Introduction to Probabilistic Methods in PDE
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Lecture 40
Brownian Bridge

In the last tutorial session, we have seen this system of stochastic differential equations. And then we have I mean demonstrated that how the v, the second component behaves over time and then this system of equation actually, you know is used for modeling asset prices, because this v stands for volatility fluctuations which you know moves you know, within some range not a hardbound range, but there is a mean reverting dynamics of the process v. And then this S is model for asset price dynamics in financial market.

Okay, today, we are going to see another stochastic differential equation for which also we have some very nice geometrical pattern and for which you would be able to interpret that we can actually explain the pattern from the equation also.

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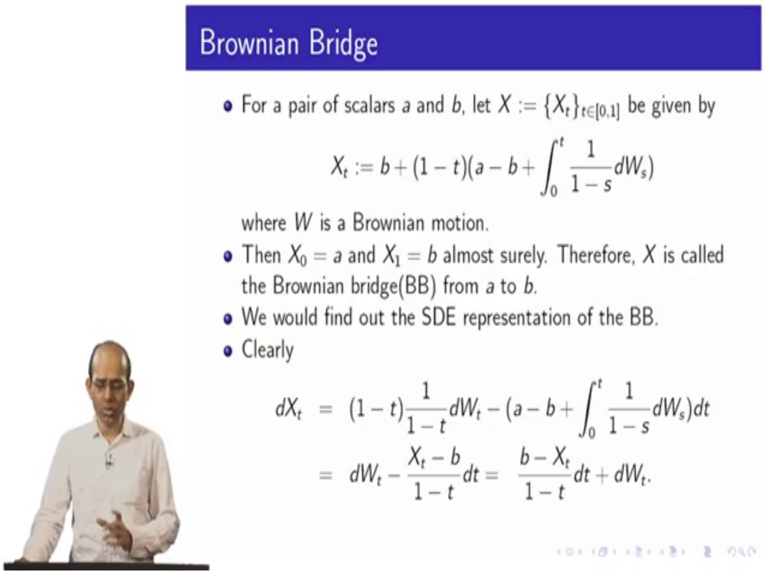

Okay so, let us start that. So, this is Brownian Bridge, so what is Brownian bridge? So, for a pair of scalars a and b let X which is a stochastic process define for time only between 0 and 1 be given by this equation. So, and then we would call that X is a Brownian bridge from a to b. So, what is the equation, let us look at the equation more carefully. X t is defined as b plus

1 minus t times a minus b plus integration 0 to t, 1 over 1 minus s dW s, okay where W is a Brownian motion.

So here, let us understand that, how does this process look like at the end points first? So, here t is running between 0 and 1, so endpoints are time 0 and time 1. At time 0, we can put 0 here. So if you place 0 here, this integration vanishes. On the other hand here, I would get 1 minus 0, so that is 1 only. So, I would get a minus b here, but this would not be there.

So, a minus b but plus b already present here. So, b plus a minus b would give me a, that means at times 0, X 0 is equal to a. Now, if we want to understand that, how does it look, you know at time t is equal to 1, then we have to place t is equals to 1 here, here. However, for this whatever it is, so t is equal to 1 but here t is equal to 1 makes everything 0 correct, 1 minus 1 will be 0.

So, this part vanishes, this part would not be there and we get to b, correct. So, X t is equal to b then X of 1, so X 1 would be b. So, that means this process at time 0 is a, at time 1 is at b okay. And this process is called Brownian bridge between a and b. Why do we call it Brownian? Because it is coming from Brownian motion that is giving there randomness and why is it bridge because it is bridging between these 2 points a and b.

However, these 2 points could be at the same points also. So, it will make bridge. Okay now, we would like to understand or derive what stochastic differential equation for this process. So, this is the definition of the process, but we would see that you derive actually the stochastic differential equation for which this is a solution.

So, here this is the comment that X 0 is going to a, X 1 is equal to b almost surely, so this is called Brownian bridge from a to b. So, we would find SDE representation, but that is very easy. So, what we do? We write down dX t here, so differential we take from both sides. So, dX t is equal to, so this is constant, so it vanishes.

I would get from here minus dT and then this product correct so, minus dt times this part and 1 minus t times again the differential of this that is 1 over 1 minus t dW t. So, that we write down here. So, first we write down this thing that it is 1 minus t times, so here 1 over 1 minus t dW t and then minus sign here the whole thing a minus b plus integrations 0 to t, 1 over 1 minus s dW s and the here from here minus dt. So dt and minus sign is here, okay just taking differentials of both sides I get this.

Now just simplification, so here we can understand that this term can be rewritten as X t minus b minus 1 minus t, okay so that we do here. So, this a minus b this thing is X t minus b divided by 1 minus t. So X t minus b 1 minus t divided by minus by 1 minus t. Here minus sign is there, so I get this b minus X t divided by 1 minus t.

Okay, so, dW t, so from here we get dW t here minus X t minus b divided by 1 minus t dt and then adjusting the negative sign, we get this very nice equation okay, very short equation dX t is equal to b minus X t divided by 1 minus t dt dW t. So, in this equation, we see 2 terms.

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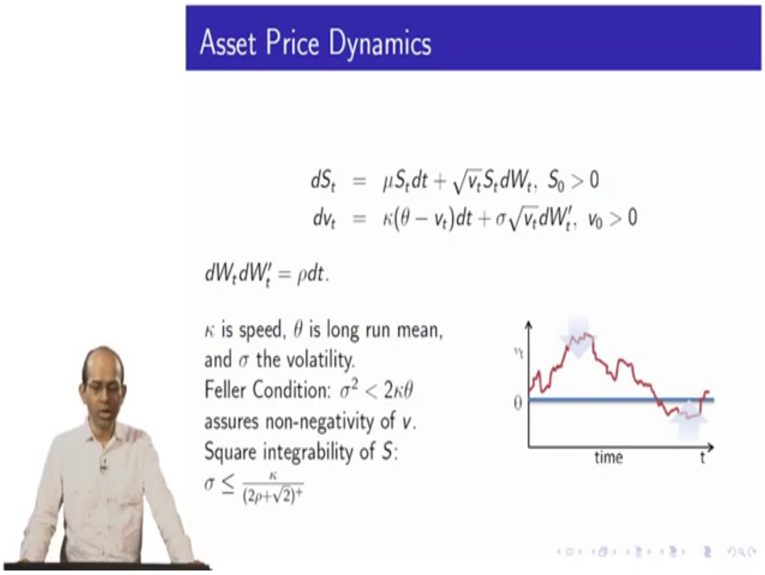

And we relate that we also have obtained that 1 term is dt term, another is dW t term, okay. So, in general ODE, you just say that dX dt is equal to f of X comma t okay and then that denominator dt if you write down that means you write want to see ODE as integral equation then you get dX is equal to f of X t dt, okay.

But here you have extra additional term this Brownian motion that is giving you the randomness uncertainty and that is why we are calling this stochastic differential equation.

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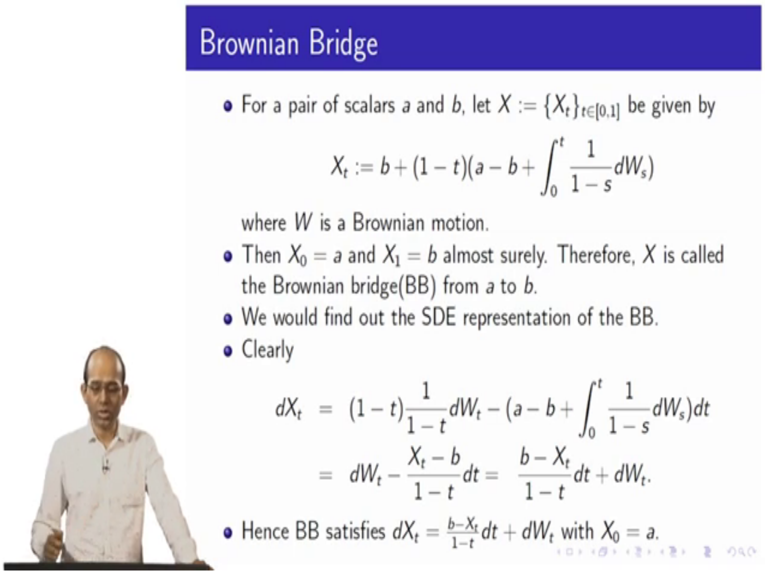

Okay so, here also we are getting that, that dX t is equal to b minus X t divided by 1 minus t dt plus dW t. So, that is the equation and where t is ranging from, you know 0 to 1 and initial condition is a. So, this is the SDE for Brownian bridge from a to b. Now, we are going to study few more other properties of Brownian bridge.

So, imagine that I have a Brownian bridge from a to b where a and b are not equal this b could be larger than a or could be smaller for without losses let us assume that is larger than a. Then this you know X t as it shows here it is continuous because it is integration with respect to Brownian motion that integral we have already seen that it is continuous martingale correct.

So, it is continuous and this is continuous, so this is actually having continuous path almost surely. So, if the process goes from a to b, then any point between a and b say for example c should be touched by the process at some time tau. So now, whatever the random time tau at which the process hits the level c which is in between a and b, so from the time onward if I look at the process okay, then this is like connecting c to b okay, is that a Brownian bridge?

Answer is not pretty yes because after all Brownian bridge by definition should be between time 0 to 1, here time is maybe smaller than 1. So, one can ask that okay, if I change time in appropriate manner, so that after scaling of the time I get from that point, tau to 1 is like, you know, 0 to I mean like 0, 1 interval, if I do this mapping.

After that mapping, can I make it a Brownian bridge? The answer is yes, we can do that. So, that is a thing I am showing it here. So, we do not need anything other than just manipulations just calculus. So, I find that okay that is very interesting example for tutorial that one would be confident there, how does this manipulations do happen and whatever we have learnt earlier can be used here to achieve this, okay.

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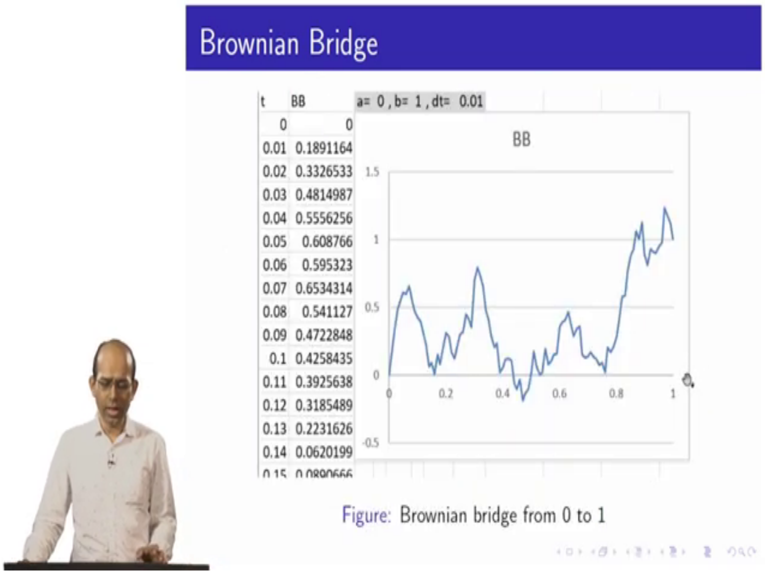

So, before that let us see bridges, you know this pattern and maybe simulation, if I have time I would come back to the simulation how to simulate, that means how did I actually compute this particular realization of Brownian bridge. So, here this is a Brownian bridge from 0 to 1, so, a here is 0, b here is 1 okay and time is horizontal axis and then this is actually, I mean simulated from the SDE.

Okay the stochastic differential equation, what we have obtained from that we have stimulated. One can actually stimulate from their direct definition also, but here I have simulated from the SDE to check that how nicely it is coming. Okay so if I have time I would come back to the simulation procedure also.

So, let us go back to our question that from point c to b, so it is at time tau, how or what is the appropriate time change so that the new process can also becomes under the new scaling becomes a Brownian bridge. So that was the question.

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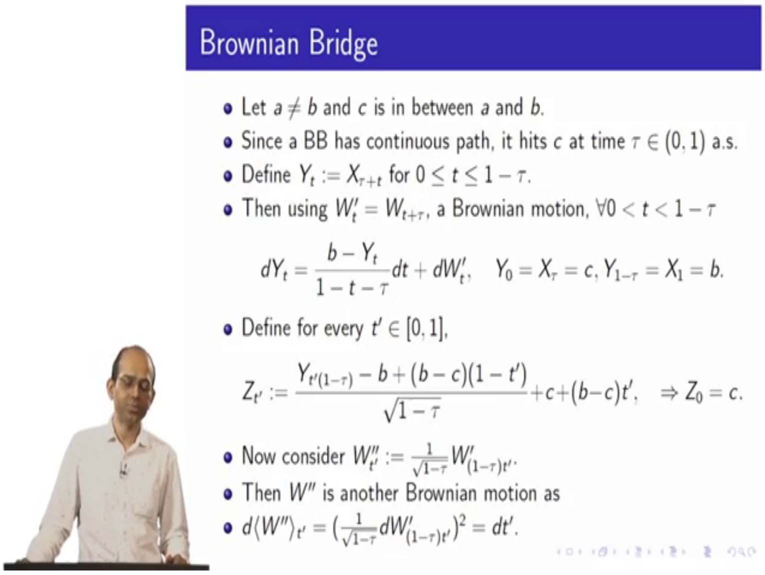

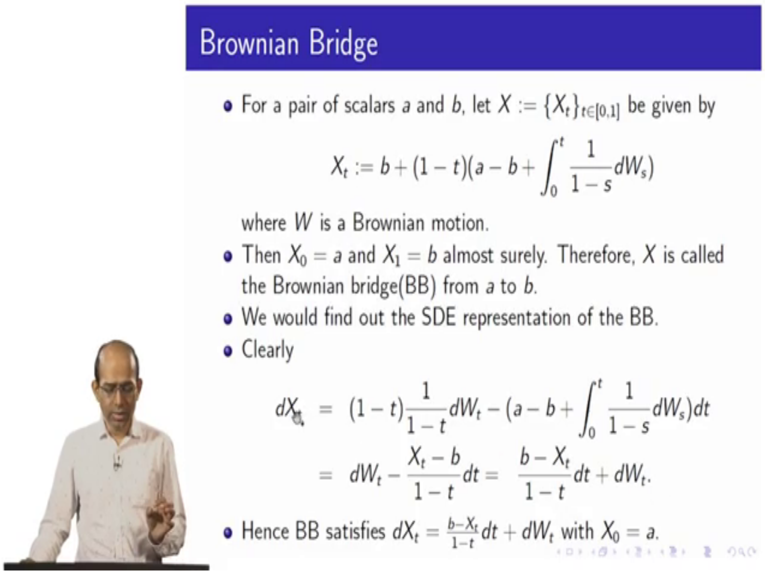

So, assume that a is not equals to b and c is in between a and b. So, since Brownian bridge has continuous path, it hits at, you know c at some point tau, so hitting time tau, which is of course between 0 and 1, almost surely it would hit here and then we define Y t, okay so this is doing, you know step by step time change is equal to defined as X tau plus t.

That means Y 0 is equal to X tau. So, Y is equal to X tau and then t is running between 0 to 1 minus tau, I cannot talk about t more than 1 minus tau because that is not defined because when t is equal to 1 minus tau, I get X tau plus 1 minus tau that is X 1, okay X is not defined for a time more than 1, so anyway time is here between 0 to 1 minus tau.

So, that shows us the reason that we need to a scale the time, okay. The shifting does not help the scaling is required, okay we do that. So, we define one Brownian motion W prime t which is W t plus tau. So, it is just shifted process, so from tau onward.

So, here t is between 0 to 1 minus tau okay and then here, what we do? We take dY t is equal to b minus Y t divided by 1 minus t. So, this Y t as I have defined here would satisfy this is SDE. So, this we can obtain directly from the SDE, what we have obtained here. So, dX t is equal to b minus X t by 1 minus t dt plus dW t here. So, t plus tau is replaced by this thing Y t correct.

So, if I replace t by t plus tau all the cases, so here I would get 1 minus t minus tau, here I would get W t plus tau, here also t plus tau, but X t plus tau is Y t. So, that we are going to use so you would get that 1 minus t minus tau and W t plus tau is W prime t, so dW prime t, and here and the numerator you would get b minus X t plus tau but X t plus tau is Y t.

So, it get in a numerator b minus Y t. So, we get this new SDE for you know, for this process Y, where Y 0 is equal to X tau goes to 0, X tau. And X tau is c by definition of tau. And then Y 1 minus tau is X 1 because you know, 1 minus tau here is X 1and X 1 is b by definition of X.

Okay so, we know pretty sure that we understand now Y is just you know, starting at time 0 there at c and at 1 minus tau, it is a random time there it is touching b but still it is random correct. So, it is not exactly like a Brownian bridge, you have to do something more. There is something more comes here. So, this is our proposal that let us define a process, new process Z.

So, Z of t prime where t prime is between 0 and 1, Z of t prime is defined as this fraction plus c plus b minus c times t prime. This fraction in a numerator we have Y t prime 1 minus tau, see here that earlier, the time interval on which y is defined has the length 1 minus tau that is possibly less than 1 okay. So, but t prime is between 0 to 1, so what we are doing is the t prime into 1 minus tau we are taking.

So, when t prime is 0, this is Y 0, when t prime is 1, this you know this time would be 1 minus tau. So, the whole, you know process Y, you know that is scaled into this. However, we also need to adjust this little more, so Y t prime into 1 minus tau minus b plus b minus c times 1 minus t prime divided by square root of 1 minus tau. So, this depends on tau of course, I mean that is expected.

So, here tau appears plus c plus b minus c, t prime. It looks little complicated, but it is not that complicated actually this is the natural thing to do actually, because I would like to match it at time 1 to b, so I have subtracted here and then later I have added also because when t prime is equal to 1, you will get b there from that.

So, let us just analyze, what did we obtain here? So, put t prime is equal to 0, both sides, I got Z 0. So, Z 0 is equal to Y 0 minus b plus b minus c. So, this b and this minus b would cancel, so I would get Y0 minus c. Y 0 is anyway c. So, Y 0 minus c is 0, so numerator is 0 okay. On the right hand side, I have, here I have get t prime is equal to 0 that this factor would vanish, so I would get c. So, Z 0 is equal to c.

Now we see from this definition, what is Z 1? So, we would put t prime is equal to 1 here, if we put then I would get Y of 1 minus tau, that is b anyway, so b minus b, that is 0. And then t prime is equal to 1, so 1 minus 1 is 0, so numerator is 0. And then here t prime is equal to 1 if I put then I would get b minus c, but this c and this minus c would cancel each other so I would get b.

So Z of 1 is b. So, Z is connecting c to b it during the time interval 0 to 1 okay. So, surely this is a very nice candidate for Brownian bridge but any process which connects you know, the 2 points at time 0 and 1 is need not be a Brownian bridge. I mean it should have the similar statistical property also. So, that is the thing we need to check okay. And that is a thing for that we actually have appropriately chosen that denominator, okay.

So, let us see that. So, to prove that this process Z is indeed a Brownian bridge from c to b, what are you going to do? We are going to show that these Z satisfies the stochastic differential equation. So, that stochastic differential equation I mean, this characterization we have already obtained that if a process satisfies this SDE then it is a bridge between a to b, a appears here b appears here.

So, this is the characterization we would consider and then, so now for proving that we need a little more. So, we define another Brownian motion W double prime. So, that is defined as W double prime X t prime is defined as 1 over square root 1 minus tau, W prime of 1 minus tau times t prime okay. So, here this why is it a Brownian motion? I mean that is subject to, you know proof, we are going to prove that.

But before that let us see that what happens that here I have changed the time since tau is less than 1, so 1 minus is smaller than 1. So, this process, you know runs little slowly okay. However, we also when you divide it, you know by square root of 1 minus tau smaller number. So, that means we have actually scaled up I mean this fraction is more than 1 okay. So, to prove that this is a Brownian motion what we are going to do is that first we check whether it is a continuous process? Yes, it is a continuous process, the way it is defined since W prime is continuous this is continuous.

And next is that what is its quadratic variation? So, then we are using Levis characterization of Brownian motion that if the quadratic variation is identity function then it is a Brownian motion. So, dW double prime t prime, so that we obtain as we have seen in the last tutorial session that to compute it, we are going to just take the corresponding, you know different square of the differentials for competition.

So, here 1 over square root of 1 minus tau and here is the differential is dW prime 1 minus tau t prime and the whole square. So, then from here this part I would get 1 over 1 minus tau square of the square root is 1 over 1 minus tau. And for this the quadratic variation of W prime appears and that would be exactly you know since this is Brownian motion it will be 1 minus tau times t prime okay, dt prime.

And this 1 minus tau what would appear and here in the denominator also you have 1 minus tau, they would cancel each other and what would remain is just dt prime. So, that justifies as a W double prime defined this way is also a Brownian motion. Okay so this is something we are going to use to obtain an SDE for the process Z, okay.

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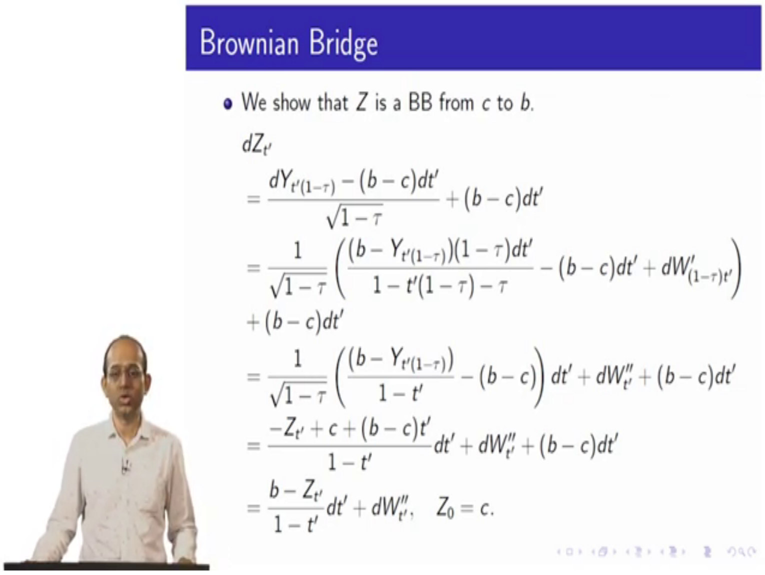

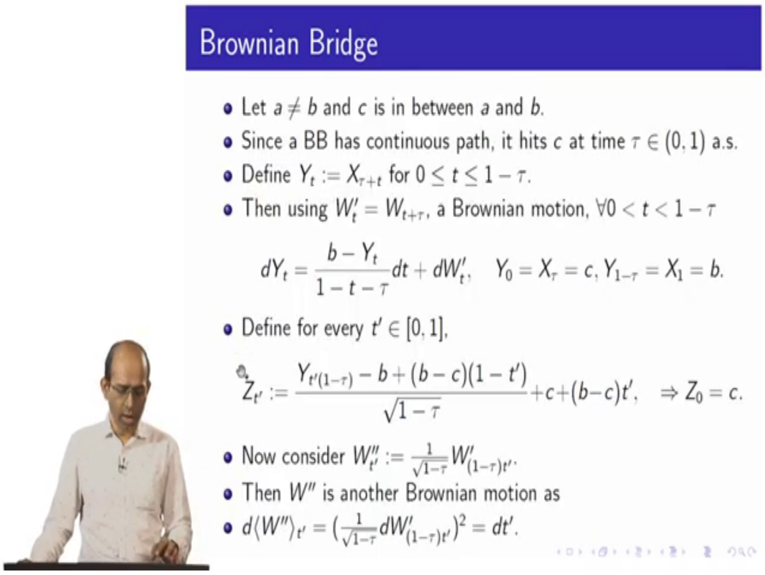

So, this is the way we do, so here we show that Z is a Brownian bridge from c to b. So, we show that okay Z satisfy dZ t prime is equal to b minus Z t prime divided by 1 minus t prime dt prime plus dW double prime t prime and Z 0 is equal to c. So, this thing okay I am going to explain it first I see that let us look at the result, end result is that Z satisfy this SDE.

So from definition not only definition after the definition we derived that is a characterization of the Brownian bridge using stochastic differential equation. So, there what we have seen is that, if some process satisfies this equation, then we say that this is a Brownian bridge, the continuous solution of this equation would the Brownian bridge and is starting from c to b, the bridge from c to b.

So, the proof starts here, that we are having the Z t prime is equal to this thing and we take the differential of both sides. So, when we take the differentials, you know this would not appear here, but the dY would appear here from here minus dt prime times b minus c would appear here, from here dt prime b minus c would appear.

So, b minus c dt prime appear from that side and here dY t prime, etc and minus b minus c, dt prime okay. So, I have from here there I have obtained this equation. Now, from here we do more you know, analysis here. So, what we do is that we use the formula for SDE for Y. So, Y itself satisfies one SDE that we have written here. So, we substitute that part.

So, here what dY t we have we substitute this by the part on the right hand side. However, here it is written in terms of t but there it is different. So, that we have to appropriately change okay. So, here you have t prime into 1 minus tau.

So, here everywhere, wherever I see t I should write down there t prime into 1 minus tau, t prime into 1 minus tau, etc okay. So, here W prime t prime into 1 minus tau that I need to write down here, so that I do here, so b minus Y t prime into 1 minus tau, and here 1 minus t was there, so I am writing 1 minus t prime into 1 minus tau here and then, so dt I should write 1 minus tau dt prime.

And here so these things and from Brownian motion part I should write dW prime 1 minus tau t prime. So, these are the things we just do. And then these b minus c dt prime I just have written here, sounds good.

Now, next I need to simplify this okay. So first simplification is that we use the definition of W double prime that W prime 1 minus tau t prime is, you know W double prime. So, 1 over this fraction is also there, this is W double prime, so that we use here 1 over square root 1 minus tau is already present and this part is there.

So, these together would give me dW double prime, t prime, from here I am going to get it. And so from here this side, what we are going to get is b minus Y t prime 1 minus tau see here, a very nice thing has happened, because 1 minus tau and here minus t prime 1 minus tau, so I can take 1 minus common. If I take 1 minus tau common then it will be 1 minus tau into 1 minus t prime. So, 1 minus t prime stays here and that 1 minus tau cancels with this 1 minus tau.

So, after this simplification, I would get b minus Y t prime 1 minus tau divided by 1 minus t prime okay and here minus b minus c, so, dt prime okay. And then this part I have already seen this term is this term okay as before. Okay so now we back substitute Z, because for Z we have one equation, so this is a formula for Z in terms of Y.

So, here in the denominator, I have 1 over square root of 1 minus tau, and numerator we have this Y term minus b plus b minus c times one minus t prime. So, these are things, but here, when we actually write down this also in the numerator, I would get b minus c into 1 minus t prime as it appears here. So, this would appear on the in the numerator because 1 minus t prime is here, correct.

If I put it in above, I would get that and then this you know Y and minus b also is here. So, only the sign is negative, here I had plus b minus c into 1 minus t prime here we get minus b minus c into 1 minus t prime and only the sign is negative. So, I get minus Z t prime from there, but there are some other terms this term, so this term would go here, so minus Z t prime plus c plus b minus c t prime.

So, plus c plus b minus c t prime okay from here we are getting this numerator. However in the denominator we have 1 minus t prime and this square root of 1 minus tau would not appear because that is that was also already there, right, we have taken care, okay.

So, only 1 minus t prime would remain here and numerator would be this and dt prime as it is and this part was as before I am just rewriting this thing. Now here it is just a matter of you know, manipulation because here we have b minus c dt prime. And then if I put it together, so we take dt prime common, then I would get b minus c into 1 minus t prime from there also. So, the plus sign here, b minus c into 1 minus t prime, but minus sign is there.

So minus t prime into this thing would cancel with this okay. So, what would get that plus b minus c to appear here, but c plus c is here, so this plus c and that minus c would cancel, so I would get only b here. So, b minus Z t prime would remain in the numerator and denominator is 1 minus t prime dt prime and I would have only this more term dW double prime t prime.

So, by this we have finished the proof that the Z satisfies this stochastic differential equation, and therefore Z is a Brownian bridge from c to b okay, thank you.