Introduction to Probabilities Methods in PDE
Professor Anindya Goswami
Department of Mathematics,
Indian Institute of Science Education Research Pune
Lecture 43
Stochastic differential equations: Existence (Part 01)

Now, we come to the proof of existence, so for that we use kind of Picard's iteration method.

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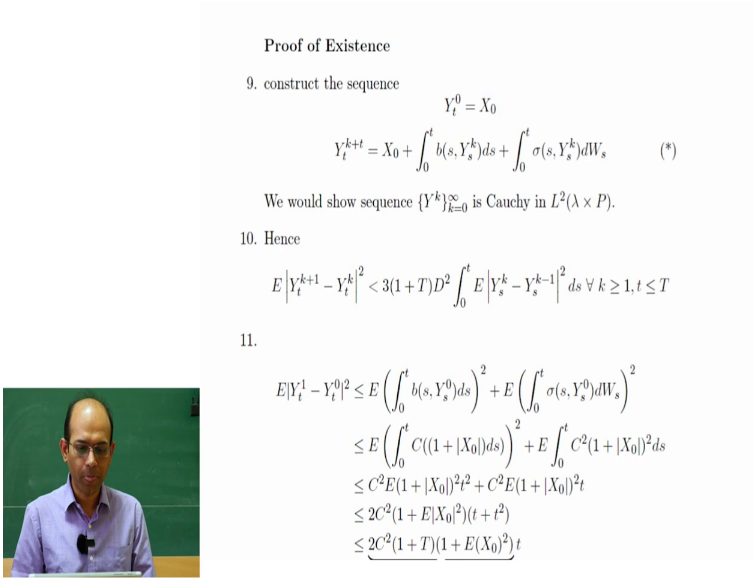

So, the iteration starts from 0, so Y is 0 is the 0eth iteration that is X 0. So, for all t it is X0, just a constant random variable. Now, we define iteratively the sequences. So, we are actually producing a sequence of processes. Y 0 is just a constant random variable, Y k plus 1 t is defined as X0 plus integration 0 to t b of s comma Y k s ds plus integration 0 to t sigma s Y k s d W s .

So, this is the way we define. So, we define for k is equal to 1 first, there is a mistake typo, Yt of k plus 1, this is 1 not t. So, k is equal to 0 case. So, this is known because above it is there. So, we define Y k plus 1 and iteratively we define the whole sequence of processes. We would show that these processes, this sequence is Cauchy in L2 lambda cross P.

We must talk because here a process is a function of omega and t together, so there are two different independent variables t and omega. So, here it is a sequence of we would show there is a sequence which lie in the L2 space of where measure is the product measure, the

Lebesgue measured lambda on the time interval and P is the probability measure and then this sequence is Cauchy in this Hilbert space L2 lambda cross P.

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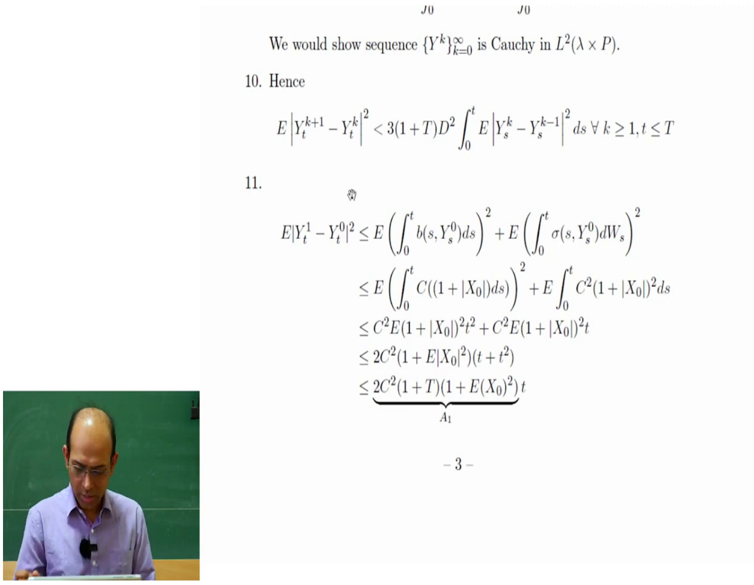

So, since it is a complete space, so one would be able to show that this is a Cauchy sequence that would ensure existence of a limit, and that limit would be a candidate for the solution. And then we would show that candidate is indeed a solution.

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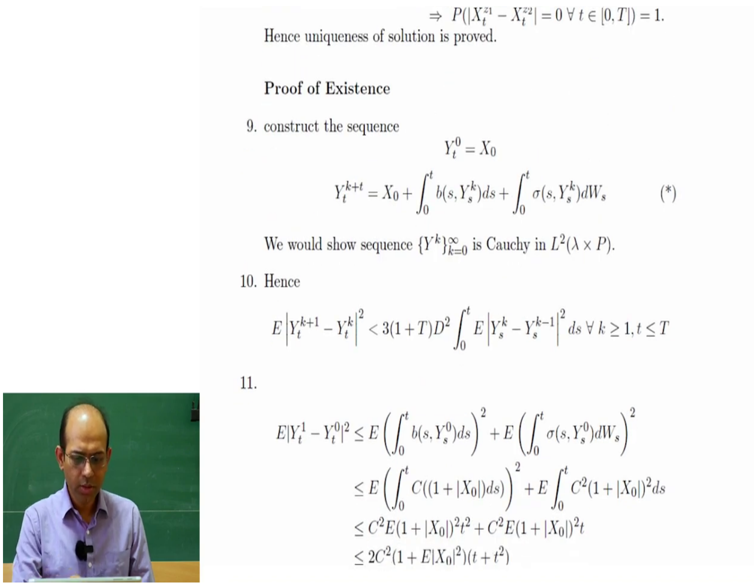

So, what we do is that Yk plus 1 minus Y k t expression of this whole square mod of this whole square this is, so here we see from above. So, exactly you remember, the earlier how did you do here? So, when we assume the existence and we are trying to prove in uniqueness, we had three different terms, exactly here also when we write down Y k plus 1 minus Y k for Y k t I would again get X0 plus 0 to t integration b of s Y k minus 1 s then, left hand side is Y k right hand side Yk minus 1 would appear.

So, those terms would be there. So, Y k plus 1 minus Y k at time t these difference can be written as three different differences, but X0 and they are also X 0 would cancel, so, two different differences would survive. So, from this difference, again going to apply the Ito's isometry and for these difference again we are going to apply Holder’s inequality to obtain again exactly similar manner three times 1 plus capital TD square integration 0 to t expectation of Y k s minus Y k minus 1 s whole square ds .

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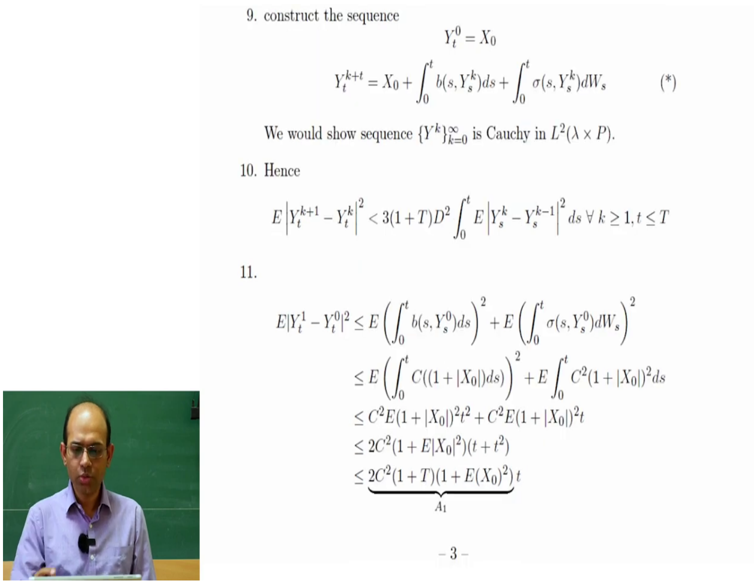

So, this we would get for all t 0 to capital T and k for all k we are going to get this. So this iterative scheme we would require again and again so, for our ease of reference, we are going to call that star. So whenever we would like to recollect this equation, we would call that star.

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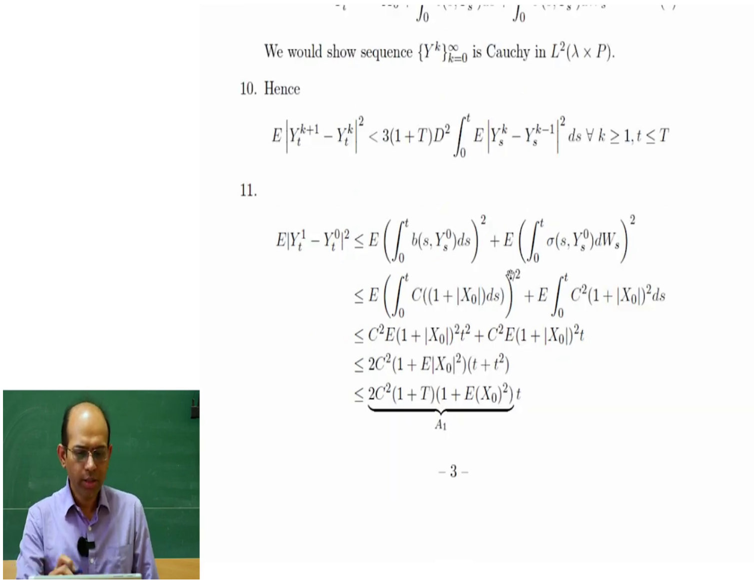

So next what we do is that we start with say k is equal to 0 yes. So here we see that you have this expectation this k plus 1 minus k here k and k minus 1, but here some integration is also there. So, here when we substitute k is equal to 0 here, so here we would get, I mean, Y 0 is X 0 and here say Y minus 1 I mean that is nothing is 0 I mean we did not define anything that we assume to be 0 here. So, here for k is equal to 0 expectation of Y 1 t minus Y 0 t whole square, so this whole square is what?

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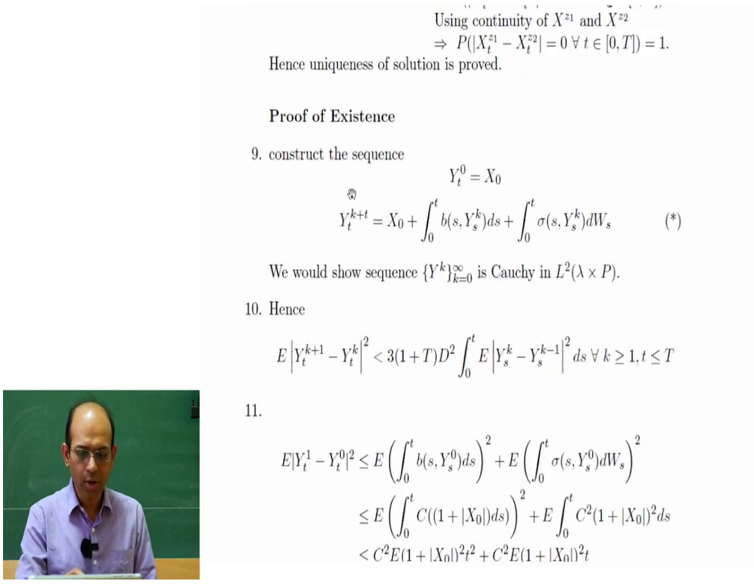

Y 1 t is from here, X 0 plus in 0 to t b s X 0 d s plus 0 to t sigma s X 0 d W s, but Y 1 minus Y 0 would be just sum of these two terms. So 0 to t b of s X 0 d s, plus 0 to t sigma s X0 d W s and then whole square and we are taking expectation. So, here this is a random variable, which depends only on X 0 here this is a stochastic integration, which depends on X 0 and Brownian motion. However, this sigma since it is at most linear growth, what we have obtained there, so it is bounded for these let us revisit here.

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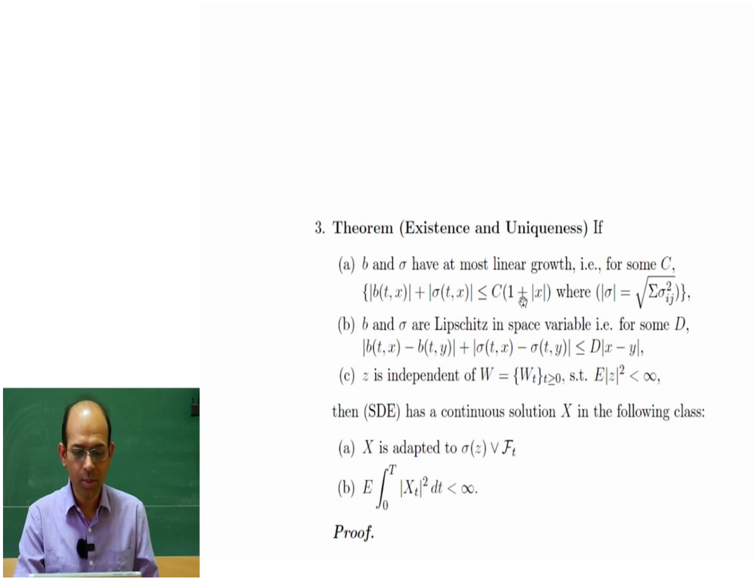

So here we have obtained this. So when this x is not changing, so it is basically bounded. So sigma 0 to capital T it will be bounded if you fix X and that is the reason that so this part would be a just a bounded random variable, fixed random variable, and but s is changing, so it is like a stochastic process but the sigma is bounded because this part is fixed bounded by C basically, C times 1 plus mod of X 0 so, then it is a Martingale part so, then expectation of this 0 expectation.

So, expectation of this random variable multiplied with this 0 to this part. So, that expected in the product we can take conditioning with respect to the filtrations that are generated by X 0, then this part would come out of that insight conditional expectation and then we are going live with expectation of this term which would be 0. So that is the way we can establish that this is less than or equal to this sum of these two terms, the squared terms would appear, the cross term would not appear at this stage.

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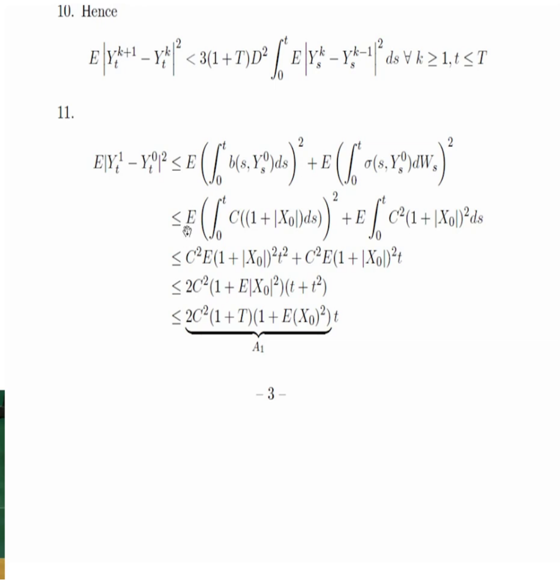

10. Hence

$$E\left|Y_t^{k+1}-Y_t^k\right|^2 < 3(1+T)D^2\int_0^t E\left|Y_s^k-Y_s^{k-1}\right|^2 ds \ \forall\, k\geq 1, t\leq T$$

11.

$$\begin{aligned}E|Y_t^1-Y_t^0|^2 &\leq E\left(\int_0^t b(s,Y_s^0)ds\right)^2+E\left(\int_0^t \sigma(s,Y_s^0)dW_s\right)^2\\ &\leq E\left(\int_0^t C((1+|X_0|)ds)\right)^2+E\int_0^t C^2(1+|X_0|)^2 ds\\ &\leq C^2E(1+|X_0|)^2t^2+C^2E(1+|X_0|)^2t\\ &\leq 2C^2(1+E|X_0|^2)(t+t^2)\\ &\leq \underbrace{2C^2(1+T)(1+E(X_0)^2)}_{A_1}\,t\end{aligned}$$

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And here next we get that this term we are going to use exactly as before the Holder is no, not as before, here, what we are using, we are using the at most linear growth of the function b here also we are using at most linear growth of function sigma. So, here b is less than or equals to dominated by affine linear functions C times 1 plus mod X 0 because Y 0 is exactly X 0 for each and every s and here this sigma would be replaced by a C squared times 1 plus mod X 0 whole square ds.

So here at this stage, we could avoid the square of the integration so the square is inside integration here. So, here again we are going to use that C square is outside. So, we are going to use Holders inequality to get t squared times 1 plus mod X 0 whole Square, and here C squared times 1 plus mod X 0 whole square t. So, 1 plus mod X 0 whole Square is common here.

So that we can further right down 1 plus expectation mod X 0 square times 2, but this C square is already there, and this t square and t together is t plus t square. And then this term here from that we are going to take small t common, what remain is 1 plus small t that we are going to dominate by 1 plus capital T. So, you are going to get 2 C square 1 plus capital T into 1 plus expectation X 0 squared times small t. This whole quantity which is independent small t we are going to call as A 1 this parameter.

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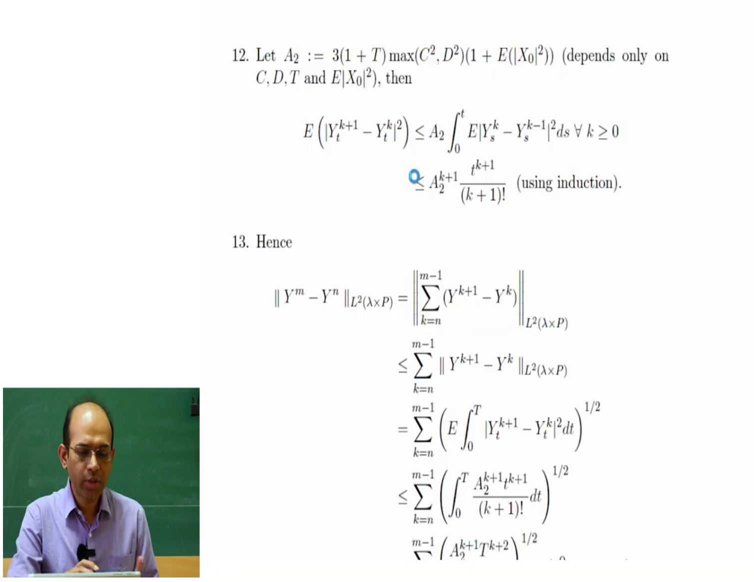

Now we define A 2, A 2 is 3 times 1 plus capital T times maximum of C squared D squared 1 plus expedition of X 0 square that we consider, why do we consider this? Because this is coming from exactly as before earlier.

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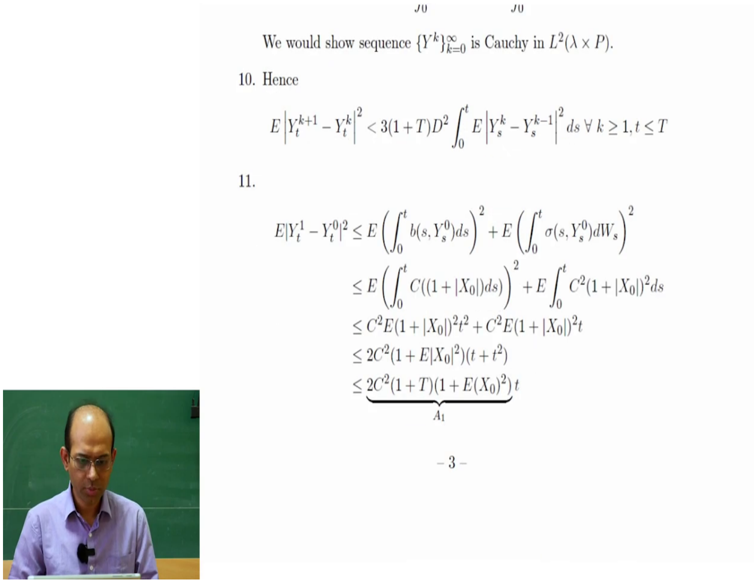

So, we had obtained this equation in the 10th bullet that Y k plus 1 minus Y k whole square 3 times 1 plus T D square and here we have C squared etc. So, to get together everything, so, we are taking you know maximum of these things.

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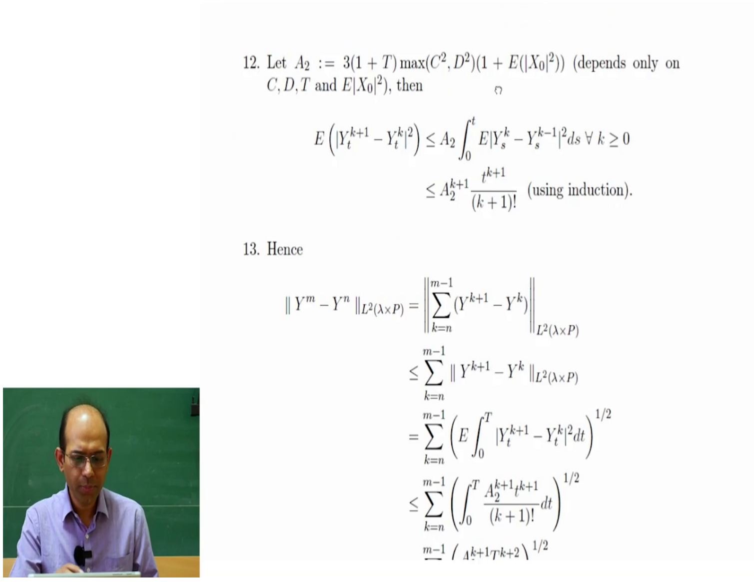

So when we obtain A 2 this way, then we can write down one particular inequality, which works for all k starting from 0.

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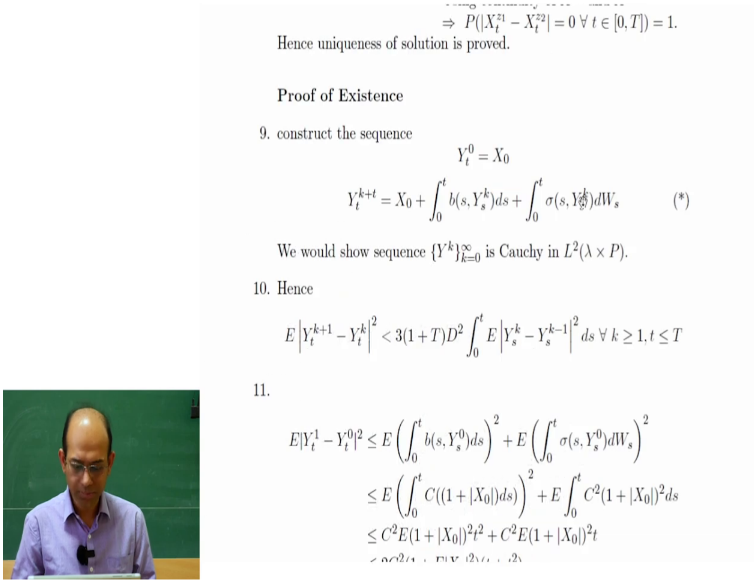

Remember earlier, we had this for k is equal to 1 or more than 1 but now here I can write down using this thing. Here we can write down k is equal to 0 onward, that this condition holds. So, this condition is very nice looking because here say whatever I have on my left hand side I have on the right hand side also, but instead of k plus 1 I have k is I have k minus 1.

So, here we can now use induction. So again here also we can replace by you know the k minus 1 k minus 2 etc. But then integration, there were so many different integration to come. So here would get k different integrations. And if we perform that, we would get t to the power of k plus 1 by k plus 1 factorial and A 2 to the power of k plus 1, every time we are going to replace that inductively we are going to get this quantity as our upper bound.

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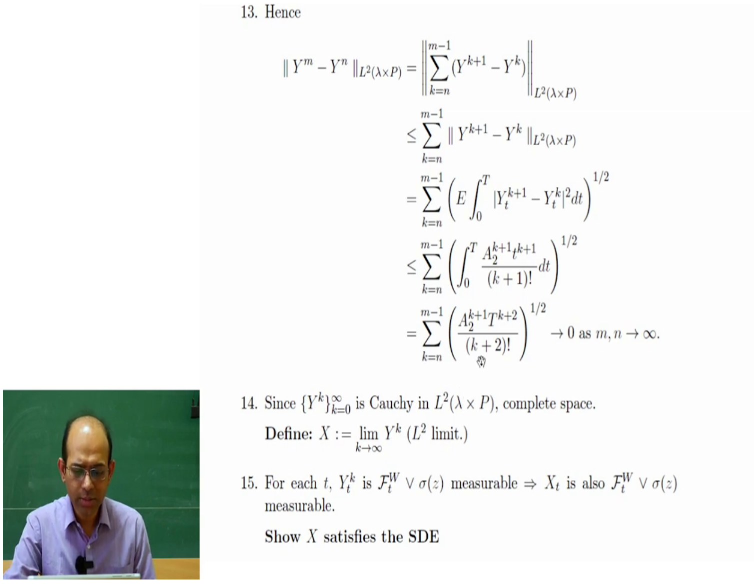

So, this is really helpful, because this is going to say that , this whole quantity is like you know, some poly I mean, bounded by some polynomial in t, . On the other hand, small t is bounded by 0 to capital T, right. So, now we consider Y m minus Y n. So, what is this? the m and neth element of this. So, this term also, this is important to note that this term also appears in the expansion of exponential function.

So, this is general k plus 1th term of exponential function A2t, e to the power A2t correct? So and that series converges, so we are going to use that fact here at this stage. So, norm of L2 norm of Y m minus Y n is equal to here that we can write down as telescopic sum, like you know m I choose here more than n. So, Y m minus Y you can write Y m minus Y m minus 1, Y m minus 1 minus Y m minus 2 et cetera, et cetera till Y n plus 1 minus Y n so that way.

So, that way we have written telescopic sum and then that is less than or equal to Sum of norms, so norm of sum is less than to some of norms triangular inequality and then we are writing down this norm in its full form that expectation of this integration of the square of the difference and then outside square root.

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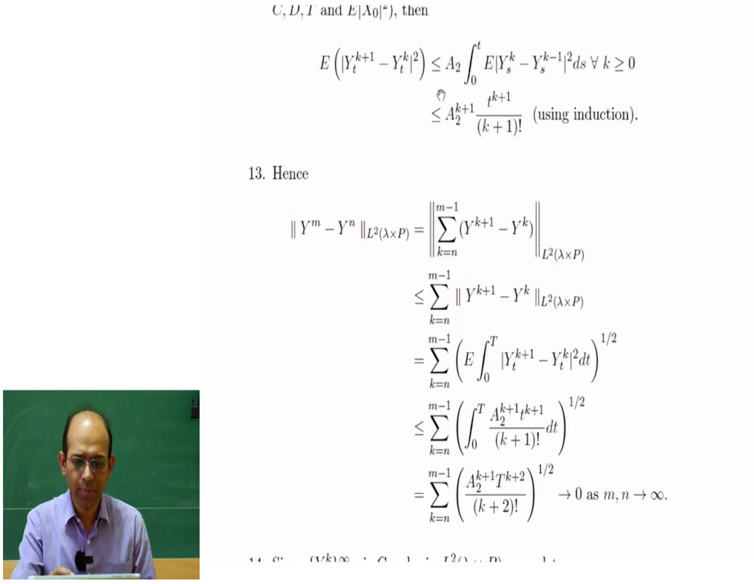

And this is the place where we are using the above obtain the upper bound A 2 to the power of k plus 1 t to the k plus 1 by k plus 1 factorial here we are using it and then we just perform the integration when you do that we get t to the power of k plus 2 by k plus 2 but k plus 2 and k plus 1 factorial would give us give us k plus 2 factorial so we would get this. So, this term is you know, just 1 over A 2 you can have A 2 to the k plus k plus 2 appear. So this term would converge to 0 got it, because as you know this sum you know, as n and m tends to infinity this would go to 0.

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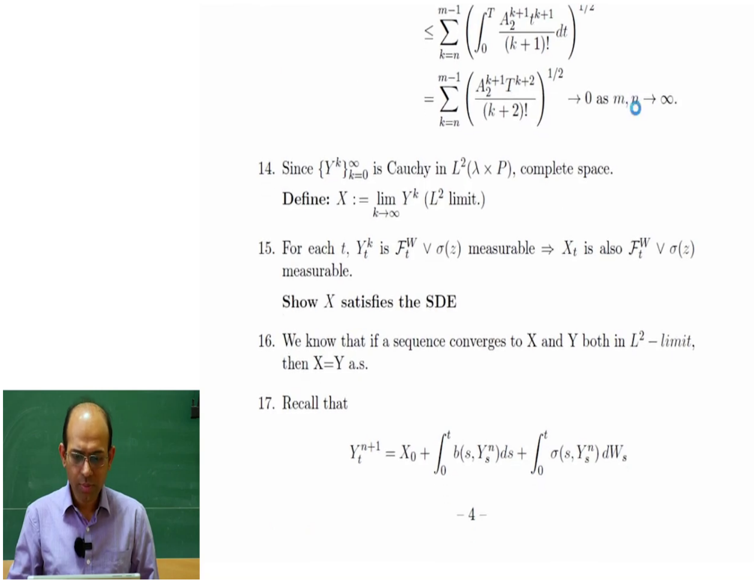

So what we have obtained? We have obtained that Yk is Cauchy in L2 lambda cross P because there Y m Y minus Y n the distance the norm goes to 0 as n m tends to infinity, . So since it is Cauchy sequence so it has some limit, so this L2 limit in this space we call that as X, so this X is our candidate solution of the equation.