Introduction to Probabilistic Methods in PDE

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Lecture 04

Random Variable

The title of the course is Probabilistic Method in PDE, in this course what we are going to do is we are going to see the fundamental basic tools of probability mostly those are stochastic processes and stochastic calculus and then we are going to say how those would be used to solve some PDE problem.

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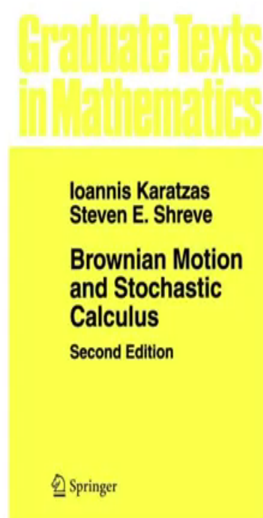

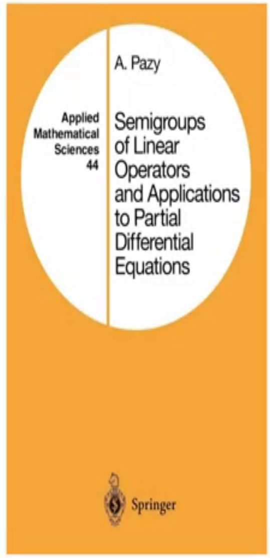

So, for that I am going to use a particular book mostly that is the Brownian Motion and Stochastic Calculus by Karatzas and Shreve, there be another book that is on Semi-group theory written by A Pazy. So, more than half of the course would be based on few chapters of this book. However I will not start from the very beginning of the book because the beginning is dedicated for on stochastic process some fundamental basic things. I would be rather little fast on few topics, first few chapters. I would like to keep this course as self-content as much as possible.

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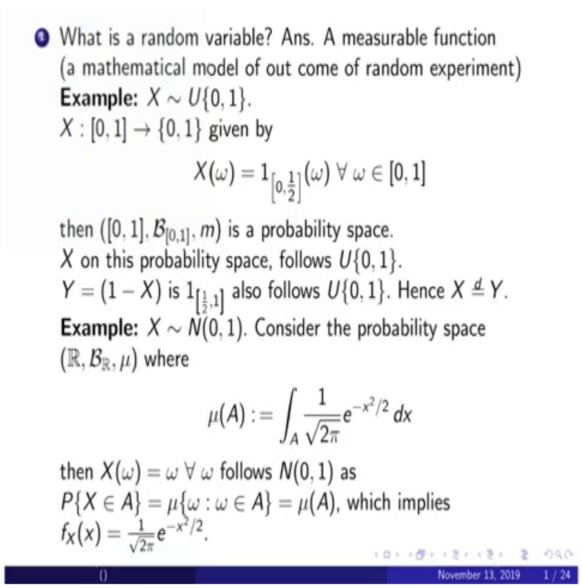

So, let me open the first slide, in the first slide we are going to see what do you mean by random variable, that is the first thing we start for in a course in probability theory. So what is a random variable? When we ask this question in a statistics course the answer is, result of a random experiment, outcome of an random experiment. However, when we talk about the mathematical probability theory, then we take the mathematical model of this random variable.

So, random variable is of-course outcome of a random experiment but instead of model, what is random experiment mathematically we just model this outcome as a measurable function in a probability measure space. So for our course whenever you say that X is a random variable, we mean X is a measureable function define on a probability space. So, let us start with some various simple example. If I say so, this example is saying that X and this symbol is denoting the follows that means X as a distribution, U and then a set containing only two element, 0 and 1.

So, what is this U denoting? U denotes uniform, so I mean here X is a random variable which follows uniform distribution on the binary set 0 and 1. That means that it has equal probabilities at point 0 and point 1. Or in other words X is a random variable which takes value 0 with probability half and 1 with probability half. So, this is distributional description of a random variable.

And that type of random variable you can find in nature say for example tossing an unbiased coin and head stands for 1 and tails stand for 0. Then you can actually replicate such random variable. But how can we mathematically model that random variable. So let us see how are we modeling that random variable mathematically. We are taking a function X from interval 0 1 to the set binary set 0 and 1. Left hand side you have the closed interval 0 and right hand side you have binary set 0 and 1.

And this is given by this function is X of omega is equal to indicator function of the interval 0 to half and this indicator function evaluated omega. So, now right hand side there is only two values 1 and 0, it takes value 1 when omega is inside the interval 0 and half and it takes value 0 when omega is outside this interval 0 and half, correct? Then this indicate a function gives you 0.

So, here precisely this X is a candidate for this example but let us see whether this really puts half of probability is for 0 and 1 further we need to mention, what are the weights and what is the probability, so here we specify that the probability space. Now we have specifying, so 0 ,1 interval is the non-empty set on that we are considering the sigma algebra, (())(05:29) sigma algebra on closed 0 1.

And m stands for the lebesgue measure, m stands for the lebesgue measure, so now this is a measure space and then total measure of the set 0 1 is 1 because lebesgue measure of 0 1 interval the length is just 1, so measure is 1. So it is a finite measure space where total measure or the set is 1 so it is a probability space. So, by saying that I am also recollecting the definition of probability measure. So, here we have a probability measure space and we have a function X, define on 0 1, this way.

And this function is of-course a Borel measurable function, why is it so because it is indicator function of a Borel set, closed interval 0 to half is a Borel set. So, this random variable this measurable, this function is measurable with respect to the measure space this. So, here as I have noted what is my random variable, random variable is nothing but a measurable function, so X is a really a measurable function here.

Now, we calculate the probabilities, what is the probability that X is 0? What is the probability that X is 0, so for that we are going to find out the set, the measure of the set of omega where X becomes 0, that is nothing but compliment of 0 to half inside this 0 1, so that is open half closed 1 that has measure lebesgue measure half, the length is half, so probability X is equal to 0 is half here.

So, this actually proves that the construction what we have done X here is indeed a random variable whose distribution is uniform on 0 1. Now this random variable has a popular name amongst statistician we call that Bernoulli trail. So, let us consider another function Y, one can say that can we have another measurable function having the same distribution, the answer is of course yes, there actually plenty of such, so this is only one another example.

Y, it is defined as 1 minus X so, 1 minus X is indicator function of here there is a mistake, it should be open half one closed. So, that random variable follows uniform 0 again, due to the same reason. So, here X and Y are two different measurable functions and however they are equal in distribution, fine. Now, we go for little more details of some other random variables, I mean we are taking normal random variable.

So, earlier it was Bernoulli trail which is a discrete random variable, now we are going to talk about normal random variable which is a contrast random variable. And this example I would consider to illustrate two other different ways to find out two different two different way of modeling a given random variable. So here this symbol says X is a distribution which follows normal random variable 0 1, that we mean 0 and variance 1.

We consider the probability space the real number set of real numbers Borel sigma algebra on R and Mu a measure which is defined following way, Mu of set A where A is a Borel set is defined as integration over the set A of this integrant. This integrant is 1 over square 2 Pi e to the minus x square by 2 dx. Now, if you consider this measure space, we can as the (())(10:11) probability space, yes because Mu measure of the whole R is integration of this function over whole R and that integration is 1. So, it is a probability space.

Now, question is that what should be my random variable, correct. So, what should be my that measurable function, so that under this probability space it is normal random variable with mean 0 variance 1. So, here we do that, we take X of omega is equal to omega just identity function and then the probability of X belongs to A this is equal to therefore Mu of because of probability is now for us Mu here.

So, this is nothing but Mu of omega such that omega is in A, because X omega is omega here. But that is nothing but Mu of A which implies that this probability density function of this random variable is this. Because Mu of A is this right hand side is nothing but this integration, left hand side I have probability X belongs to A, right hand side I have integration of a function on the set A. So this left hand side I have A here, right hand side I have A here, so that implies that the probability density function of X should be this integrant.

So, what does it mean? It means that capital X is normal random variable because we know that this function is the probability density function of standard normal random variable. So, this proves that this particular choice of measureable function and this particular choice of probability space together can model a normal random distribution, standard normal distribution.

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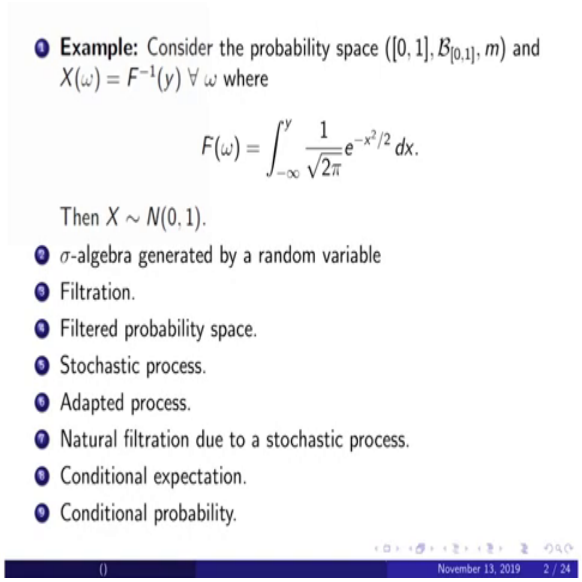

So, we go to the next slide, now as I have promised now I am showing another way of modeling standard normal distribution. Earlier what we have done, we have taken a very canonical choice of random variable, just Identity function. So, we have not included any you know information in that. However, all the normal random variables you know for the normal random variable whatever the distributions information that we have encoded in the measure itself. But here another complimentary manner. Here we take a very flat standard measure space closed 0 and interval Borel sigma algebra and 0 1 and the lebesgue measure.

So, that means this choice of measure space is nothing incorporate any information on normal random variable. However, we chose the measurable function here so that it matches the normal distribution. So, this taken as F inverse of Y for here this be Y, for all Y, for all Y real where here also it should be Y. So, f of y is equal to minus infinity to y 1 over square 2 Pi e to the minus x square by 2 dx.

So we define this way, then also we can prove that X follows normal random variable, how do you prove that? Because, we just calculate the probability of capital X less than or equal to Y. So, probability X less than or equal to Y is same as, so when I say X I mean F inverse of y so, sorry here actually here it should be F in omega, so here omega but here y. So, F inverse of omega is less than or equal to Y that would imply that Y is less than or equal to F of omega. Why? Because F here is a monotonic function its invertible.

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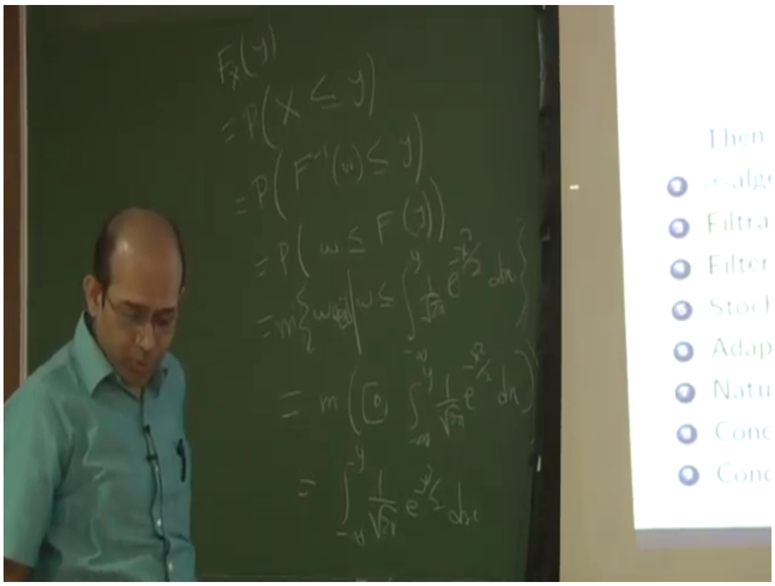

Its probability X less than or equal to Y is same as probability F inverse omega less than or equals to y it is probability omega less than or equal to F of y is equal to probability omega such that, so what is F of y? F of y is so, I write down here clearly set of all omega such that omega less than or equal to minus infinity to y 1 over square root 2 Pi e to the minus x square by 2 dx.

So, here P is nothing but the lebesque measure m, lebesgue measure m so here it is n measure so lebesgue measure of this set omega so, here omega is between 0 to 1 so, omega is between 0 to 1 so, here we want to find out the lebesgue measure of that. So, it is nothing but lebesgue measure of closed interval 0 to minus infinity to y 1 over square root 2 Pi e to the minus x square d by 2 dx. So, this intervals lebesgue measure that is the length.

But the length is exactly this value, because this is positive number, is it this value this minus infinity to y 1 over square root 2 Pi e to minus x square by 2 dx and what is this? So, this is the probability density function of normal random variable, so this is nothing but the CDF of normal random variable, so this if I mean this is that normal random variable CDF so, here this was my X less than equal to y so this is CDF of random number X, so this matches. So X has the same distribution of n, n is normal random variable.

So, this is the proof that if I consider this pair, this particular probability space and this particular measureable function then also I can model a normal random variable.