Probabilistic Methods in PDE
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Module 08
Lecture 42
Stochastic differential equations Uniqueness

So, today we are going to discuss stochastic differential equation. So, first let me describe what we mean by a stochastic differential equation.

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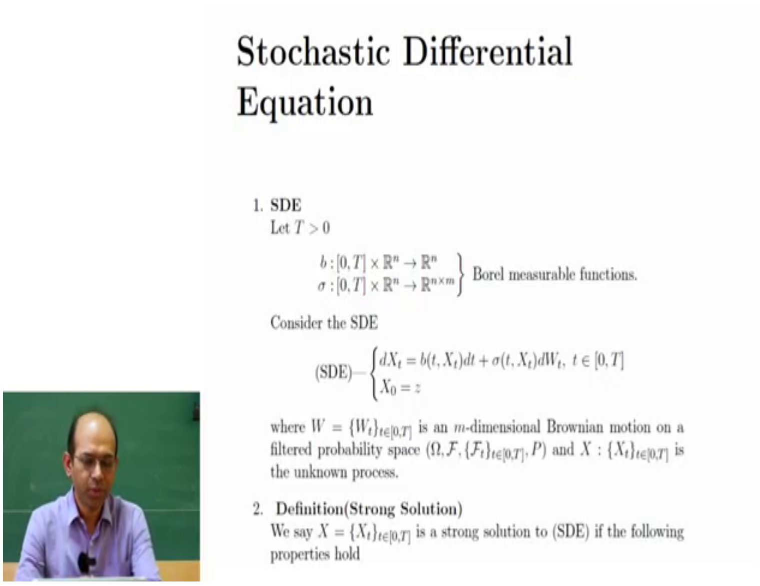

Let capital T is positive. So, here we are describing only finite horizon stochastic differential equation that can later be generalized for close 0 to infinity, but for our discussions we are taking only a finite time interval. So let capital T is positive and imagine that we have b and sigma given, b is called the drift coefficient that is a function of t and the space variable. So, here space variable is coming from n dimensional Euclidean space and b is the function of t and x.

And giving another n dimensional vector, on the other hand sigma is taking value t and space variable x and giving rise to a matrix n cross m matrix. And these two functions are assumed to be Borel measurable functions. We consider the stochastic differential equation, we call these equation as SDE for our repeated reference that dX t is equal to b of t comma X t dt plus sigma of t X t dW t, such equations should be understood that there are some integration sign involved.

So, you can see that X is the unknown here which appears on the both sides of the equation. And that unknown if you integrate that sigma of t of unknown and with respect to Brownian motion that integration 0 to t and here also b of t X t dt 0 to t that unknown and then what are we going to get is exactly X t minus X 0. So, that is the way one has to understand a stochastic differential equation. So, this is true for all small t belongs to 0 to capital T so, here initial condition is X 0 is equal to z.

Here in this, the W as I have mentioned earlier is a Brownian motion m dimensional Brownian motion here. Here sigma is a matrix n cross m matrix and this W is m dimensional Brownian motion. So, in this integration what we are going to get an n dimensional vector. And here this Brownian motion is defined on a filtered probability space omega, F, F t, where t belongs to 0 to capital T and P is the measure. So, here for this I mean if X is unknown process now, let us discuss that what do we mean by a solution of this equation, there are different meaning.

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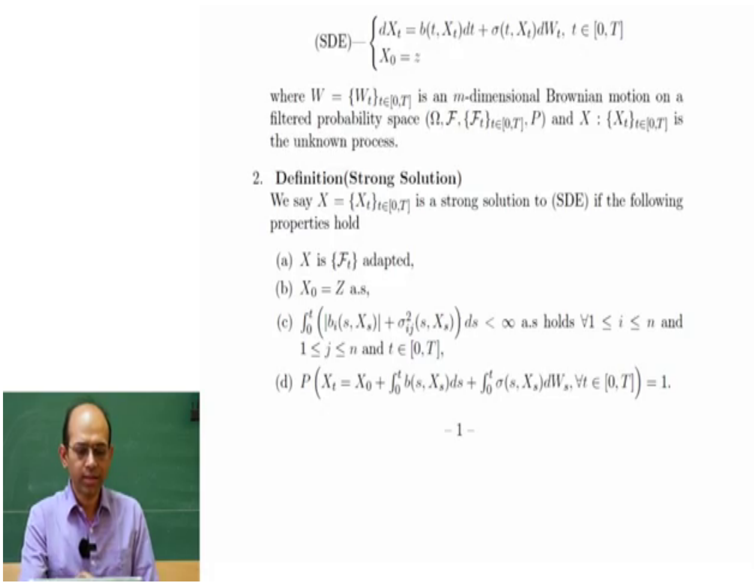

So, we are going to first talk about here strong solution, we say that X is equal to X t belongs to 0 to capital T is a strong solution to the SDE if the following properties hold. First thing is that X is F t adapted, the same F t. So, in principle, I am assuming that this X is defined on that probability space what is given here. So, as W is given to us on this probability space in the filtered probability space, so we are looking for X on this.

So, X is adapted to this filtration, and X 0 is z so this is small z so you type and integration 0 to t, b i is X s plus sigma i j so i j is the i jth element of the matrix sigma, sigma square i j. X s is d s so this modular stands for just the Modular, this is not the because this b i is not the vector b i so, this modular. So, this is finite for all t belongs to 0 to capital T and here as we have written i and j, i is running from 1 to n and j is running from 1 to m.

So that is the condition we want here and since the random variable we mean that this is finite almost surely with probability 1. Now, we call this x as a strong solution if this is the main thing what we must get so that X t is equal to X 0 so, X is unknown which satisfies this equation X t is equal to X 0 plus integrations 0 to t b of s X s d s plus integrations 0 to t sigma s X s d W s.

So, this is true for all t between 0 to capital T and you would observe this true with probability 1. We would see this as probability 1 so if we get such process X in this property space Omega, F , F t then we are going to say that that process X is the strong solution of the given SDE.

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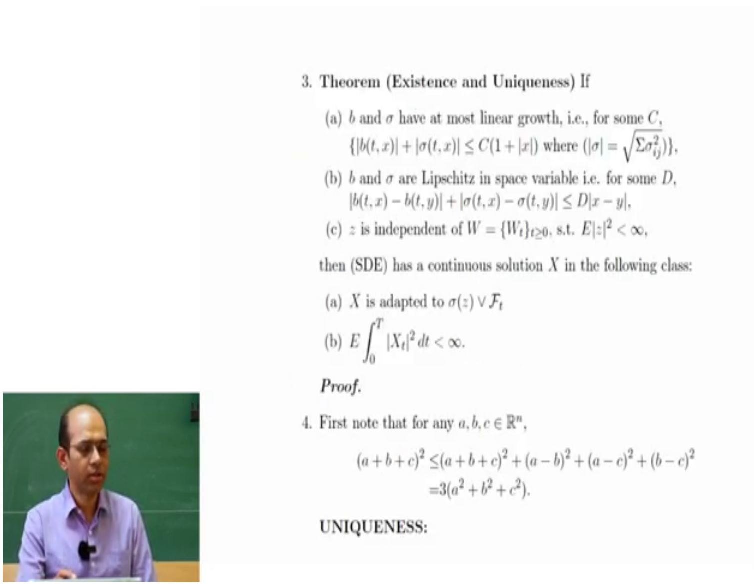

Next we state this theorem of existence and uniqueness of SDE. Imagine that b and sigma whatever given before has some special property under certain special properties b and sigma then only we can assure existence and uniqueness of the solution and it is very natural because even you do not have Brownian motion component, then you just get ODE, we know for ODE also we cannot assure existence and uniqueness of solutions unless we put certain conditions of the coefficient for example, that Lipchitz continuity of the coefficient b that is required for uniqueness as one does not get.

So, we here are listing down those conditions. So, if b and sigma have at most linear growth that is for some constant C mod of b t x so, this norm actually Euclidean norm this although I am using just a mod sign so sign is the norm of this vector and this norm of sigma t, so this mod is stands for this following functional that square root of summation i, j sigma square i, j. So, b t x plus mod sigma t x is less than or equal to C times 1 plus mod x so, like b and sigma need not be a bounded function, it could be unbounded but it would be dominated by one linear affine linear function.

So, when such function any function satisfies this type of conditions, we call them that they have at most linear growth condition. So B and sigma they have at most linear growth condition. Now, this Lipchitz continuity we must have that we must impose this type of conditions. So, B and sigma are Lipchitz continuous functions in space variable that means with respect to this x, so here we are writing what does it mean?

That means that for some D some positive D we can have such type of inequality that mod b t, x minus b t, y plus mod of sigma t x minus sigma t y so, this is less than or equal to capital D times mod of x minus y. So, this implies that these are continuous in this variable, normally this is Lipchitz continuous because D is fixed, D does not even depend on t, here also C does not dependent on t. So, this is called global Lipchitz condition.

And z that random variable for initial condition is assumed to be independent of the Brownian motion and also assume that it has finite variance expectation mod z square is finite is an L 2. Why are you doing that because we want to obtain solution in the L 2 space? So the conclusion of this theorem is that that SDE has a continuous solution X in the following class that X is adapted to.

So here I am writing that sigma algebra generated by z and also the filtration F t because I know that when we talk about F t, we just talked about that Brownian motion is adapted to F t that does not mean that z would also be measurable with respect to F 0 say. So to mention that we write down this that sigma algebra generated by F t and z both together.

So, in this occasion, it is also important to mention that often we start with the just the filtration generated with the Brownian motion, the usual filtration that the Brownian motion we take and then for each t, W t and we take the filtration generated by W t and then that may not satisfy the usual condition as we have seen in towards the few of initial lectures that we augment few more sets, we call those negligible sets, I mean sets which are subsets of 0 measures so that we get a complete so that the measure on those sigma algebras become complete and also we take care so that it becomes right continuous.

So, all those things we do I mean most of the cases people start with just the filtration generated by the Brownian motion, but then augment with some more sets so that it satisfy the usual hypothesis. So, for that one says in a simple word that the usual filtration generated by the Brownian motion so, that is often obtained here. So, we do not bother much because we are assuming that that is given to us, a filtered probability space is given to us and the Brownian motion is given to us and then X is a process in the same filter probability space and then it satisfies that equation.

Also it has continuous path and it has this L 2 norm finite, this is X t square integration 0 to capital T expectation, this is finite. So, this is actually the L 2 norm with respect to the product measure, the Lebesgue measure on 0 to T and the probability measure on the Omega. So the product measure of these two measures is taken here, and this is the L 2 norm under integral of X under that measure.

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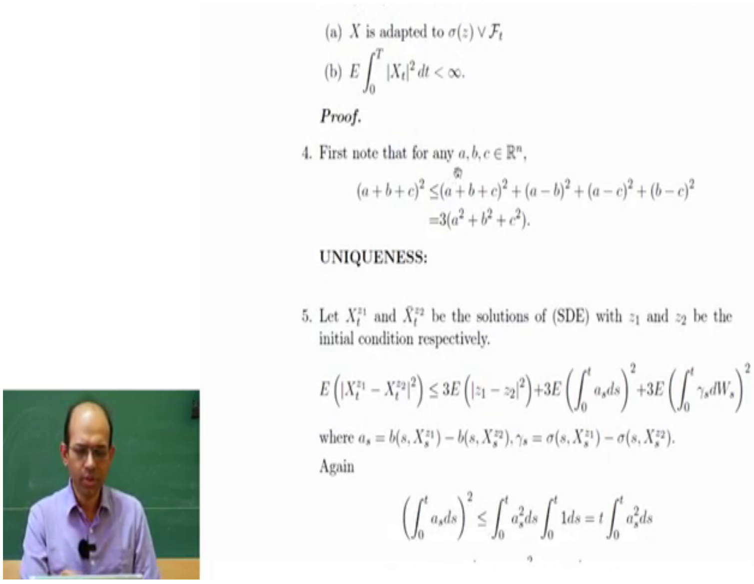

So the proof starts here, for the proof we need certain things, so I am just writing this is trivial inequality. So, when you have a, b, c vectors in R n, then square like dot product a plus b plus c square is less than or equal to a plus b. So, c whole square plus a minus b whole square plus a minus c whole square plus b minus c whole square that you can see because these two terms are same, but you we are adding some more few more things.

So this inequality is trivial and then if we break these things we would see there that is equal to three times a square plus b square plus c square. So we are going to use this inequality to break some terms in parts. So here we write down X superscript z 1 and subscript t, here X superscript so the bar is type of X superscript z 2 subscript t so, what do they signify? So, this signifies the solution of the SDE where the initial random variable is z 1, here this is initial random variable is z 2, solution of SDE with z 1, z 2 be the initial conditions respectively.

Then, we consider this L 2 norm with respect to probability manager, so t is fixed here, expectation of X z 1 t minus X z2 t whole square so that we consider. So that is less than or equal to from the above inequality here because here are these two are these vector. So, this vector can be written as solution of SDE, and right hand side of SDE has three different terms. So here this also has three different terms.

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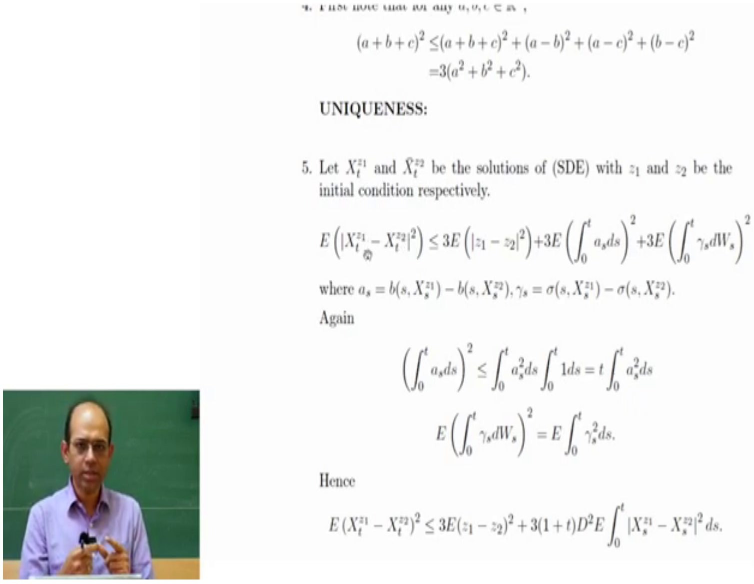

And then if we take the subtractions term by term wise, then there will be one term z 1 minus z 2 would appear here, then plus then integration with respect to dt term that difference would appear here. So which is like b s X z 1 s minus b s X z 2 s that term would appear integration. And then there would be another term that dW the Brownian motion integration term would appear where sigma s z 1 s minus sigma s z 2 s, this thing would appear.

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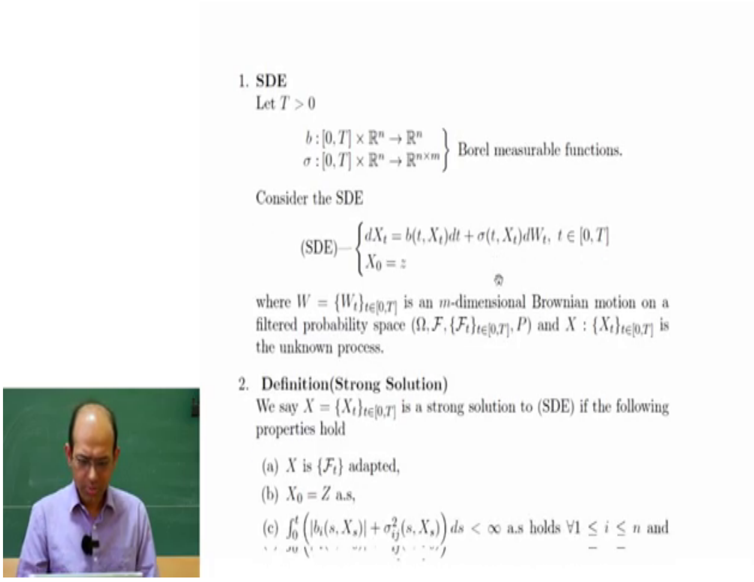

So here the difference would appear here the difference would appear so this difference would appear so this is better to show.

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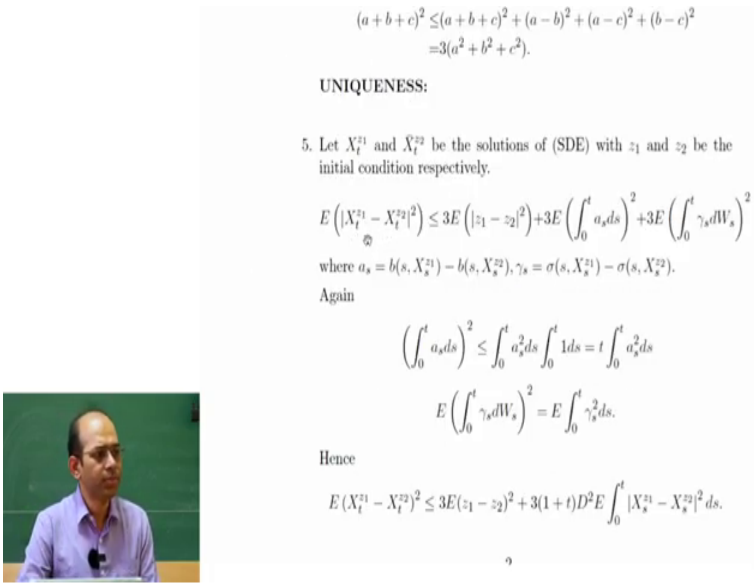

So those three terms would appear here, and then square is there. So a plus b plus c whole square is less than or equals to three times a square plus b square plus c square that we are going to apply here. So, it is less than or equal to three times of z 1 minus z 2 square like a would be my z 1 minus z 2. And then three times that what is this, this is b of X z 1 s minus b s X z 2 s so this difference would appear here.

So, now the whole thing is integration so the square is outside of the integration plus three times expectation of this gamma. What is this gamma? That is difference of this sigma for where this X is with z 1 initial where X is with z 2 initial, so whole square, the square is outside the integration. So, this part we are doing to establish uniqueness.

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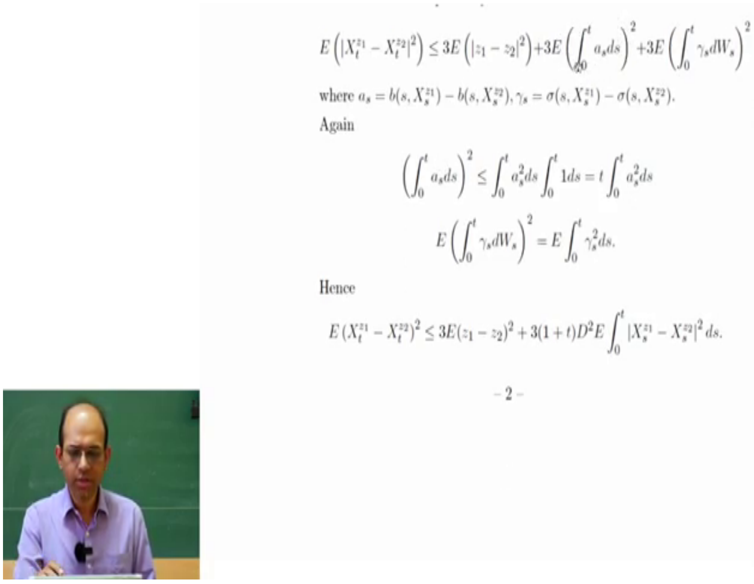

So, here since we are doing uniqueness so, for the time being we assume existence. So that is why we can write down X z 1, X z 2 because we are assuming that solution exists, we are just showing that it would be unique. So, now, we first consider this term, integration is ds so whole square is there we use Holder's inequality here, we can write down a s as one time a s, so that whole square is less than or equal to a square ds and one ds o to t and this part is t, integration 0 to t is a square d s is here.

Now for this term here we use, Ito's isometry 0 to t Gamma s dW s whole square is equal to 0 to t Gamma s square where expectation is there, so, gamma s square ds. So, we are using Ito's Isometry to get this. So, both the places we have obtained integration of sub square terms. The square outside integrations we could avoid now, which is inside. So, we are applying these estimates here at this stage.

So, this left hand side I have written exactly the same, but right hand side I would change now, so this first term is as it is, three times expectations z 1 minus z 2 whole square. Here for this term, we got t times this a square, where a square is like this difference square and we have Lipchitz continuity of b on the second variable. So, this difference for Lipchitz continuities coefficient, D would appear therefore, so this square would be less than or equal to D square times the difference square.

So, I would get t times D square of this expectation of X z 1 s minus X z 2 s square d s coming from this. However, from here we would have this thing gamma s square, where gamma s also involves these sigma difference and here we again apply the Lipchitz continuity property of sigma on the second variable.

So, this square of the difference would be less than or equals to d square times again this square term. So, from here we are going to get this one times here and from here we are going to get t times this thing. So, this two terms would contribute to this two terms so, 1, 2, 3 so three terms we have obtained.

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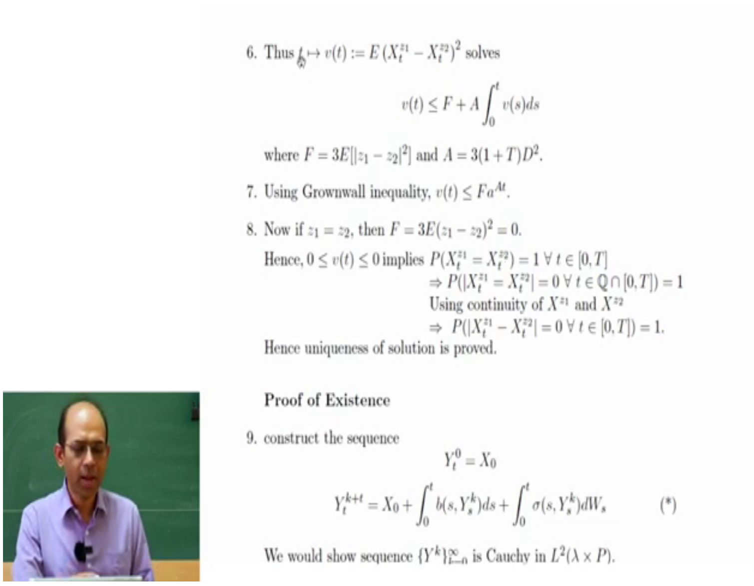

Now we observe this map t to v t, what is v t? v t is defined as expectation of X z 1 t minus X z 2 t whole square. This term anyway appear on the left hand side of the above inequality. So, here this term appear, expectation of X z 1 t minus X z 2 t whole square, this term also appears here. So, if you take expectation inside, can you do that? Yes, you can do that because the inside is non negative, so you can apply Tonelli's theorem to take it inside. So, this expression this is a function of t and that same function appears here also, but there is an integration involved here.

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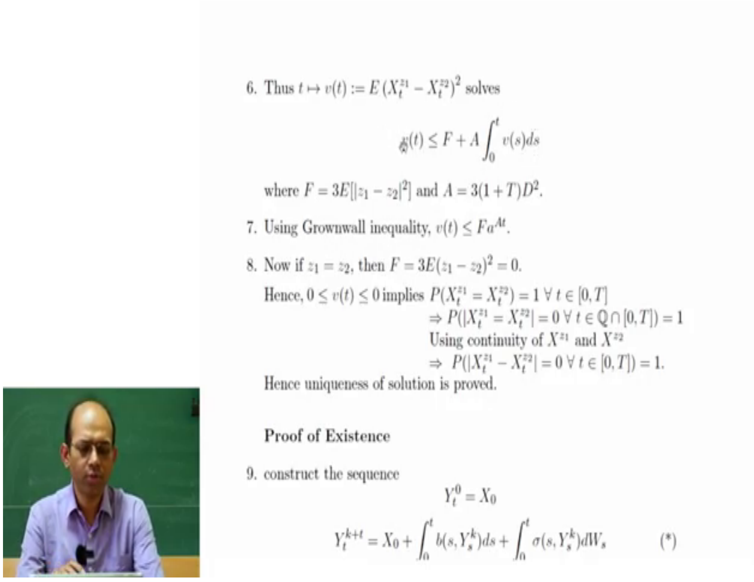

So, if you write down that function as v, which we do now so v t then the above thing is v t is less than or equal to F plus A times integration 0 to t v s ds, where F and A are constants which do not depend on time t. So, here we write down, F is equal to 3 times expectations z 1 minus 1 2 whole square from above and capital A is three times 1 plus capital T D square, I have small t there, but that was depending on small t, but I would like my capital A to be independent of small t.

So I replace small t by capital T since less than equal to sign so you can do that, because anyway this is a non-negative quantity so I can replace by a larger quantity capital T. So, this is precisely the inequality for which Grownwall's inequality can be applied. When we have such a function v which satisfies such type of inequality, then of course, F and A are constant and then we can obtain that v which satisfies this inequality which satisfies this type of v t is less than or equals to capital F times e to the power of A t.

So we are going to use this condition here what we can obtain from here since it is e to the power A t so we understand that this is less than or equals to capital F e to the power A capital T so, we obtain that this is finite, so it gives a bounded function. So now, next what we do is that if z 1 is equal to z 2, so then what happens to F then F becomes 0 and then v of t is less than or equals to v of t is anyway non-negative because it is expectation of square of some random variable is less than or equals to 0 times e to the power of A t.

So it is here on the right hand side so v becomes 0 function v becomes identically 0 function. So, what does it imply? That implies that if your initial conditions match, then your solutions would also match. The probability of X z 1 t is equal to X z 2 t, you would observe that is they are equal with probability one for each and every t. For each and every t we would see that they are equal so, next what we want to show, we want to show that they are unique not only in modification sense but indistinguishable sense also.

So what does it mean that here, we can put these for all small t belongs to 0 to capital T inside the probability, but it is not clear how here what we do is that, since it is true so we can take only the set of rational numbers in 0 to capital T. So then the null set on which this is not obeyed for t in rationals, those are countably many 0 measure sets, so union of that would again be 0 measure set.

So, complement of that 0 measure set would be that probability 1. So, here we write down that that X z 1 t is equal to X z 2 t is equal to 0 for all t rational that probability would be 1, you can do that. Now, we use the continuity property, because we assume the existence of continuous solution of the SDE at this stage, the proof of existence would come later. So these two are continuous, since they are continuous and they match on set of rational numbers and rational numbers are dense in real. So we would get that these two be equal for all t in 0 to capital T. The probability would be 1, so this ends the uniqueness property of the solution.