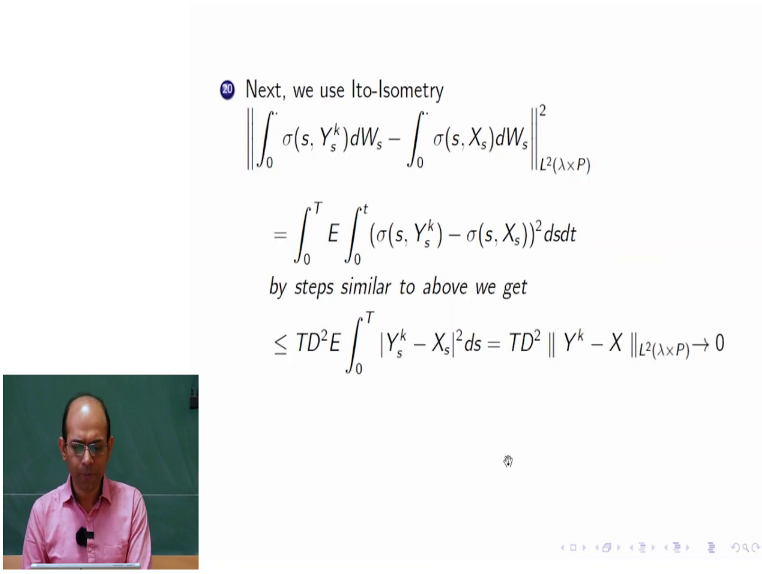

Okay. So square, so, that goes to 0, that goes to 0, so the whole thing goes to 0. So, what you obtain is that in that iteration term right hand side those 2 integrals converges to the integral with respect to X process. So, hence Yk converges to X which solves the SDE for every t omega from this except possibly for some 0 measure set So, lambda cross P in 0.

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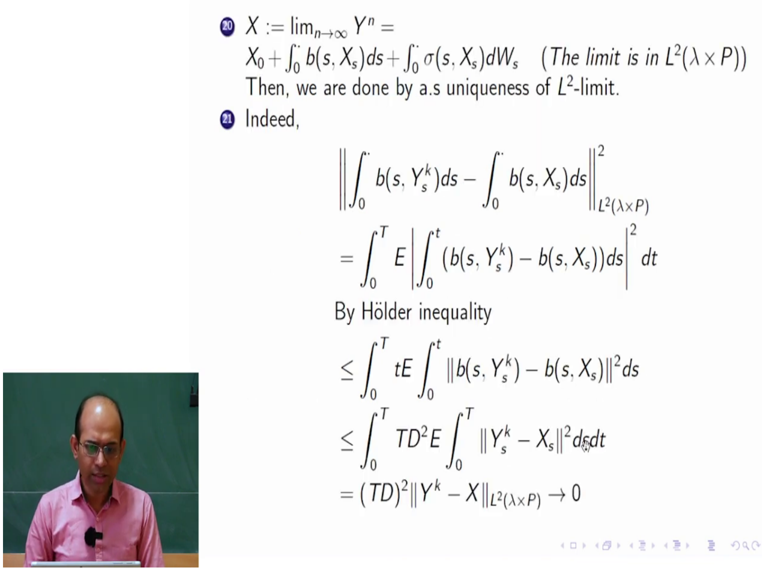

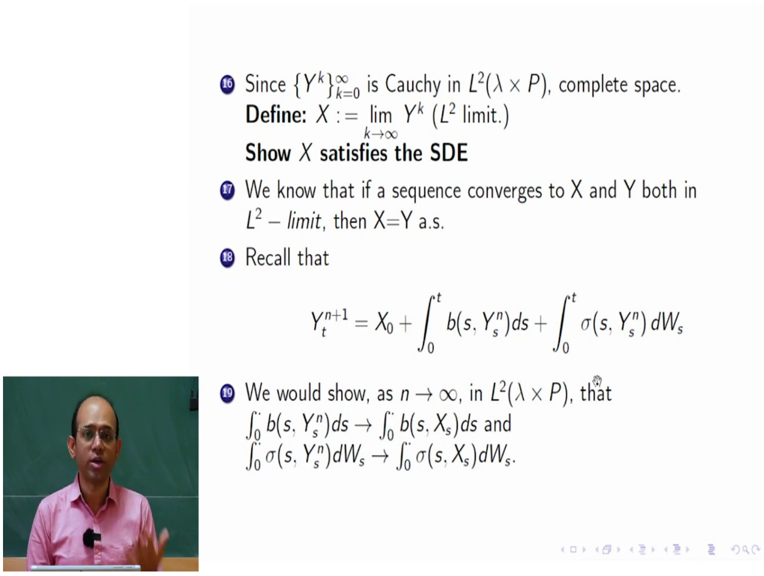

Why is it so, because, you know both sides because this side, you know, converges to the integration with respect to X these things this side also converges to X in the same norm so, both and both are the same sequences. So same because they are equal, so a sequence converges to do different functions, in the same space, so then they should be in the same equivalence class that means the left hand side is same as right hand side almost everywhere.

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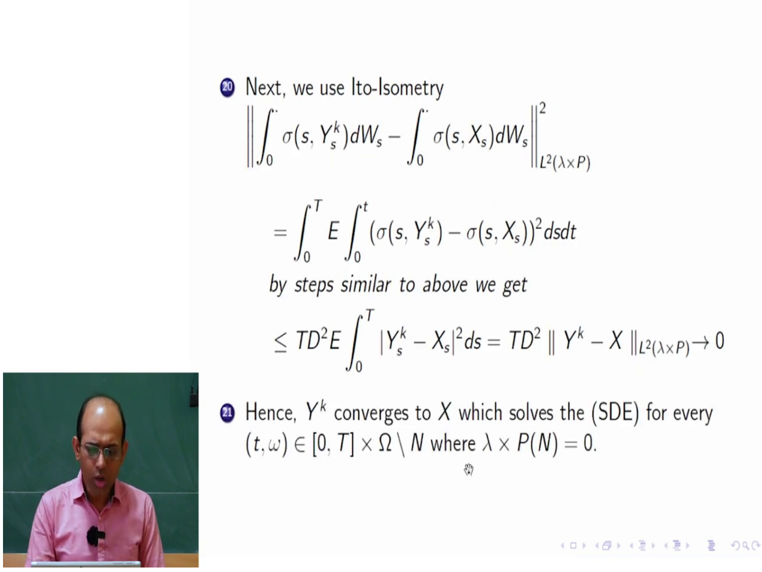

So, what did we get then? We got that X satisfies that SD for almost every T and omega. So, that means it might not satisfy the SD for some combination of t omega that said I am writing as capital N. however, the measure of the capital N will be 0, is it clear? The measure of that will be 0. So, here by arriving at it we are not yet done with the proof of the theorem because there it says that for each and every t it should be true, so we need to do little more work here.

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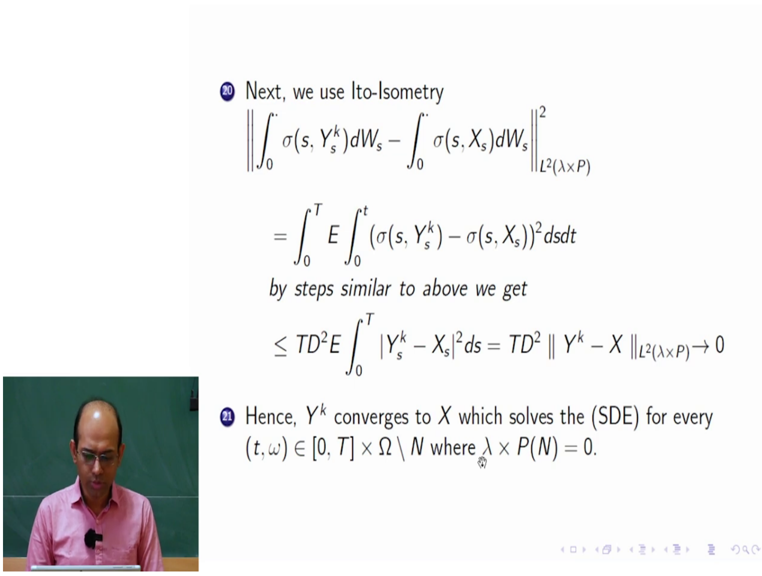

So here we introduce some notation. So, given that for that N, this notation N restricted t, so, we call the t section. N this restricted t we denote this notation to denote the t section. What is t section? That set of all omegas smaller omegas such that t, omega belongs to capital N. Now since N is a 0 measure set, I cannot just say that, this could also be 0 measure set that for a particular t section, That set may not be a 0 measure set however, that measure will be nonzero. Only for 0 measure t basically.

So I would say, the measure of t such that the probability the N restricted to t is not equal to 0 that is equal to 0. Why can you do that? Because if it is not the case then N would have a positive measure. Imagine that if it is not 0 positive that means you have got some positive measure of Set of t on t section of N is positive measure.

So you would be able to get one nice, you know, subset of N which you would have positive measure, but N has zero measure. So, we have this. So, now we define it. So, why are you doing all these? Because you know, we I mean, we need to show that Xt solves SDE for each and every t, but we are not yet there. So that is why we are classifying these two parts for…

So now we have considered those t, those time points t such that the t section of N is zero. So, that is most of the cases there I am writing tilde as X, so X tilde is a new process, which is same as X. So most of the cases X tilde is like X, but when the t section is nonzero, then I take X tilde as this X0 plus integration 0 to t, bs Xs ds plus 0 to t sigma s Xs dWs why are you doing that? Because the good thing is that X is already defined, defined. So, then I take this and then our goal was to show that X tilde solve that SDE.

Now, even for some earlier point is X tilde was different from Xt,, still I can replace here X by X tilde because the time points where X was different from X tilde that was 0 measure set so there I can therefore change this, so I would get X tilde is also solving the X tilde here. So I would be able to get that, that is the idea. And we would put X tilde 0 is equal to X0 if that is not coming naturally. So if required we will also put that.

So now we say since we know this, so then for each and every t positive. So, this is the thing what I have already explained that we can write down X tilde t is equal to X tilde 0 plus integration 0 to t bs X tilde sds plus 0 to t sigma s X tilde s dWs. So, I would be able to get it for almost every omega why?

Because here we started with say we have fixed one omega and then I mean we have started with t and then we have checked that whenever this is the case we have put these Xt values and then these values so what we have obtained is X tilde t, it actually is a satisfied you know I can replace X by X tilde here by doing that, it might not agree with this only at time point finite only zero measure sets by however that integration would be the same because it is integration.

with respect to Lebesgue measure here and the Brownian here. So, that is the thing we obtained here. Now, this is just a candidate. So we have many things to do now, first thing is that we need to show that this is continuous solution? So this t to X tilde t is continuous. And here what we obtain is for each t this is true with probability 1.

But we need something stronger, we need that this is true for all t and property of that is 1. Basically we mean to say prove the property that X tilde satisfies the SDE for all t the process and probability of that is 1 so that is the thing we need to prove. So it is like a like a modification kind of thing, but we need kind of indistinguishability.

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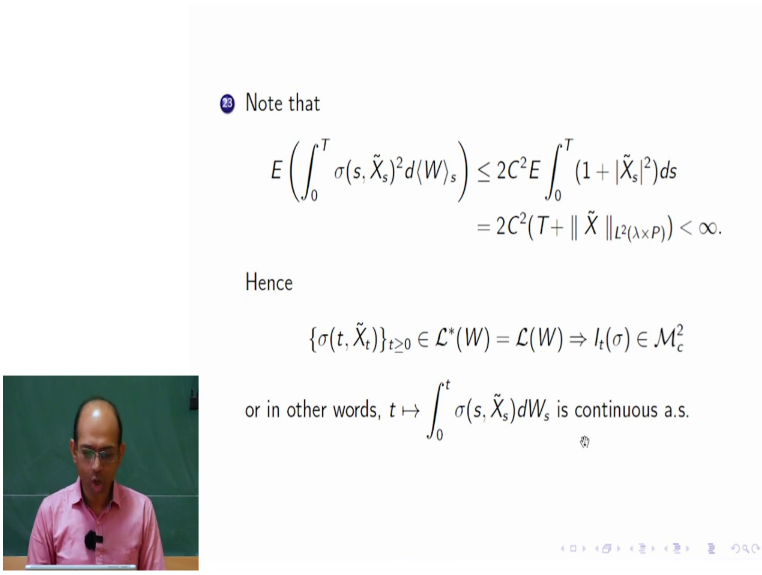

So what we do is that we note that the expectation of integration 0 to capital T sigma s X tilde s whole square quadratic veriation of Brownian motion that is ds actually so I am writing this way. So is less than or equal to, so here we are using the linear at most linear growth property of sigma. So then I would get 1 plus mod X whole square, but then I would I do not, I like to take square insights, I will get 2 here.

So, two times square expectation of 0 to capital T 1 plus mod X tilde s square ds. And then this part, we recognize that this is nothing but again this L2 norm of these X tilde 1 plus X tilde but 1 I would take outside so I would get T here. So two times square times capital T plus this thing, however, X tilde is in this space. So this norm is finite. So you get finite thing. So why are you doing that?

Because we do this to use the theorem, what we have done earlier that if a stochastic processe is such that I mean, this whole thing is finite, then this is in the class of L star of W, of this measure and then this integration of sigma with respect ot the martingale W would be a square integrable continuous martingale. So, since this is finite, so, we are using a theorem what we have already visited in first, second week or maybe yeah, so, when we have introduced a stochastic integration.

So, using that we know that this process is in L star of W, but W is you know the quadratic relation of W is dT time itself which is absolutely continuous with Lebesgue measure continuous function and therefore L star omega is same L of omega and from there we know that integration, so it is a short, very short notation actually basically I want to write down the integration of this thing with respect to dWt, that stochastic process 0 to t would be a square integral continuous martingale. So, what we have done is that t to this map is continuous almost everywhere, so this map is continuous almost surely.

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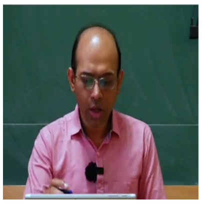
- Let Ω^* (where $P(\Omega^*) = 1$) be the set s.t.
  $\forall\, \omega \in \Omega^*, t \mapsto \int_0^t \sigma(s, \tilde{X}_s(\omega))dW_s(\omega)$ is continuous.
  Let
  $$C'(\omega) := \sup_{t\in[0,T]} \left\| \int_0^t \sigma(s, \tilde{X}_s)dW_s \right\| \text{ due to a.s. continuity}$$
  $$P(C' < \infty) = 1$$
  Now,



$$\| \tilde{X}_t \| \leq \| \tilde{X}_0 \| + \int_0^t \| b(s, \tilde{X}_s) \| ds + C'$$
$$\leq C'' + C\int_0^t \| \tilde{X}_s \| ds \text{ (where } C'' = C' + \| \tilde{X}_0 \| + CT)$$
$\forall\, t \in [0, T]$.

Now, we consider this omega star. Omega star is basically I mean probability 1 measure set and for all omega in omega star, so, this is continuous. So, we are actually picking up those omega for which this is continuous.

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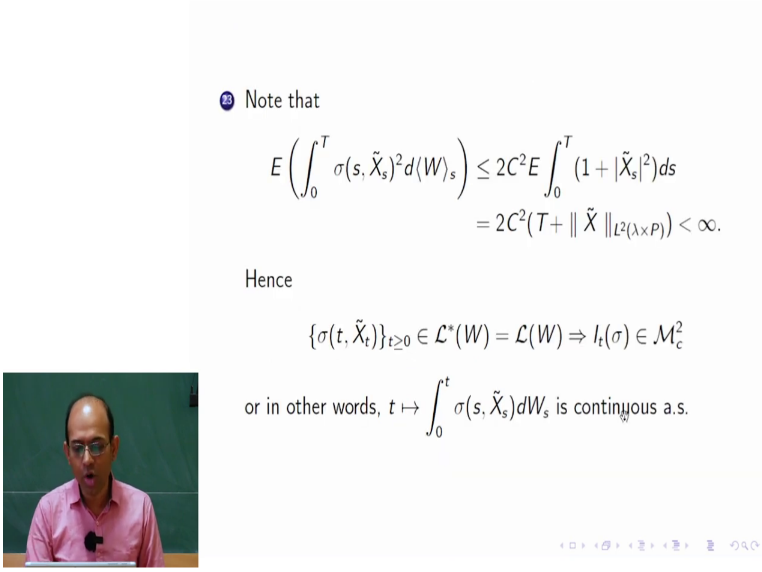

Because here we are obatining almost sure continuity, correct?

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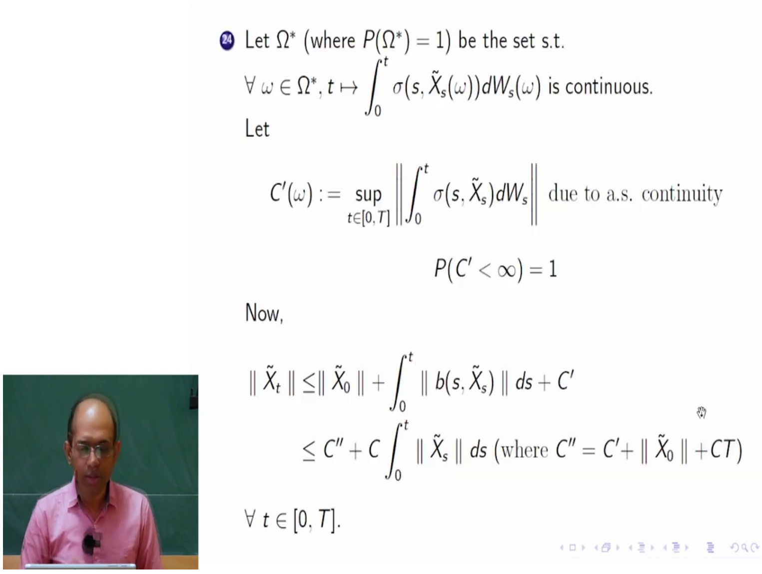

So, omega star is the space of events, the set of events for which this is continuous. Now, what we do is that we take C prime of omega which is supremum of this quantity, that 0 to t sigma s W tilde is dWs its quantity, this continuous function on compact sets. So, this supremum is finite.

So, we would get that C prime would be finite for every omega from omega star. So, C prime will be finite with probability 1. So, whenever Omega comming from omega star I would get

C prime some finite number that may not be bounded. C prime needs to be a bounded random variable but is finite random variable that much we can say.

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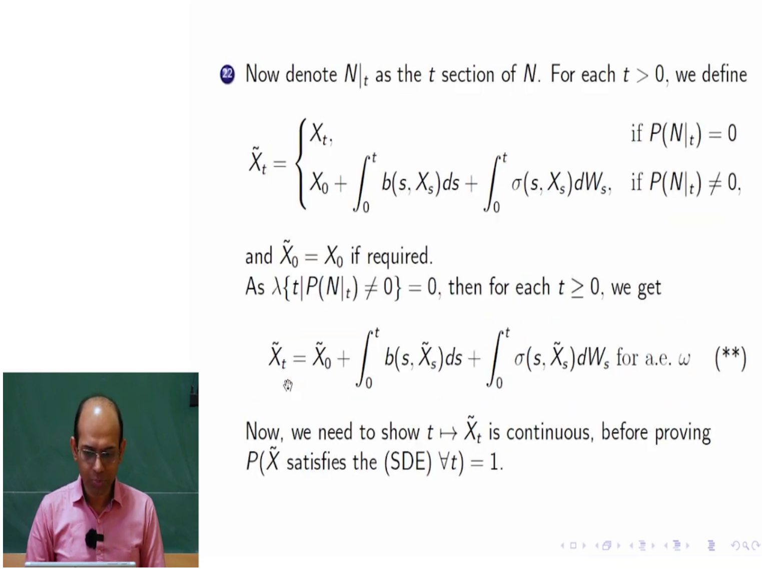

So now we talk about what is X tilde that so here we are using this equation this star star so use this right because we are going to use this equation time and again so we are giving names just double star.

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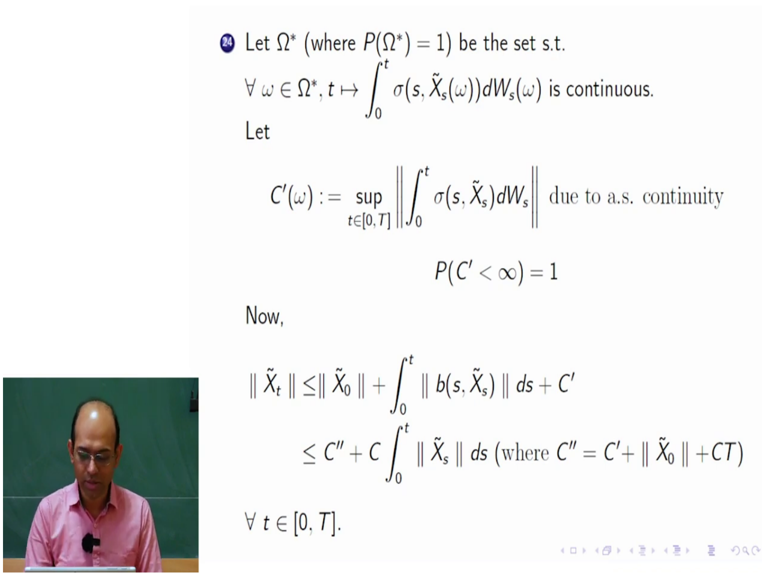

So, there we use triangle inequality. So, norm of X tilde t less than equals to X tilde 0 plus 0 to t this thing plus C prime why? because this whole thing is less than or equals to C prime and then this together the C prime and norm X tilde 0 and from here again using the at most linear growth of b I would get this is less than this to some 1 plus something and then I would get some constant CT everything together. I would write down C double prime. So this whole thing is less than equals to C double prime plus C times of integers 0 to t norm of X tilde sds.

So here same X tilde appears both sides here we have t here we have s 0 to t ds,, we have got these inequality. So this is dictating us that, this is the place where we can use Gronwall's inequality here. This is true for all t in 0 to capital T. So, earlier if I would not have taken X tilde but X I could not have said that, correct? Because it was, the equation was true only almost everywhere, correct. But here we know that that is double star is true for all t. So that is why you can write down this.

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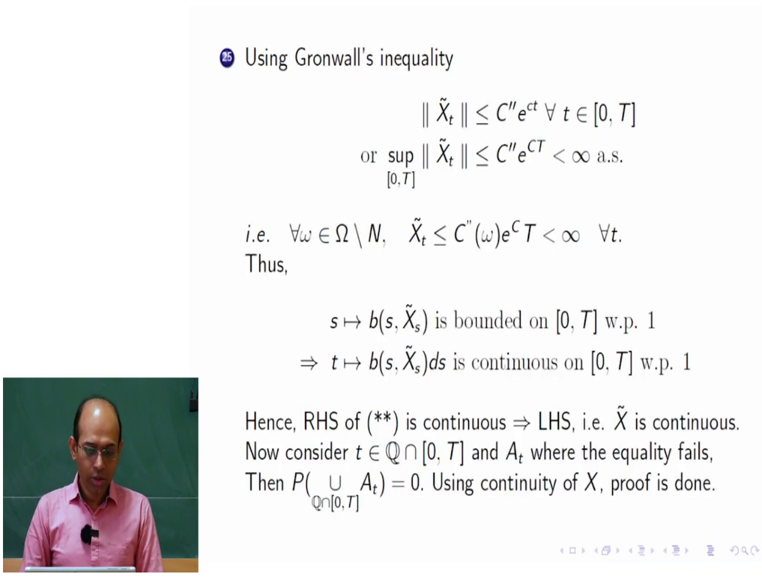

So now we can use Gronwall’s inequality. So to say that, this X tilde is less than or equal to C double prime e to the power of CT.

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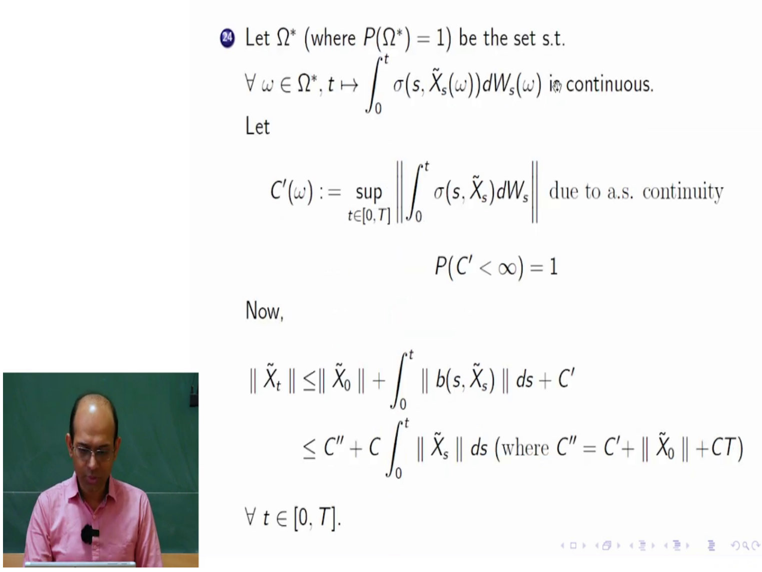

So this C and the C double prime into C.

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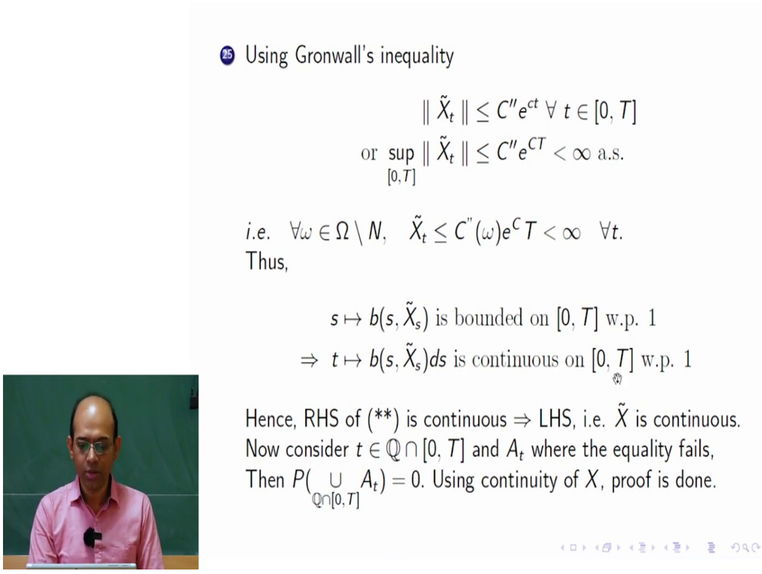

So this is true for all t in zero to capital T. Capital T is finite. So the supremum over all these things is finite that we are getting almost surely. So like in omega star whenever omega in omega star we are getting this. So we are getting this is finite, so now for all omega in omega minus set minus N. So, this N is like you know I mean different from the earlier in basically. So, I should have done omega and omega star here.

So, X tilde t is less than or equal to C double prime omega e to the C this is a typo C capital. So, this line there are many typos actually so, whatever is written here I am just writing this again and all. So, so thus now we consider s to b s X tilde s this map. So, what can I say about this map now? With X tilde turns out to be finite, since it is bounded, turns out to be bounded, for almost every omega.

So for, almost every omega is bounded and b has at most linear growth property. So, this whole this map on 0 to capital T would be bounded, is bounded with probability 1. So, here also there is one type of integration 0 to t this would be integration 0 to capital T, t to that path. So, since it is an bounded integrand on this interval, so, then t to integration 0 to t this as a function of t this is continuous. So, you get continuity of this on this interval 0 to capital T with probability 1.

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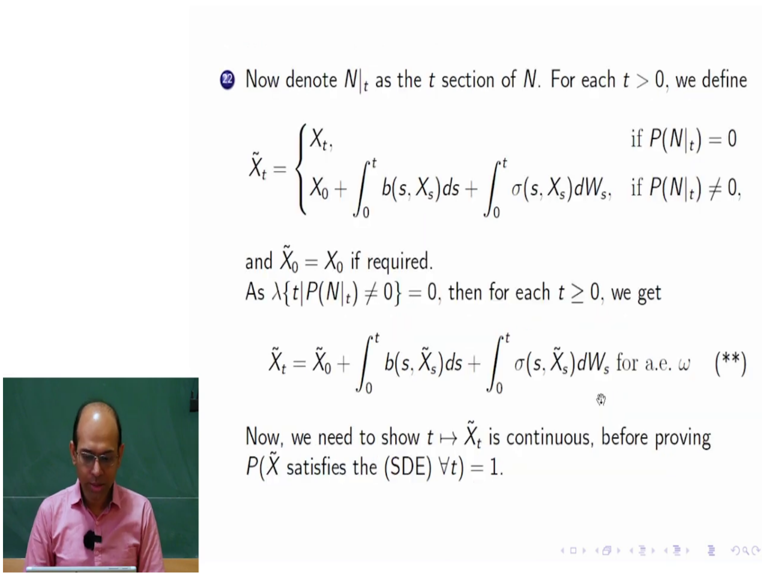

So, here we look into this double star equation here that means, we have obtained that it is continuous with respect to t continuous here, so right hand side is continuous function of t. So, left hand side is also functional of t. So, we prove that X tilde is continuous in t so, this is proved but now we have to prove this.

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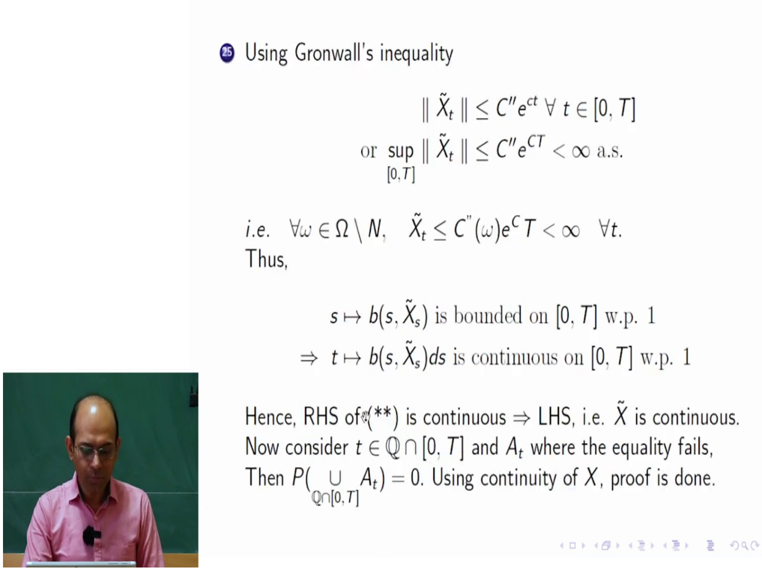

So, now here what we do is that so, here whatever I told here it is written that right hand side a double stras is continuous. So, the left hand side which is X tilde is continuous. Now, we consider the time points t which is a rational number in zero to capital T in that close set all rational numbers. So, this set is countable set this set is a countable set. So, for each such set point t I know that double star is true.

So, double star is true for almost this thing with probability one that means with some small set 0 measure set apart from that is 0 measure set for that particular t which is confirmed this countable set. This equation in this double star is true. We call that At So, At is that is 0 measure set for a particular t which belongs to here such that in complement of At that equation will be true.

So it is the space of fields. The union of all At is 0 because countable union of 0 measure set is 0. So, what we can do is now that we can write down the double star follows, for all t in this set the event of this event has probability 1. So, probability of the event that double star.

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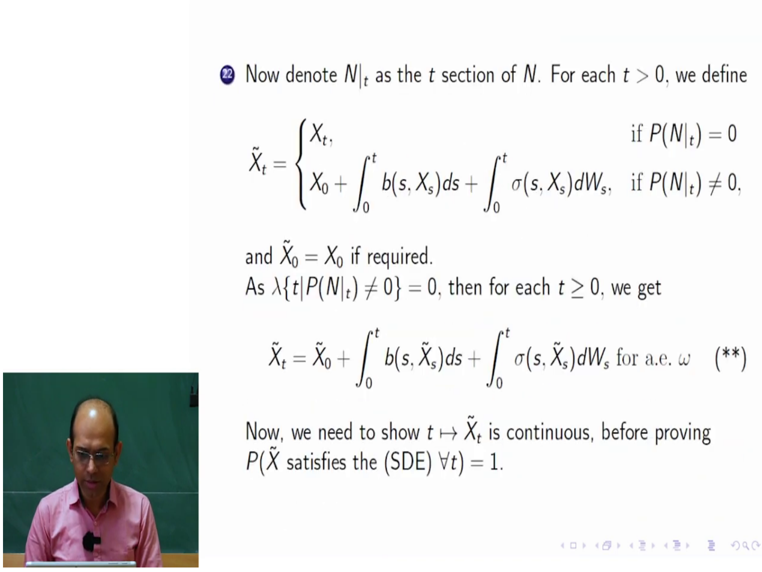

What is the double star, the probability of the event that this equation is true for all t rationals in 0 to capital T that is one? Now, if that is the case, then we use that Xt is continuous So, since it is continuous on both sides is a continuous, so both sides are matching, so the left hand side and right hand sides are matching on a dense subset, uncountable many, so rational number is dense in r.

So left hand side and right hand side are matching or a dense subset and both left and right hand side as a function of t are continuous functions. So they must match on the whole interval 0 to capital T. So using the continuity of X tilde we achieve that X is the solution Like the X tilde satisfies SDE for all t is equal to 1, clear.., thank you very much.