Probabilistic Methods in PDE

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Module 08
Lecture 41

Simulation of stochastic differential equations

So, today we are going to see some simulations of stochastic differential equation. Since, here the students have various different background. So, I would not assume any expertise in any particular coding language. So, what we are going to do is that we would use just a spreadsheet, which everybody knows to some extent, and just using the spreadsheet, we would see that how to simulate a stochastic process from a stochastic differential equation.

So when we want to do that, we first need to simulate a random variable, because stochastic differential equation involves random variable like Brownian motion. So, we would see first how to simulate a Brownian motion, before that, one must also see how to simulate a normal random variable because Brownian motion has independent increments and the increments are normal random variables. So, we proceed in the following manner.

We would first see that how to simulate a uniform random variable on interval 0 to 1 that is actually a default function given in almost every language. So, spreadsheet also has one particular library function, but for generating normal random variable there is no particular library function but by composing two different library functions we would be able to do that? And then from that, we would see how to simulate a Brownian motion. And then we would see that how to simulate a stochastic process from a given S D E.

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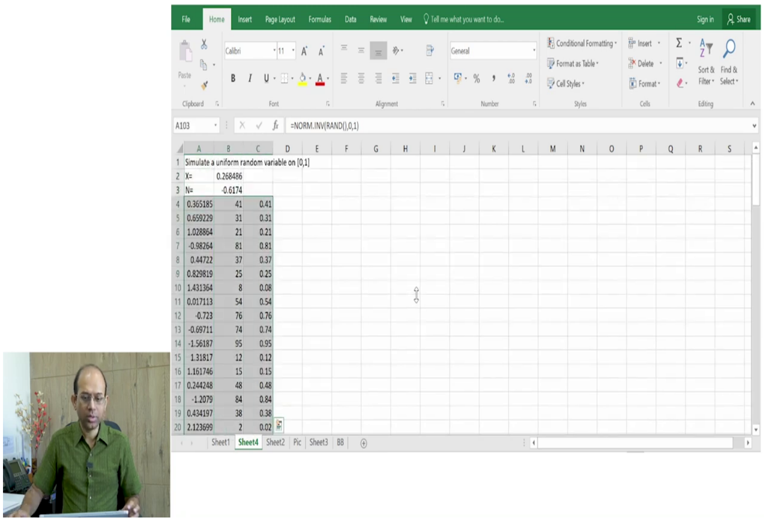

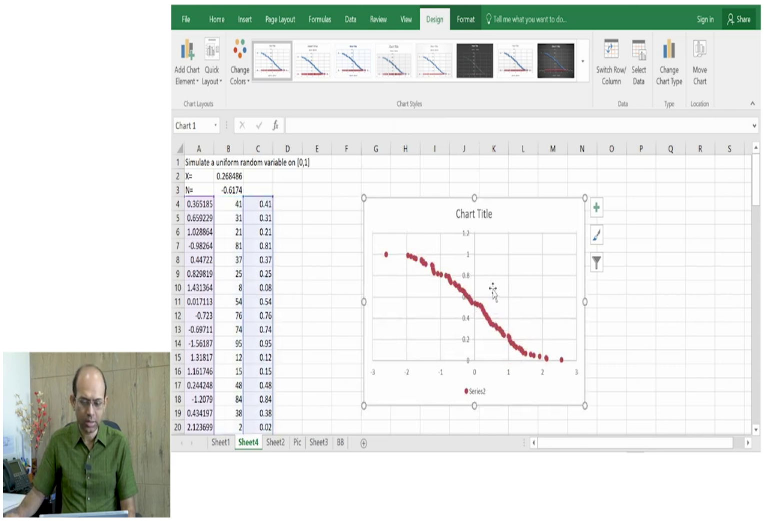

So we start from here, so this is a new sheet where we would like to see that how to simulate so let me write down, simulate a uniform random variable on closed 0 1. So here, I call that X is equal to so here write is equal to sign R A N D, and then write down this parenthesis and press enter, what we get is a number between 0 and 1. So now if you press F 9 that is calculation key then it would recalculate however, for some technical reason I would not press the F9 key, I would rather use delete button. So, I would place my selection to some blank cell and press delete button.

So, we are going to see different realization of the random variable. So all the time we are getting a number between 0 and 1. And these numbers are random sample from the population of which has no distribution uniform on interval 0 1. So, one can ask that how can I be sure that this is really coming from uniform distribution, that is also not difficult one can actually have instead of one particular say 100 particular 100 cells the same random variable, so, and then one can plot histogram or say empirical CDF. So, from that one can see that it is coming from that distribution.

So, we are going to do that latter for some for uniform we will not do that because it is coming from the library function. So, now we take it for granted that it is coming from uniform random distribution on interval 01. So, now we take n, n is a normal random variable so that we would like to generate, so what we do is that we take inverse of the CDF of normal random variable and that we compose with this uniform random number.

So, then the result what we are going to get would be a random variable, but that would have distribution according to standard normal random variable. So, let us see that so, here we see normal inverse so, this is the inverse of CDF. Here one should pick up this probability to be this so inverse, and mean to be 0, and standard deviation is say 1 standard normal you want and then you close.

So, now here what we are getting here is a random number, not between 0 and 1 but so, here this one could be any real number between minus infinity to plus infinity and however it is coming from the population, this is random sample from the population of normal distribution with mean 0 and variance 1. So, let us see various different realizations. So far it is all positive because the way we have defined when capital X value that means this second row this thing is half or more than half then this n is non-negative.

However, if this is less than half then you see that n is negative. And that is quite expectable the normal random variable is half of the time with mean 0 is half of the time positive, half of the time negative. So we see that sometimes it is positive, and sometimes it is negative. So now here there is the point when one can ask is it really coming from normal random variable? Let us see the distribution.

So for that what we do is that we would generate one, array of this, so we start here, so is equal to normal inverse and instead of probability I write down here directly Rand, so this Rand is the uniform distribution from 0 to 1 and then mean is 0 and variance is 1, so we are getting some number and then I drag it for 100 such things, so I go to so I just coming for 4 a 103 and from here to here I select and then I copy this formula down. So, we have now 100 different realization of normal random variable with mean 0 and variance 1.

Now here we would like to see that what is the CDF, we can plot histogram or something, but here we plot. So here what we do, we take rank, rank all this thing so this number, than this whole list whole array, so that is giving me the rank of this number. And then I would like to keep this array as it is so we put dollar before the row number, row name so that if I drag the formula handle, the list selection does not change.

So, now we are done with the putting formula, now we drag it so, this is the rank of these numbers. Now if we just plot, so here these numbers are between know 1 to 100 because these are ranks correct, these are ranks but if I want to draw it versus so, like 0 to 1 then you have to divide by 100.

So, let us do that is equal to this divided by 100 so that now it is ranked between 0 to 1, not between one to 100, so here we drag the formula down. So, we have got this thing so now, we just insert a scatter plot. So here this thing I do not need, so, I just delete this thing so, this thing I need, so it is coming the reverse direction ranking so that it looks like CDF, I should put ranking from ascending order so, they need to look like proper CDF so that we do here.

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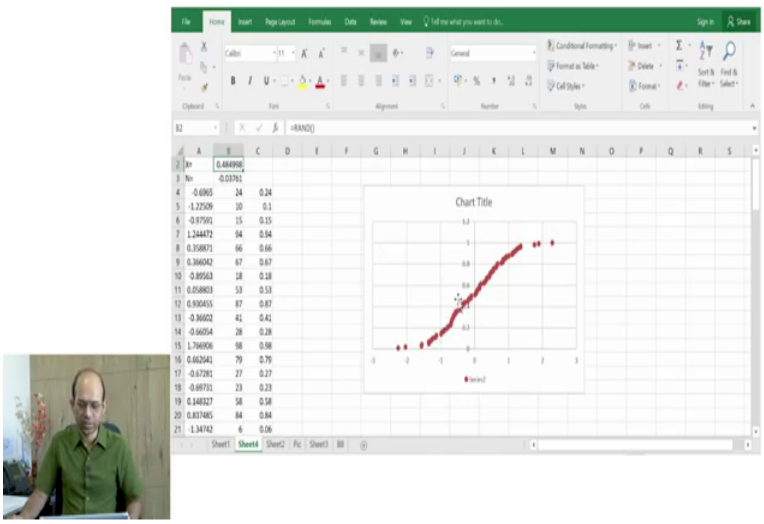

Now it looks like a proper CDF so, see that it looks like a CDF, the cumulative distribution function of normal random variable. So with mean 0 and variance 1 here, I mean we did not have any realization beyond minus 3 and plus 3 and then here it looks like this. So, this made us quite confident that what we have simulated using a normal CDF coming from normal random variable so I delete this thing.

So now if I want to simulate a Brownian motion using this random variable what are we going to do? We are have to you know add small-small normal random variables as increments and then proceed. So that we do here, so here first we do the time axis for example, this my column is my time axis, so I write down time.

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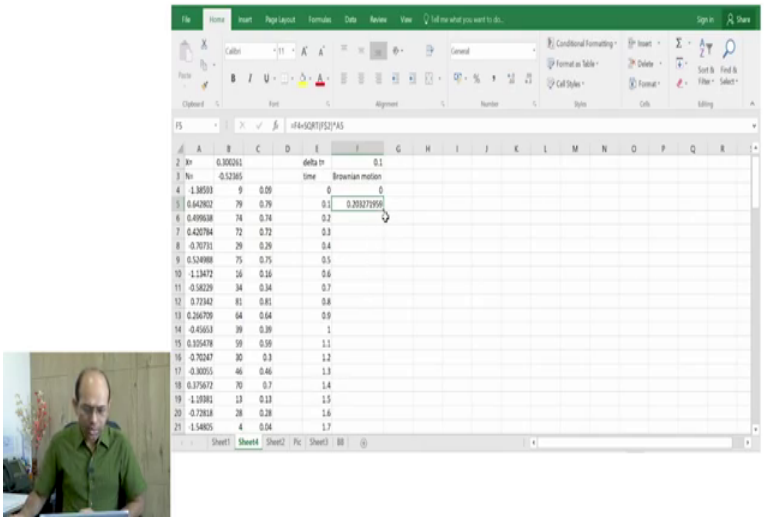

So and then increment granularity you have to choose so let us choose that say 0.1. So, delta t is equal to say 0.1, so then time starts with say, so 0 is actually Brownian motion is 0, so we do not need to simulate anything so times 0 and this is the Brownian motion, so now here the next time step would be first is 0, next step is equal to the above 1 plus this delta this thing. However, we do not need one to move these delta t this value should remain as it is as we move the formula handle.

So here we would go to 100 steps so, we go at 103 and then we create the time axis. So time axis is done, so here last number should be like 10, 9.9, plus number is 9.9 so time axis is done. Now for Brownian motion, what we need to do is that we need to recall the definition of Brownian motion, there is the increment size is delta then the Brownian motion increment is normal random variable with mean 0 and variance Delta. So here the normal random variable what we have obtained here had variance 1.

So if we multiply with the standard deviation with this, then we are going to get so what is the standard deviation? Variance is delta standard which is square root of delta, so here delta is 0.1. So what I do is that is equal to so Brownian motion is incremented so early thing whatever I had plus so now square root, so s q r t square root of this number F 2, so square root closed multiplied with this normal random variable what we have done already in the A column. So that would give me the Brownian motion value at times 0.1.

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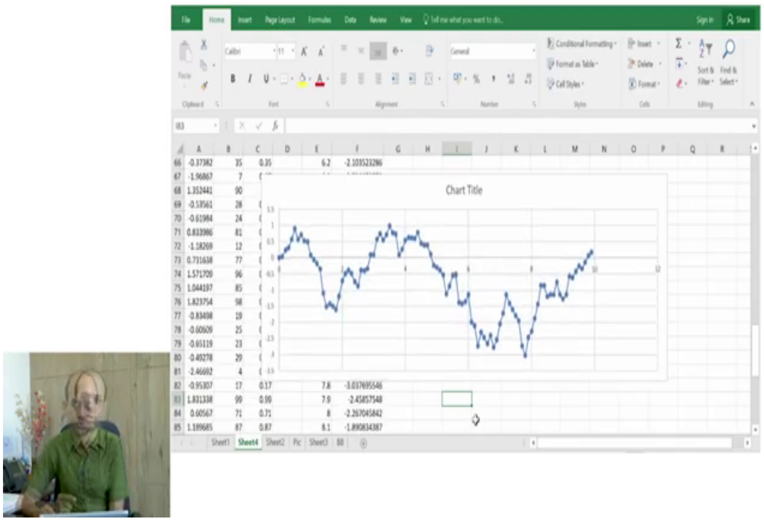

Now what we are going to do, we are going to get the whole realization so that I have selected F column 100 meny and then we press down. So, what we have obtained is the simulation of Brownian motion here , but we would like to check that how does it look like so, we are going to do that, we are going to plot it, we are going to insert one scatter plot here. So, that is giving me a particular realization of Brownian motion.

And if we recalculate, so we are going to get another sample of Brownian motion. So, let us see so I go to a blank cell and press Delete So, that would request spreadsheets to recalculate the sheet so we would get a fresh set of new random numbers would be generated and then from that whole path would be generated. So, we see that here we have 0, so at times 0, Brownian motion is 0, but then when we run several times, so here it is going negative but sometimes it goes also positive so like this goes positive.

So, this is actually not precisely a Brownian motion, this is more or less a random type of thing. However, here this is an approximation Brownian motion, because here with this granularity of 0.1. So, but if you want better accuracy, you should have much smaller granularity, instead of 0.1 maybe 10 to the power minus 3 or something. However, I mean so, so these are the way to simulate a Brownian motion.

So now we go to the case when we have a stochastic differential equation, so in the last week's lecture we have seen Brownian Bridge. So that satisfies a certain stochastic differential equation and then we have also seen a particular realization of the solution of that is D and Brownian bridge has a very nice path that particular relation you have seen that it was going from 0 to 1 and then so, we would like to see that code how to do that. So I come to Brownian Bridge, so this page.

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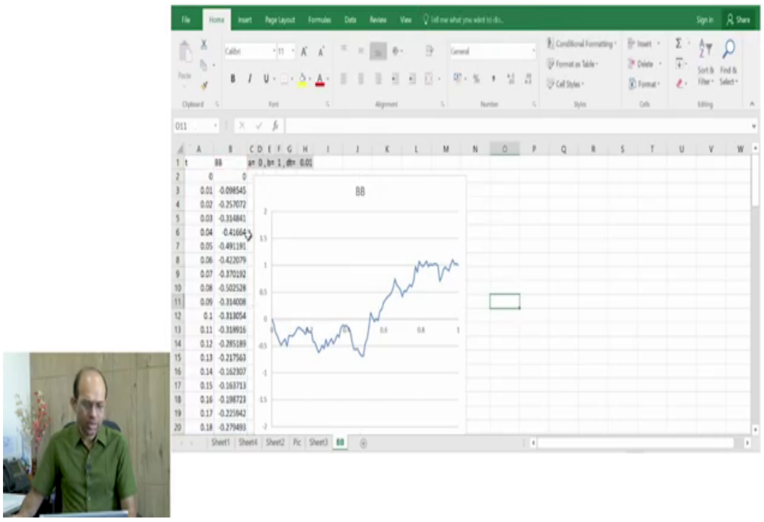

So here just as before, what we have done we have taken a time axis so this column A is time axis. So here increment is dt, dt is equal to 0.01 increment, and here so we have 0.01, 0.02, 0.03, et cetera. We have done it for 102 so that we get exactly 101 number of points so that last number is 1. So we have partition of 0 to 1 with 101 number of points or in other words 100 number of sub intervals.

So here what we are simulating is the Brownian Bridge from point 0 to point 1, so, this is point 0, this is point 1, so see 1 is here 0 is here. So, this is the time axis horizontal axis is the time axis and here vertical axis is the Brownian bridge value. So, let me do this so that it looks better. Now we before explaining the formula here. So, let us see how realization changes if we ask the sheet to recalculate.

So, here it changes, so the breach point 0 and 1 is pinned here. So sometimes Brownian Bridge is also called pinned process, so pinned here and we see that for various relation we are seeing various different paths connecting 0 and 1. So this plot clearly shows that the code whatever we have written, so that is doing what it is supposed to do. So let us see the code here, so this code is very simple.

It is just a spreadsheet application so now we recall the stochastic differential equation that is d X t is equal to Xt divided by 1 minus t dt plus dWt. So, what we do is that here the increment is dXt so we look at the increment, so whenever I write down increment so this is like every cell is increment plus the earlier time so earlier values. So earlier value the first one is 0, so 0 plus so B 2 is 0. So, it depends on many things it depends on B 2 0 and also it depends on the B value and A value.

So, B value is here 1 and A is 0. So, but actually I mean A does not appear in the SDE appears in the initial condition. So here, F 1 is basically B value minus B 2, B 2 is this earlier state. So this is one minus x divided by 1 minus A 2, A column is time column so it is 1 minus t. So, numerator you have 1 minus X t and denominator you have 1 minus t and then h 1 is the dt 0.01 dt term is green colour.

And then you have a Brownian motion increment so as we have seen earlier in the earlier sheet, the normal random variable inverse CDF of the inverse of that, and then this Rand so that is going to give me the probability of that and then with mean 0 and variance square root of this dt, so that is Brownian increment so, this whole quantity is the DBT, it stands for DBT. So that is the formula here and then as we go down we see that just started for B2, B2 becomes B3 and A2 becomes A3, and all other things are as it is.

So that with respect to time 1 minus Xt divided by 1 minus t that only changes with time t and dBT is the same increment however, independent realizations. So, all the time a different cell calls for the same library function Rand, one gets independent realization of that uniform random variable on 0 1 so that is the way it is done here. I think I have pretty much explained how to simulate a Brownian bridge from the stochastic differential equation.

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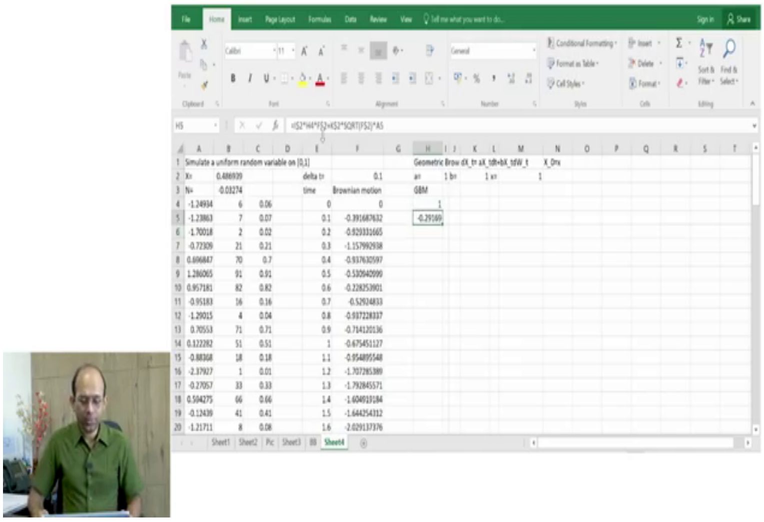

Next we would see simulation of another stochastic differential equation. So let us simulate geometric Brownian motion. So that we can do actually on this sheet itself, but let us go to the earlier sheet 4, where we have simulated the Brownian motion. So the geometric Brownian motion already we know the definition. So let us recall, so geometric Brownian motion that satisfies the following stochastic differential equation that is d X t is equal to particular values a, a times Xt dt plus b times Xt dWt.

So that is the equations stochastic differential equation for geometric Brownian motion. So as before, we have to specify what value of a and b we are going to select here, so let us write down those, so say a is equal to say we take say 1, and b is equal to we also take these say 1, and then dt we also need to specify. Say it is still the same dt as 0.1 so same dt we take and then here we would like to write down this Xt that is geometric Brownian motion Xt so I write down GBM

So, now GBM has actually this SDE has a strong solution and the strong solution has also a closed form. We know that we can actually simulate from the closed form solution itself. However, for many SDE one does not have closed form solution so we were actually trying to learn how to simulate from the SDE itself. So here we would just simulate from this SDE so this is a starting point I should mention what should be this so, starting point say X 0 is equal to say, can I put 0?

No, I cannot put 0 because if it starts from 0, right hand side becomes 0 because a times 0 dt plus b times 0 dwt, so dX would be 0 so increment would be 0 throughout the time horizon, so the whole process will be just trivial 0 process. So, that is the reason that we will not put 0 but some x. So, a is going to say 1, b is equal to 1, small x is equal to say also that we put at 1. So now we have fixed all the parameters what we need, now we can so let us make it smaller. So, here is equal to first initial point would come from here x is equal to 1.

Now d X so that is equal to a times but a is I 2 but these parameters we do not want to move so we will put dollar before the row number and then times X t just earlier time that is this quantity plus b, b is K 2. So K 2 times dWt. So dWt here already we have come up with the increments Brownian motions. So these are the increments square root s q r t square root of dt, dt is same as here, multiplied with this standard normal this thing, so you have got a realization here.

So, we got this value here. So as before, so here we have done see for Brownian motion, the earlier value plus square root of F 2 plus A 5, E into A 5. A 5 is the standard normal variable, so here we are doing exactly the same thing here. So, for these Brownian motion part is concerned that K 2 is b times square root of F 2 times A 5. So, this part is as before, but only one thing is that here we have a drift term that is coming in the beginning so, that is a times xt dt so, I forgot to write down dt here. So good that I have revised so dt is again this thing with if F 2 put a dollar now it looks good.

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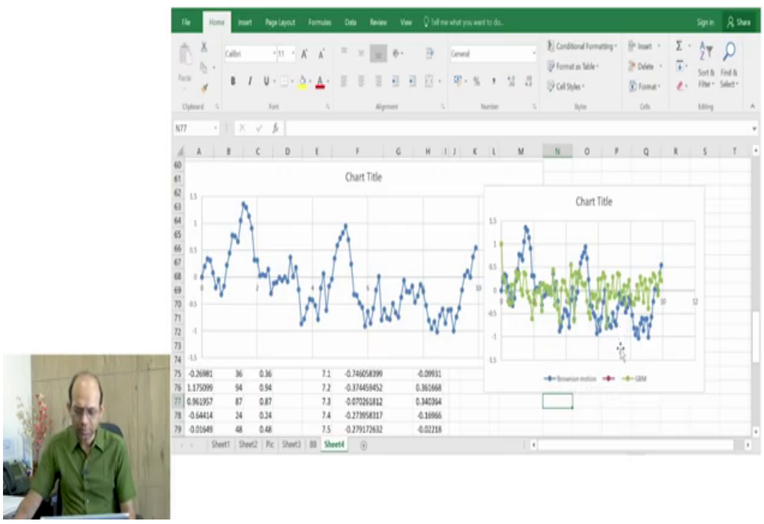

So, we are going to plot 100 such realization successively. So that we do here down good, so we go back above and we try to plot. So we try to plot this thing by inserting this plot so this plot came very nicely. So here we I mean the Brownian motion, so earlier plot of Brownian motion appears here and here we get the plot of Brownian motion and geometric Brownian motion where the green thing is a geometric Brownian motion and the blue curve is the Brownian motion, so let us put this little above.

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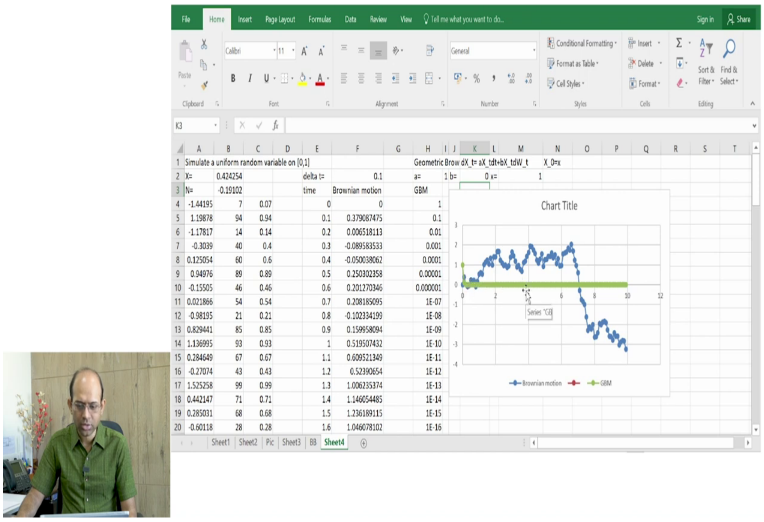

So, now one can ask that what would happen if I change those parameter a and b. So, of course I mean then the curve would look different. See if my b is very small then the Brownian motion effect would be minimal so the variation would also be very small. On the other hand, if b value is very large than the variation fluctuation would also be quite large.

So, let us do this small trivial experiment here since b is very small 0.001 very small value. See this green thing is almost it is like flat, it is not fluctuating at all. So, on the other hand if I put B is equal to say 0 directly then it is just exponential decay, it is like ODE solve solution ODE from 1 it converges to 0.

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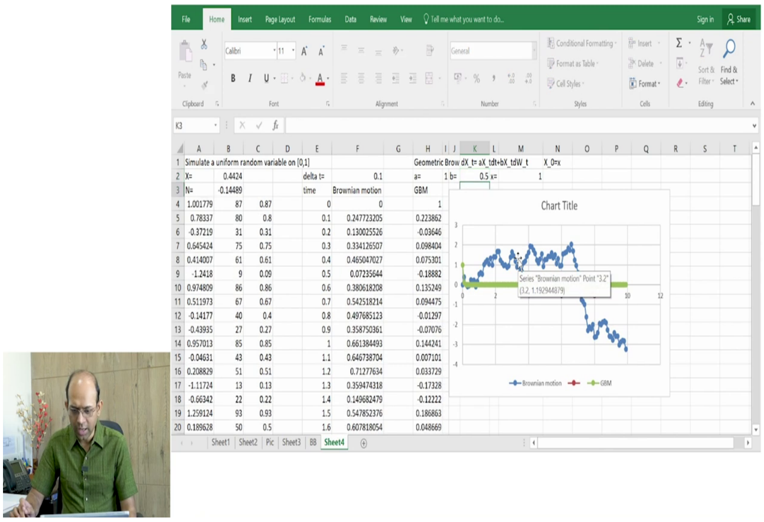

Now, if my b is not small but not large, in between say 0.5 then we see that compared to the Brownian motion the blue line, the GBM fluctuation is less but it is very close to 0.

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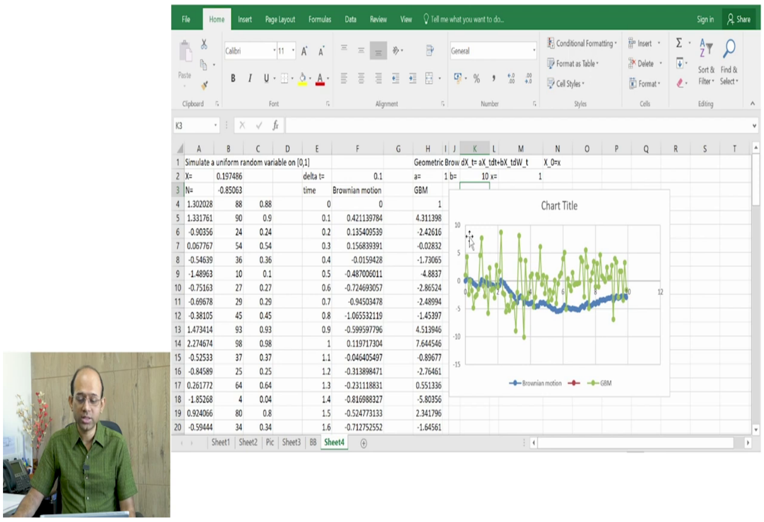

However, now, we ask the other side of question that of when b is very large, say I put b is equal to 10. Then we see that due to the rescaling of this graph, this blue line looks flat which is Brownian motion and green line is going away the variation.

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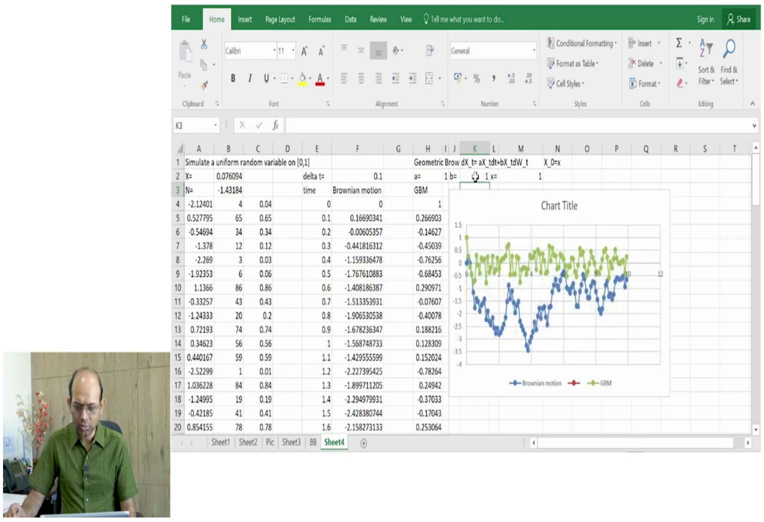

So now we come back to say b is equal to one case. And then we talk about how does the solution behaves by changing a value.

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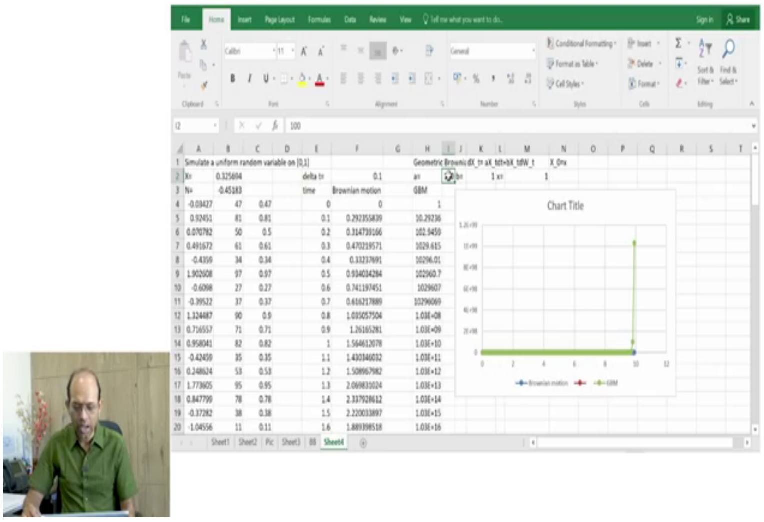

So a is 1 here so, instead of 1 if I put a large value say 100 so then we see that it becomes very very large so that is too large I mean like exponentially it is increasing or hardly you can see anything, so we put not this much larger say only the value 3 or maybe say 10.

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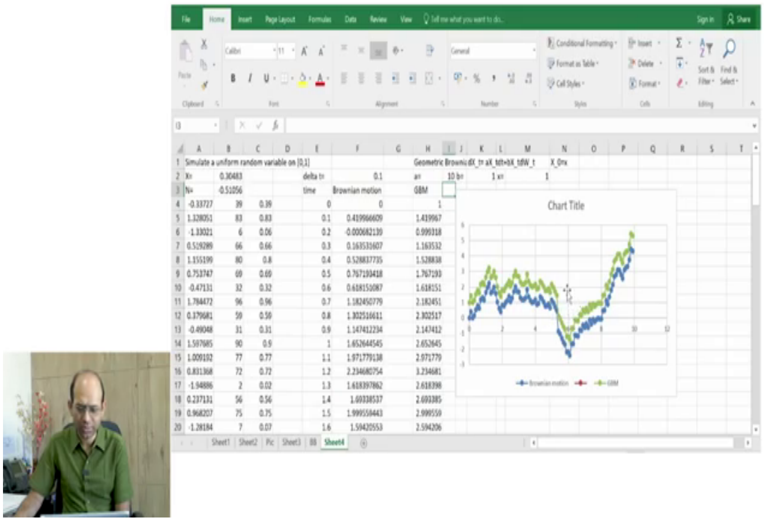

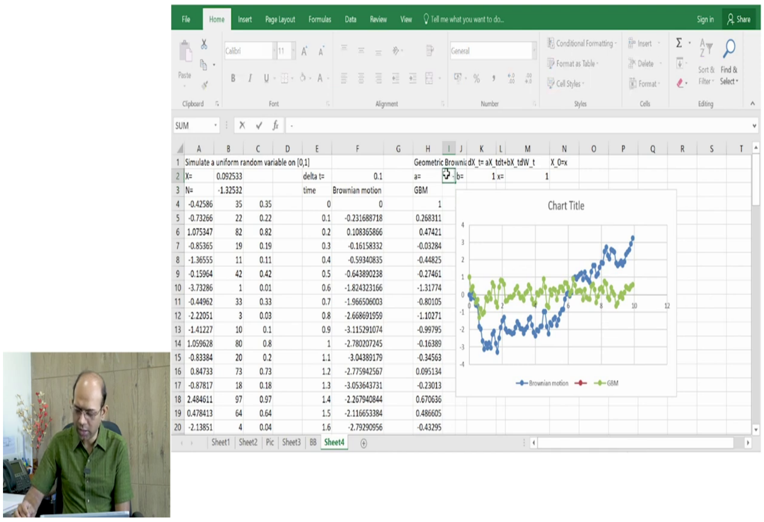

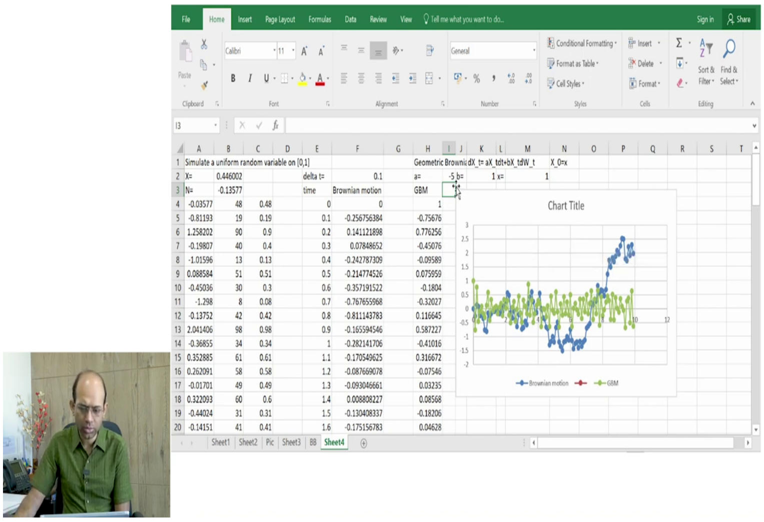

Simulate a uniform random variable on [0,1]
Geometric Brownian dX_t= aX_tdt+bX_tdW_t
X_0=x
delta t=
time
Brownian motion
GBM
Chart Title
Brownian motion
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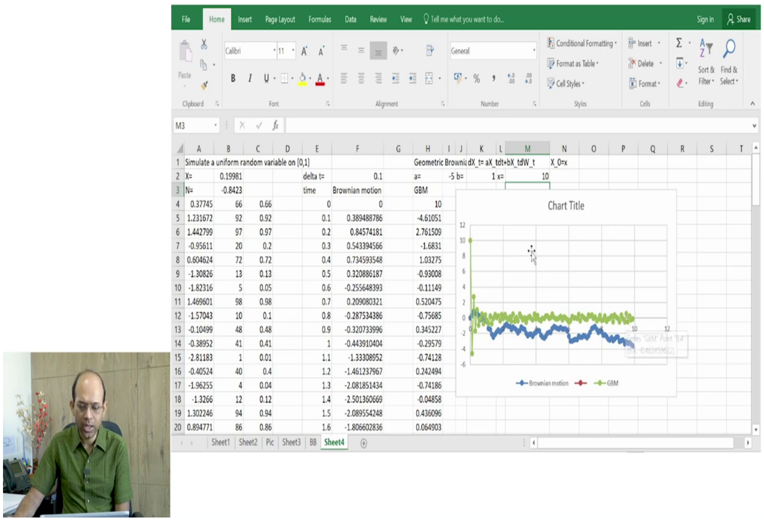

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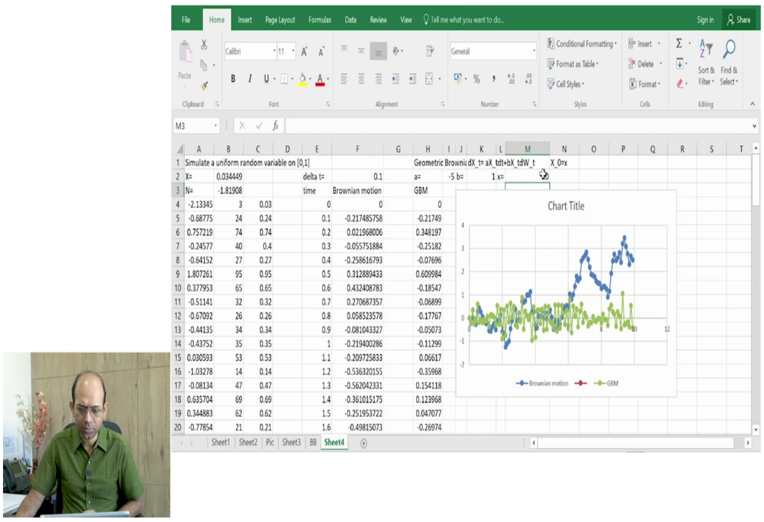

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Brownian motion
GBM
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Sheet2
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Sheet3
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Sheet4

So, when I put a value is equal to say 10 then we see that I mean it is like a matching magically you know both are like going like parallel kind of things, . However, if we put the two smaller values say 5, then we see that, this coming like this. So, now we look at the negative values, so for negative values it should go down to 0, so that is happening actually it is going down to 0.

So now we have a pretty good understanding that how it moves. Now putting x value is 1 say starting from above say 10. So then it starts from above and then slowly it comes to 0. On the other hand, if x value is just 0 as we expect it should be flat so here we are not getting exactly flat thing. I think I have I missed one very important thing that is why which was a dangerous typo equal to this plus, it was a dangerous typo.

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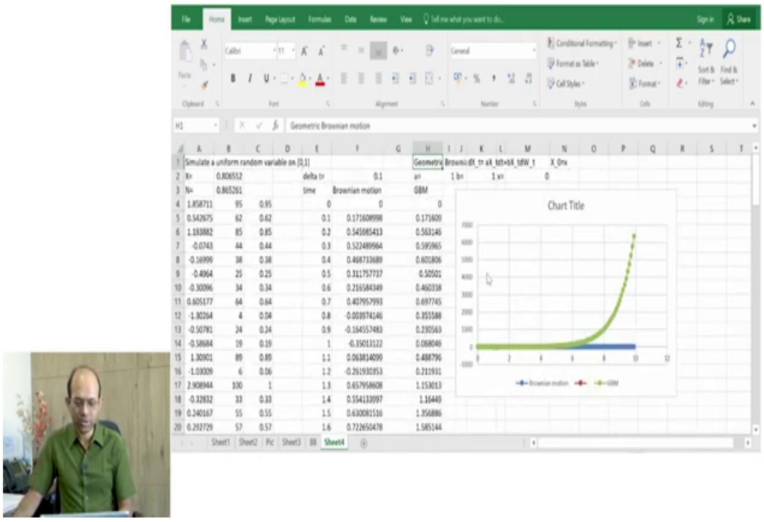

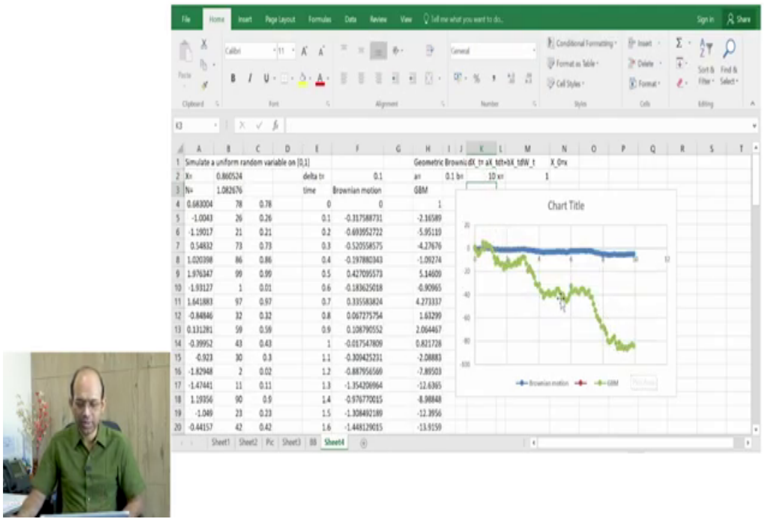

So, now it is a proper geometric Brownian motion, earlier we had a dangerous typo. So, here now, if we put x is equal to 0 it is mostly 0 and then some fluctuation is appearing here, but if you put x equal to 1 so, let me place a small value of a and b 0.1, b is equal to 0.1 so then as before you know that small variation, so Green line is almost same and then large variation for 10 so, Green line is varying very largely. On the other hand for a and b is 1 so, it is like going almost like parallel kind of thing. And then if x is equal to not 1 but starting from small numbers, 0 so then looks like this. Thank you very much.